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A New Class of Third Derivative Fourth-Step Exponential Fitting Numerical Integrator for Stiff Differential Equations

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Abstract

In this paper, we construct a new class of four-step third derivative exponential fitting integrator of order six for the numerical integration of stiff initial-value problems of the type: $y' = f(x, y),\ y(x_0) = y_0$. The implicit method has free parameters which allow it to be fitted automatically to exponential functions. For the purpose of effective implementation of the new proposed method, we adopted the techniques of splitting the method into predictor and corrector schemes. The numerical analysis of the stability of the new method was discussed; the results show that the new method is A-stable. Finally, numerical examples are presented, to show the efficiency and accuracy of the new method

Keywords: Third derivative four-step, exponentially fitted, A-stable and stiff differential equations.
MSC2010: 65L05, 65L06

1 Introduction

In this paper, we consider the initial value problems of the form:

$$y' = f(x, y),\ \ y(x_0) = y_0, \tag{1.1}$$

whose solutions exhibit a pronounced exponential behaviour. For the numerical solution of the initial value problem above, classical general–purpose method of Runge-Kutta and Linear Multistep methods cannot produce satisfactory results due to the special structure of the problem. According to [1], the possible ways to construct the numerical methods adapted to the character of the solutions can be obtained by using exponential fitting techniques. So in this work, we constructed a four step third derivative exponentially fitted scheme to solve stiff initial value problems in ordinary differential equations efficiently. Many new methods have been developed in recent years, which satisfy certain stability requirements. The A-stability (stability property) is desirable in formulas to be used in the solution of stiff systems of differential equations especially from chemical kinetics and the discretization of partial differential equations. [2], proved that A-stable linear multistep formulas must be implicit, its maximum order is two and of those of second order, the one with the smallest truncation error coefficient is the trapezoidal rule. In nearly every linear systems of differential equation, which have widely disperse eigenvalues, high order A-stable formulas are particularly appropriate since they allow integration to proceed with a larger step size. Thus, the need to develop a high order A-stable implicit multistep formula, which uses linear combinations of derivatives higher than the first, give rise to the development of multiderivative multistep formulas. [3], and [4], pointed out that Multiderivative methods give high accuracy and possess good stability properties when used to solve first order initial value problems in ordinary differential equation. A-stable multiderivative multistep formulas designed for solving stiff systems of differential equations includes, [4], [5], [6], [7], [8], [9], [10] and [11].

The idea of using exponentially fitted formulas for the appropriate numerical integration of certain classes of stiff systems of first order ordinary differential equation of the form in (1.1) which was originally proposed by [12], is to derive integration formulas containing free parameters (other than the step length of integration) and then to choose these parameters so that a given function e^u, where u is real, satisfies the integration formula exactly.

[12] derived three 1-step integration formulas with orders ranging from 1-3. Their results revealed that for all choices of the fitting parameter u, their formulas are A-stable.

2 THE GENERAL MULTIDERIVATIVE MULTISTEP METHOD

The general multiderivative multistep method is given by,

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = \sum_{j=1}^{s} h^j \sum_{i=0}^{k}\gamma_{j,i} f_{n+i}^{j-i}, \quad n = 0,1,2, \tag{2.1}$$

where f_{n+i}^{j} is the j^{th} derivative of $f(x,y)$ evaluated at $(x_{n+1}, y_{n+1}), \alpha_i$ and $\gamma_{j,i}$ are real constants with $\alpha_k \neq 0$ and y_{n+1} is the appropriate numerical solution evaluated at the point x_{n+i}. In order to remove the arbitrary constant in (2.1), we shall always assume that $\alpha_k = +1$, and $\sum_{i=0}^{k}|\alpha_i| > 0$ and $\sum_{i=0}^{k}|\gamma_{j,i}| > 0, \quad j = 1,2,\dots,s$.

3 DERIVATION OF FAMILY OF FOUR STEP EXPONENTIALLY FITTED METHODS

To develop a four-step, third derivative multiderivative exponentially fitted formulas. (i.e.$k = 4$ and $s = 3$). For this purpose, (2.1) is reduced to,

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\gamma_{1,i} f_{n+i} + h^2\sum_{i=0}^{k}\gamma_{2,i} f_{n+i}^{(1)} + h^3\sum_{i=0}^{k}\gamma_{3,i} f_{n+i}^{(2)}. \tag{3.1}$$

From (3.1), we obtain

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i} + h^2\sum_{i=0}^{k}\phi_i g_{n+i} + h^3\sum_{i=0}^{k}\omega_i v_{n+i} \tag{3.2}$$

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and

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k+1}\beta_i f_{n+i} + h^2\sum_{i=0}^{k}\phi_i g_{n+i} + h^3\sum_{i=0}^{k}\omega_i v_{n+i}, \tag{3.3}$$

where $\beta_i = \gamma_{1,i}, \phi = \gamma_{2,i}, \omega_i = \gamma_{3,i}$. The implementation of the proposed formulas involves a pair of formulas, that is predictor and corrector formulas. Thus, (3.1) serves as the predictor and while equation (3.2) serves as the corrector. where,

$$\begin{cases} f_{n+i} &= f(x_{n+i}, y(x_{n+i})) = y'_{n+i} \text{ and } g_{n+i} = f'(x_{n+i}, y(x_{n+i})) = y''_{n+i} \\ v_{n+i} &= f''(x_{n+i}, y(x_{n+i})) = y'''_{n+i} \end{cases} \tag{3.4}$$

are respectively the first, second and third derivatives of y_{n+i}. When we are deriving exponentially fitted multistep methods, the approach is to allow both (3.2) and (3.4) to posses free parameters order than the mesh size $'h'$ which allow it to be fitted automatically to exponential function.

4 DERIVATION OF ORDER SIX FORMULA

To develop an exponentially fitted method, we choose one free parameter each for both the predictor and corrector formula. The free parameter is used to annihilate the leading error term of the method.

The derivation of predictor-corrector integration formula of order 6 involves two stages. In the first stage, we derive the order six predictor formula by setting $\alpha_1 = \alpha_2 = \alpha_3 = 0, \alpha_4 = 1, \beta_1 = \beta_2 = \beta_3 = 0, \beta_4 = t, \gamma_1 = \gamma_2 = \gamma_3 = 0, \omega_1 = \omega_2 = \omega_3 = 0$ in (3.2) to obtain the following six set of simultaneous equations.

$$\begin{cases} \alpha_0 + 1 = 0 \\ 4 - \beta_0 + t = 0 \\ 8 - 4t - \gamma_0 - \gamma_4 = 0 \\ \frac{32}{3} - 8t - 4\gamma_4 - \omega_0 - \omega_4 = 0 \\ \frac{32}{3} - \frac{32}{3}t - 8\gamma_4 - 4\gamma_4 = 0 \\ \frac{128}{15} - \frac{32}{3}t - \frac{32}{3}\gamma_4 - 8\omega_4 = 0. \end{cases} \tag{4.1}$$

Solving (4.1) simulteneously we obtained $\alpha_0 = -1, \alpha_4 = +1, \beta_0 = 4 - t, y_0 = \frac{28}{5} - 2t, y_4 = \frac{12}{5} - 2t, \omega_0 = \frac{16}{5} - \frac{4}{3}t, \omega_4 = -\frac{32}{15} + \frac{4}{3}t.$

By substituting the values of the parameters into (3.2), we obtained the predictor formula as:

$$\begin{cases} y_{n+4} &= y_n + h((4-t)y'_n + ty'_{n+4}) + h^2\left(\left(\frac{28}{5} - 2t\right)y''_n + \left(\frac{12}{5} - 2t\right)y''_{n+4}\right) \\ & +h^3\left(\left(\frac{16}{5} - \frac{4}{3}t\right)y'''_n + \left(-\frac{32}{15} - \frac{4}{3}t\right)y'''_{n+4}\right) \end{cases} \tag{4.2}$$

Now, for exponential fitting purpose, we apply (4.2) to scalar test function

$$y\prime = \lambda y, y(0) = 1, \tag{4.3}$$

to obtain:

$$\frac{\overline{y}_{n+4}}{y_n} = \frac{1+(4-t)u+\left(\frac{28}{5}-2t\right)u^2+\left(\frac{16}{5}-\frac{4}{5}t\right)t^3}{1+ut-\left(\frac{12}{5}-2t\right)u^2-\left(-\frac{32}{15}+\frac{4}{3}t\right)u^3} = R(\overline{u}). \tag{4.4}$$

For the purpose of stability analysis, we obtain the free parameter t from (4.4). But $\frac{\overline{y}_{n+4}}{y_n} = e^{4u}$ so that (4.4) yields,

$$t = \frac{1-e^{4u}+4u+\left(\frac{28}{5}+\frac{12}{5}u^2+\left(\frac{16}{5}-\frac{32}{15}e^{4u}\right)u^3\right.}{(1-e^{4u})u+(2+2e^{4u})u^2+\left(\frac{4}{3}-\frac{4}{3}e^{4u}\right)u^3} \tag{4.5}$$

Again to obtain the corresponding order 6 corrector formula, we obtain seven set of simultaneous equation from (3.4) as follows.

$$\begin{cases} \alpha_0+1=0 \\ 4-\beta_0-\beta_4-r=0 \\ 8-4\beta_4-5r-\gamma_0-\gamma_4=0 \\ \frac{32}{3}-8\beta_4-\frac{25}{2}r-4\gamma_4-\omega_0-\omega_4=0 \\ \frac{128}{15}-\frac{32}{3}\beta_4-\frac{625}{24}r-\frac{32}{3}\gamma_4-8\omega_4=0 \\ \frac{32}{3}-\frac{32}{3}\beta_4-\frac{125}{6}r-8\gamma_4-4\omega_4=0 \\ \frac{256}{45}-\frac{128}{15}\beta_4-\frac{625}{24}r-\frac{32}{3}\gamma_4-\frac{32}{3}\omega_4=0. \end{cases} \tag{4.6}$$

We impose the same condition as in predictor, and in addition, we let $\beta_5 = r$ as free parameters $\alpha_4 = +1$, the values of the unknown parameters are obtained from (4.6) as, $\alpha_0 = -1, \alpha_2 = +1, \beta_0 = \frac{2}{5}-\frac{689}{64}r, \beta_4 = \frac{18}{5}+\frac{625}{64}r, \beta_5 = r\gamma_0 = -\frac{8}{5}-\frac{6155}{256}r, \gamma_4 = -\frac{24}{5}-\frac{5125}{256}r, \omega_0 = -\frac{8}{5}-\frac{2475}{128}r, \omega_4 = \frac{8}{3}+\frac{1125}{128}r$.
When these values are substitute into (3.4), we obtain the fifth order corrector formula as,

$$\begin{cases} y_{n+4} = & y_n + h\left(\left(\frac{2}{5}-\frac{689}{64}r\right)y'_n+\left(\frac{18}{5}+\frac{625}{64}r\right)y'_{n+4}+ry'_{n+5}\right) \\ & +h^2\left(\left(-\frac{8}{5}-\frac{6155}{256}r\right)y''_n+\left(-\frac{24}{5}-\frac{5125}{256}r\right)y''_{n+4}\right) \\ & +h^3\left(\left(-\frac{8}{5}-\frac{2475}{128}r\right)y'''_n+\left(\frac{8}{3}+\frac{1125}{128}r\right)y'''_{n+4}\right) \end{cases} \tag{4.7}$$

By applying (4.7) to test function (4.3) we obtain,

$$\frac{y_{n+4}}{y_n} = \frac{1+\left(\frac{2}{5}-\frac{689}{64}r\right)u-\left(\frac{8}{5}+\frac{6155}{256}r\right)u^2-\left(\frac{8}{5}+\frac{2475}{128}r\right)u^3+ru\frac{y_{n+5}}{y_n}}{1-\left(\frac{18}{5}+\frac{625}{64}r\right)u+\left(\frac{24}{5}+\frac{5127}{256}r\right)u^2-\left(\frac{8}{3}+\frac{1125}{128}r\right)u^3} = R(u). \tag{4.8}$$

We need to obtain the relation $\frac{y_{n+5}}{y_n}$ in (4.8) for the purpose of exponential fitting condition. We established that

$$\frac{y_{n+4}}{y_n} = e^{4u} = R(u).$$

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Also,

$$y_{n+5}y_n = e^{5u} = \left(e^{4u}\right)^{5/4}.$$

Equation (4.8) now becomes:

$$\frac{y_{n+4}}{y_n} = \frac{\left(1 + \left(\frac{2}{5} - \frac{689}{64}r\right)u - \left(\frac{8}{5} + \frac{6155}{256}r\right)u^2 - \left(\frac{8}{5} + \frac{2475}{128}r\right)u^3 + ruR(\overline{u})\right)}{\left(1 - \left(\frac{18}{5} + \frac{625}{64}r\right)u + \left(\frac{24}{5} + \frac{5127}{256}r\right)u^2 - \left(\frac{8}{3} + \frac{1125}{128}r)u^3\right)} = R(u) \tag{4.9}$$

Equation (4.9) now unites both the parameter and corrector formulas which is capable of solving stiff systems. We obtain the value of the free parameter r from (4.9) as,

$$r(u) = \frac{\left(1 + e^{4u} + \left(\frac{2}{5} + \frac{18}{5}e^{4u}\right)\right)u - \left(\frac{8}{5} + \frac{24}{5}e^{4u}\right)u^2 + \left(-\frac{8}{5} + \frac{8}{3}e^{4u}\right)u^3}{\left(\frac{689}{64} - \frac{625}{64}e^{4u} - e^{5u}\right)u + \left(\frac{5125}{256}e^{4u} + \frac{6155}{256}\right)u^2 + \left(\frac{2475}{128} - \frac{1125}{128}e^{4u}\right)u^3}. \tag{4.10}$$

5 STABILITY CONSIDERATION OF THE METHOD

To examine the stability conditions required by this method, it is expected by maximum modulus theorem that the stability function of the method given by (4.10) satisfies $|R(u)| < 1$. In order to determine the interval of absolute stability of the method, we find limits of both $t(u)$ and $r(u)$ as $u \to 0$.

$$\lim_{u\to-\infty} t(u) = \lim_{u\to-\infty} \frac{1 - e^{4u} + 4u + \left(\frac{28}{5} + \frac{12}{5}e^{4u}\right)u^2 + \left(\frac{16}{5} - \frac{32}{15}e^{4u}\right)u^3}{(1 - e^{4u})u + (2 + 2e^{4u})u^2 + \left(\frac{4}{3} - \frac{4}{3}e^{4u}\right)u^3} = \frac{12}{5}$$

and

$$\lim_{u\to 0} t(u) = \lim_{u\to 0} \frac{1 - e^{4u} + 4u + \left(\frac{28}{5} + \frac{12}{5}e^{4u}\right)u^2 + \left(\frac{16}{5} - \frac{32}{15}e^{4u}\right)u^3}{(1 - e^{4u})u + (2 + 2e^{4u})u^2 + \left(\frac{4}{3} - \frac{4}{3}e^{4u}\right)u^3} = 0.$$

This shows that $t \in \left(\frac{12}{5}, 0\right)$ if $u \in (-\infty, 0]$. Similarly, from (4.10) we have,

$$\begin{aligned}\lim_{u\to-\infty} r(u) &= \lim_{u\to-\infty} \frac{\left(1 + e^{4u} + \left(\frac{2}{5} + \frac{18}{5}e^{4u}\right)\right)u - \left(\frac{8}{5} + \frac{24}{5}e^{4u}\right)u^2 + \left(-\frac{8}{5} + \frac{8}{3}e^{4u}\right)u^3}{\left(\frac{689}{64} - \frac{625}{64}e^{4u} - e^{5u}\right)u + \left(\frac{5125}{256}e^{4u} + \frac{6155}{256}\right)u^2 + \left(\frac{2475}{128} - \frac{1125}{128}e^{4u}\right)u^3} \\ &= -\frac{1024}{12375}\end{aligned}$$

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and

$$\lim_{u\to 0} r(u) = \lim_{u\to 0} \frac{\left(1+e^{4u}+\left(\frac{2}{5}+\frac{18}{5}e^{4u}\right)\right)u-\left(\frac{8}{5}+\frac{24}{5}e^{4u}\right)u^2+\left(-\frac{8}{5}+\frac{8}{3}e^{4u}\right)u^3}{\left(\frac{689}{64}-\frac{625}{64}e^{4u}-e^{5u}\right)u+\left(\frac{5125}{256}e^{4u}+\frac{6155}{256}\right)u^2+\left(\frac{2475}{128}-\frac{1125}{128}e^{4u}\right)u^3}$$
$$= \frac{41672}{153650}.$$

Thus we found that $t \in \left(\frac{12}{5}, 0\right)$ and $r \in \left(-\frac{1024}{12375}, \frac{41672}{153650}\right)$. Now, we further verify analytically that the ranges of value of t and r represent the region of absolute stability.

u	t	r
-10	2.34314876	-0.08433447
-20	2.3707687	-0.08364902
-30	2.38033888	-0.08337217
-50	2.38812115	-0.08313357
-100	2.39403021	-0.08294685
-200	2.39700744	-0.08284767
-1000	2.3994009	-0.08276756
-2000	2.39969996	-0.08275757

Table 1: Parameter values of t and r.

From the table above, all the values of t and r are bounded within the ranges of $t \in \left(\frac{12}{5}, 0\right)$ and $r \in \left(-\frac{1024}{12375}, \frac{41672}{153650}\right)$. Thus, as u decreases the values of t and r increases monotonically.

6 NUMERICAL EXPERIMENTS

In the section, the Numerical Integrators derived are tested on several standard stiff value problems (IVPs) in Ordinary Differential Equations (ODEs). To show the effectiveness and validity of our newly derived methods, we present some numerical examples below. All numerical examples are coded in Fortran 77 and implemented on digital computer. However, for purpose of comparative analysis on the performance of the new scheme we denote AL5 as the new method, CH4, CH5-[15] method of order 4 and 5 respectively, J-K-[11], OK6-[17], AB7, AB8, NM9 represent [7], [10] methods of order 7, 8 and 9 respectively. F5-[8]) method of order 5, AF5-[13] and AG6-[6].

Example 6.1 *Non-linear stiff problems in [4]*

$$\begin{cases} y_1' = -0.013y_1 + 1000y_1y_3, \ \ y_1(0) = 1 \\ y_2' = -2500y_2y_3, \ \ y_2(0) = 1 \\ y_3' = 0.013y_1 - 1020y_1y_3 - 2500y_2y_3, \ \ y_3(0) = 0 \end{cases} \tag{6.1}$$

Step length h	Method	y_1	$y_2(1)$	$y_3(1)$
0.0625	F6	0.5882642680	1.0092344206	2.7914606478
	AL5	0.5882826880	1.0092403880	-2.7906557822
	AB8	0.5884667145	1.0090563343	-2.7919757498
0.1	F6	0.5882826884	1.0092403606	2.7914604750
	AL5	0.588282688041	1.009240388026	-2.790655782258
	AB8	0.5882826902	1.0092403584	-2.7914604809
Exact Solution		0.5882826881	1.0092403605	-2.7914604750
		Errors		
0.0625	F6	-1.8×10^{-5}	-5.9×10^{-6}	5.6
	AL5	-9.9×10^{-11}	2.8×10^{-8}	8.0×10^{-4}
	AB8	-1.8×10^{-4}	1.8×10^{-4}	5.2×10^{-4}
0.1	F6	3.0×10^{-10}	1×10^{-10}	5.6
	AL5	-5.9×10^{-11}	2.8×10^{-8}	8.0×10^{-4}
	AB8	2.2×10^{-8}	2.2×10^{-9}	6.3×10^{-9}

Table 2: Numerical result of the non-linear stiff problem.

From Table 2, it could be seen that our proposed method performed fairly well in terms of accuracy for $h = 0.0625$ when compared with other methods in existing literature.

Example 6.2 *Second order differential equation [9]. The accuracy of the new method is also illustrated*

$$\begin{cases} \frac{d^2y}{dx^2} & +1001\frac{dy}{dx} + 1000y = 0 \\ \\ y(0) & = 1, y'(0) = 1. \end{cases} \tag{6.2}$$

The system (6.2) can be rewritten as a first – order system;

$$\begin{cases} \frac{dy_1}{dx} & = y_2,\ y_1(0) = 1 \\ \\ \frac{dy_2}{dx} & = -1001y_2 - 1000y_1,\ y_2(0) = -1 \end{cases} \tag{6.3}$$

Thus we obtain a 2×2 system of stiff IVP. The eigenvalues of the Jacobian matrix of are $\lambda_1 = -1$ and $\lambda_2 = -1000$. The general solution of (4.2) is $y(x) = Ae^{-x} + Be^{-1000x}$. If we impose the initial conditions in $0 \leq x \leq 1$, the exact solution is $y(x) = e^{-x}$. The result of this problem using the newly derived methods are obtained at $x = 1$; as given in table 4.2 below.

Step length h	Method	$y(1)$	Error (y)
0.05	OK6	0.367879436	5.6×10^{-8}
	AG6	0.36787846	1.4×10^{-8}
	AF5	0.36787930	1.8×10^{-7}
	AL5	0.367879629	2.4×10^{-7}
Exact Solution		0.367879435	

Table 3: Numerical results on second order ODE at $x = 1$.

It will be observed from table (3) above, for $h = 0.05$ the numerical result of the problem considered revealed that the new method performed favorably with existing methods in the literature.

Example 6.3 ([11],[15], [17])

$$\begin{cases} y' = -y + 95z;\ y(0) = 1 \\ z' = -y - 97z;\ y(0) = 1 \\ x \in [0, 1] \end{cases} \tag{6.4}$$

The eigenvalues of the Jacobian matrix of the system are $\lambda_1 = \lambda_2$ and $\lambda_2 = -96$ with stiffness ratio 48. The exact solution is given as,

$$y = \frac{95e^{-2x} - 48e^{-96x}}{47} z = \frac{48e^{-96x} - e^{-2x}}{47}.$$

For comparison purpose, we have the following as; AB7, AB8 and NM9 to represent[7] " Two step third derivative methods order seven, eight and nine" respectively. F5 denote our proposed three-step second derivative scheme.

Method	**Step size** h	$y(1)$	**Error** (y)
J-K	h=0.0625	$0.2725503(3.0 \times 10 - 7)$	$-0.2879477(4.0 \times 10 - 9)$
CH4		$0.2735498(3.0 \times 10 - 7)$	$-0.2879471(3.0 \times 10 - 9)$
CH5		$0.27355005(3.0 \times 10 - 8)$	$-0.28794742(3.0 \times 10 - 9)$
AB7		$0.27354004(4.0 \times 10 - 5)$	$-0.28796321(6.0 \times 10 - 5)$
F4		$0.2735503(3.0 \times 10 - 7)$	$-0.2879477(3.1 \times 10 - 7)$
NM9		$0.27354004(7.9 \times 10 - 5)$	$-0.28794740(8.3 \times 10 - 7)$
F5		$0.27355003(6.4 \times 10 - 9)$	$0.28794741(6.7 \times 10 - 9)$
AL5		$0.2735501(5.0 \times 10 - 8)$	$-0.28794741(7.0 \times 10 - 10)$
J-K	0.03125	$0.27355005(5.0 \times 10 - 7)$	$-0.28794742(4.0 \times 10 - 7)$
CH4		$0.27355003(1.0 \times 10 - 8)$	$-0.28794740(1.0 \times 10 - 10)$
AL5		$0.2735501(6.0 \times 10 - 8)$	$-0.28794741(1.0 \times 10 - 10)$
AB7		$0.27354004(4.0 \times 10 - 5)$	$-0.28796321(6.0 \times 10 - 5)$
F4		$0.2735503(3.0 \times 10 - 7)$	$-0.2879477(3.1 \times 10 - 7)$
NM9		$0.27354004(7.9 \times 10 - 5)$	$-0.28794740(8.3 \times 10 - 7)$
F5		$0.27355003(6.4 \times 10 - 9)$	$0.28794741(6.7 \times 10 - 9)$
AL5	0.05	$0.2373550(1.1 \times 10 - 7)$	$-0.2879947(1.4 \times 10 - 9)$
AB7		$0.27354004(4.0 \times 10 - 5)$	$-0.28796321(6.0 \times 10 - 5)$
AF5		$0.27354738(2.7 \times 10 - 6)$	$-0.28794461(1.4 \times 10 - 8)$
Exact solution		0.27355004	-0.287947411

Table 4: Comparative analysis of result of problem 3 at $x = 1$.

As shown in the Table (4) above, the proposed method in this paper performed better than existing methods in terms of accuracy.

7 CONCLUSION

The aim of this paper was to develop numerical method which provides solution to initial value problems with stiff differential equations via exponentially fitted integrators. Numerical experiments have been carried out using appropriate step size as required by each problem. Such problems which are stiff require small step size before the solution can be smooth. In general, the results from numerical experiment so presented in this paper, show that the new method performed effectively well when compared with similar methods in the literature. Hence the aim and objective of this paper have been achieved.

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