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Coupled best proximity points of generalised Hardy-Rogers type cyclic (ω)-contraction mappings

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Abstract

In this paper, we introduce generalized Hardy-Rogers type cyclic (ω)-contraction mappings which generalizes the cyclic, Kannan, Chatterjea and Reich contraction mappings. We establish the existence and uniqueness of coupled best proximity points of such mappings in the framework of b-metric space. Some examples are presented to support the results proved herein. Our results generalize and extend various comparable results in the existing literature. An application of our result to establish the existence of a solution of a differential equation is presented.

1 Introduction

Fixed point theory provides useful techniques for solving a variety of applied problems in many branches of mathematics such as computer science, engineering, chemistry, biology, economics and statistics (see[11,25]).
The Banach contraction principle [7] is an elegant and powerful result which initiated a new area of research known as metric fixed point theory. Extensions of Banach contraction principle have been obtained either by generalizing the distance properties of underlying domain or by modifying the contractive condition on the mappings. Bakhtin [6] first introduced the concept of a b-metric space as a generalisation of metric space and then proved Banach Contraction Principle in the setup of such spaces. This served as a motivation for many reseachers to obtain results on the variational principle for single valued and multi-valued operators in b-metric spaces (see [5,8,9,12,13]).

Geraghty [15] introduced the class of mappings:

$$S = \left\{ \psi : [0, \infty) \to [0, 1) : \lim_{n\to\infty} t_n = 0 \text{ whenever } \lim_{n\to\infty} \psi(t_n) = 1 \right\}$$

and obtained an interesting extension of Banach Contraction Principle as follows.

Theorem 1.1 *Let X be a complete metric space and $f : X \to X$. If there exists $\psi \in S$ such that*

$$d(fx, fy) \leq \psi(d(x, y))d(x, y)$$

holds for all $x, y \in X$. Then f has a unique fixed point $x^* \in X$ and for each $x \in X$, the sequence $\{f^n(x)\}$ (called Picard sequence) converges to x^*.

Another generalisation of the Banach Contraction Principle is due to Bhaskar and Lakshmikanthan [16]. They introduced the concept of the mixed monotone property and obtained some coupled fixed point results satisfying certain contractive conditions. They applied their results to establish the existence and uniqueness of a solution of a periodic boundary value problem. Afterwards, several authors studied and extended coupled fixed point results in [16] to different directions (see, e.g. [1,2,23,24,26,31]).
Let A and B be nonempty subsets of a metric space X and $T : A \to B$. A point x^* in A such that $d(x^*, Tx^*) = 0$ holds is called a fixed point of T. A point x^* in A such that $d(x^*, Tx^*) = \inf\{d(a, Tx^*) : a \in A\}$, that is, $x^* \in A$ is the closest point to $Tx^* \in B$ is called an approximate fixed point of T. The study of conditions that ensure existence and uniqueness of approximate fixed point of a mapping T is an important area of research.

Suppose that $\triangle_{AB} = d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. A point x^* is called a best proximity point of T if $d(x^*, Tx^*) = \triangle_{AB}$. If $A \cap B = \phi$, fixed point problem defined by a pair (A, B) and a mapping T has no solution. If we take $A = B$, then a best proximity point problem reduces to fixed point problem. From this perspective, best proximity point problem can be viewed as a natural generalization of fixed point problem. since $d(x, fx) \geq \triangle_{AB}$ for all x in A, then the global minimum of the mapping $x \longmapsto d(x, fx)$ is attained at the best proximity point (see [14,27]). Best proximity results deal with sufficient conditions such that the non-linear programming (or minimization) problem

$$\min_{x \in A} d(x, fx) \quad \text{has at least one solution.}$$

The theory of best proximity point has proved to be durable and useful in solving real world problems in nonlinear analysis, optimization, economics, game theory, and so forth.

Best proximity point theory of a cyclic contraction map has been studied by many authors. For results regarding cyclic contractive conditions when the intersection of the sets is nonempty, (see [19,25,28]). In [14], Eldred and Veeramani extended the cyclic contractive condition above to the case when $A \cap B$ is empty and proved the existence of best proximity point. For further results in this area see [3,20,22].

Sintunavarat and Kumam [32] introduced the concept of a coupled best proximity point and proved the existence and uniqueness of coupled best proximity

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point in metric and uniformly convex Banach spaces.
Recently, Amini-Harandi et al.[4] recently introduced a new class of cyclic generalised contraction maps. They gave an existence result for a best proximity point of such mappings in uniformly convex Banach spaces.

Motivated by the work in [4] and [32], we introduce a generalised Hardy-Rogers type cyclic (ω)-contraction map in b-metric spaces and establish the existence and uniqueness of coupled best proximity point for such maps.

2 Preliminaries

The following definitions and results will be needed in the sequel.

A normed space $(X, \|\cdot\|)$ is said to be:

1 Strictly convex if for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$, we have $\|\frac{x+y}{2}\| < 1$.

2 Uniformly convex if for any ϵ with $0 < \epsilon \leq 2$, there exists $\delta > 0$ such that for all $x, y \in X$ with $\|x\| = \|y\| \leq 1$ and $\|x-y\| \geq \epsilon$, we have $\|\frac{x+y}{2}\| < 1-\delta$.

Note that a uniformly convex space X is strictly convex but the converse does not hold in general.

Let A and B be nonempty subsets of a metric space (X, d),
Definition 2.1 ([32]). An ordered pair $(A,\ B)$ is said to have the property UC if the following condition holds:
For any sequences $\{x_n\}$ and $\{z_n\}$ in A and a sequence $\{y_n\}$ in B satisfying $d(x_n, y_n) \to d(A, B)$ and $d(z_n, y_n) \to d(A, B)$, we have $d(x_n, z_n) \to 0$.

The following are some examples of pairs of nonempty subsets (A, B) satisfying the the property UC:
Example 2.2 ([32]).

a Every pair (A, B) of nonempty subsets of a metric space (X, d) with $d(A, B) = 0$.

b Every pair (A, B) of nonempty subsets of uniformly convex Banach space X when A is convex.

c Every pair (A, B) of nonempty subsets of strictly convex Banach space when A is convex and relatively compact and the closure of B is weakly compact.

Definition 2.3 ([32]). An ordered pair $(A,\ B)$ is said to have the property UC^* if (A, B) has property UC and the following holds:
For any sequences $\{x_n\}$ and $\{z_n\}$ in A and a sequence $\{y_n\}$ in B, we have

$$d(z_n, y_n) \to d(A, B)$$

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For every $\epsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that $d(x_m, y_n) \to d(A,B)+\epsilon$ for all $m > n \geq n_0$ implies that there exists $n_1 \in \mathbf{N}$ such that $d(x_m, z_n) \to d(A,B)+\epsilon$ for all $m > n \geq n_1$.

Definition 2.4 ([14]). Let A and B be nonempty subsets of a metric space (X, d) and $T : A \to B$ a mapping. A point $x \in A$ is said to be a best proximity point of T if $d(x, Tx) = d(A, B)$.

Eldred and Veeramani [14] extended Banach contraction principle to the case of non-self mappings and proved the existence of a best proximity point.
Definition 2.5([14]) Let A and B be nonempty subsets of a metric space (X, d), a map $T : A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$ is a cyclic contraction if $d(T(x), T(y)) \leq k(d(x, y)) + (1-k)d(A, B), k \in [0, 1)$.

Bhaskar and Lakshmikanthan [16] gave the following definition and proved an interesting generalization of Banach contraction principle.
Definition 2.6 ([16]). Let A be a nonempty subsets of a metric space X and $F : A \times A \to A$. A point $(x, x') \in A \times A$ is called a coupled fixed point of F if $x = F(x, x')$ and $x' = F(x', x)$.

Sintunavarat and Kumam [32] introduced the coupled best proximity point as follows:
Definition 2.7 ([32]). Let A and B be nonempty subset of a metric space X and $F : A \times A \to B$. A point $(x, x') \in A \times A$ is called a coupled best proximity point of F if $d(x, F(x, x')) = d(x', F(x', x)) = d(A, B)$.

To extend the coupled fixed point results to the case of nonself mappings, Sintunavarat and Kumam [32] modified the concept of cyclic contraction maps as follows :
Definition 2.8 ([32]). Let A and B be nonempty subset of a metric space X, $F : A \times A \to B$ and $G : B \times B \to A$. An ordered pair (F, G) is said to be a cyclic contraction if there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(F(x, x'), G(y, y')) \leq \alpha[d(x, y) + d(x', y')] + (1 - 2\alpha)d(A, B)$$

holds for any $(x, x') \in A \times A$ and $(y, y') \in B \times B$.

Geraghty [15] extended the concept of contraction mappings by replacing the contractive constant with a function and proved the following fixed point theorem:
Theorem 2.9 ([15]). Let (X, d) be a complete metric space and $T : X \to X$ a map. If for any $x, y \in X$, we have

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y),$$

where $\alpha : [0, \infty) \to [0, 1)$ is any mapping which satisfies $\lim_{s \to t^+} \sup \alpha(s) < 1$ for each $t \in (0, \infty)$. Then T has a fixed point.

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Recently, Amini-Harandi et al., ([4]) introduced the concept of a generalised cyclic contraction for a pair of non self mappings as follows:
Definition 2.10 ([4]). Let A and B be nonempty subsets of a metric space (X,d). A mapping $T : A \cup B \to A \cup B$ is a generalised cyclic contraction if $T(A) \subseteq B$, $T(B) \subseteq A$ and for any $x \in A$ and $y \in B$, we have

$$d(Tx,Ty) \leq \alpha(d(x,y))d(x,y) + [1-\alpha(d(x,y))d(A,B)],$$

where $\alpha : [d(A,B),\infty) \to [0,1)$ is any mapping which satisfies

$$\lim_{s\to t^+} \sup \alpha(s) < 1$$

for each $t \in [d(A,B),\infty)$.

If in above definition, for each $t \in [d(A,B),\infty)$, we set $\alpha(t) = k$, where $k \in [0,1)$, then T is called a cyclic contraction.

Karapinar and Erhan [19] obtained best proximity point results employing the concept of cyclic contraction analogues to Kannan [18], Chatterjea [10] and Reich [30] contraction mappings.
We extend the above definition in the following way:
Definition 2.11. Let A and B be nonempty subset of a metric space X, $F : A \times A \to B$ and $G : B \times B \to A$. An ordered pair (F,G) is said to be a cyclic generalised contraction map if for each $(x,x') \in A\times A$ and $(y,y') \in B\times B$, we have

$$\begin{aligned}&d(F(x,x'),G(y,y')) \leq \phi(d(x,y),d(x',y'))[(d(x,y)+d(x',y'))]\\&+(1-2\phi(d(x,y),d(x',y'))d(A,B)\end{aligned} \tag{2.1}$$

where $\phi : [d(A,B),\infty) \times [d(A,B),\infty) \to [0,\frac{1}{2})$ is any mapping which satisfies $\lim\limits_{(s,s')\to(t,t')} \sup \phi(s,s') < \frac{1}{2}$ for each $(t,t') \in [d(A,B),\infty) \times [d(A,B),\infty)$.
If $\phi(t,t') = k$ for each $(t,t') \in [d(A,B),\infty)\times[d(A,B),\infty)$ where $k \in [0,\frac{1}{2})$, then (F,G) is a cyclic contraction.

Olaleru and Olisama [26] recently defined and proved results on coupled best proximity points of Kannan type, Chatterjea type, Reich type and Hardy-Rogers type as an extension of the results in [19].
Definition 2.12. Let A and B be nonempty subset of a metric space (X,d), $F : A \times A \to B$ and $G : B \times B \to A$. An ordered pair (F,G) is said to be:

(K) Kannan type cyclic contraction if there exists $\alpha \in [0,\frac{1}{2})$ such that

$$d(F(x,x'),G(y,y')) \leq \alpha[d(x,F(x,x'))+d(y,G(y,y'))]+(1-2\alpha)d(A,B)$$

holds for all $(x,x') \in A \times A$ and $(y,y') \in B \times B$.

(C) Chatterjea type cyclic contraction if there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(F(x,x'),G(y,y')) \le \alpha[d(y,F(x,x')) + d(x,G(y,y'))] + (1-2\alpha)d(A,B)$$

holds for all $(x,x') \in A \times A$ and $(y,y') \in B \times B$.

(R) Reich type cyclic contraction if there exists $\alpha \in [0, \frac{1}{3})$ such that

$$\begin{aligned} d(F(x,x'),G(y,y')) &\le \alpha[d(x,y) + d(x,F(x,x')) + d(y,G(y,y'))] \\ &\quad +(1-3\alpha)d(A,B) \end{aligned}$$

holds for all $(x,x') \in A \times A$ and $(y,y') \in B \times B$.

(HR) Hardy- Rogers type cyclic contraction if there exists $\alpha \in [0, \frac{1}{6})$ and

$$\begin{aligned} d(F(x,x'),G(y,y')) &\le \alpha[d(x,y) + d(x',y') + d(x,F(x,x')) + d(y,G(y,y')) \\ &\quad +d(x,G(y,y')) + d(y,F(x,x'))] \\ &\quad +(1-6\alpha)d(A,B) \end{aligned} \tag{2.2}$$

holds for all $(x,x') \in A \times A$ and $(y,y') \in B \times B$.
Suppose,

$$\begin{aligned} \Omega &= \{\omega : [d(A,B),\infty) \times [d(A,B),\infty) \to [0, \frac{1}{6}), \\ \lim_{(s,s')\to(t,t')} \sup \omega(s,s') &< \frac{1}{6} \\ \text{for each } (t,t') &\in [d(A,B),\infty) \times [d(A,B),\infty) \ \} \end{aligned}$$

To extend Definition 2.12(HR) and to unify the comparable results in ([4,11,19, 22, 31]), we introduce a generalised Hardy-Rogers cyclic (ω)-contraction as follows:

Definition 2.13. Let A and B be nonempty subsets of a metric space X, $F : A \times A \to B$ and $G : B \times B \to A$. A pair (F,G) is said to be a generalised Hardy- Rogers cyclic (ω)-contraction if there exists $\omega \in \Omega$ such that

$$\begin{aligned} d(F(x,x'),G(y,y')) &\le \omega(d(x,y),d(x',y'))[d(x,y) + d(x',y') + d(x,F(x,x')) \\ &\quad +d(y,G(y,y')) + d(x,G(y,y')) + d(y,F(x,x'))] \\ &\quad +[1-6\omega(d(x,y),d(x',y'))]d(A,B) \end{aligned} \tag{2.3}$$

holds for any $(x,x') \in A \times A$ and $(y,y') \in B \times B$

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If $\omega(d, d') = \alpha$, then (F, G) is a Hardy-Rogers type cyclic contraction as in Definition 2.12(HR).

The following example shows that the cyclic Hardy-Rogers type contraction map (HR) may have a best proximity point but the same is not true for the generalised Hardy-Rogers type cyclic (ω)-contraction map.

Example 2.14. Let $X = [1, \infty)$ equipped with a usual metric, $A = [0, 2]$ and $B = [1, -2]$. Obviously, $d(A, B) = 1$. Define mappings $F : A \times A \to B$ and $G : B \times B \to A$ by $F(x, x') = y$ and $G(y, y') = x$, respectively. Let $\omega : [d(A, B), \infty) \times [d(A, B), \infty) \to [0, \frac{1}{6})$ be defined by $\omega(t, t') = \frac{t}{1+6t}$. If we take $d(x, y) = d(x', y')$ and $\omega(d(x, y), d(x', y')) = \frac{1}{7}$ in (2.3), we obtain that

$$\begin{aligned} d(x, y) &\leq \frac{1}{7}[4d(x, y)] + \frac{1}{7}d(A, B) \\ &= \frac{1}{3}d(A, B) < d(A, B), \end{aligned}$$

a contradiction and hence (F, G) has no best proximity point.

Now a natural question arises that under what conditions on (F, G) or ω, the existence of best proximity point of a pair (F, G) is guaranteed. Before we give answer to this question, we present the definition of a b-metric space as follows.

Definition 2.15([6]). Let X be a non empty set and $s \geq 1$ be a given real number. A map $d : X \times X \to \mathbf{R}$ is said to be a b-metric if for any $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x, y) > 0$ with $x \neq y$ and $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$.

(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a b-metric space.

If we take $s = 1$, then we have a definition of a metric space.

The following example shows that a b-metric is a generalisation of a metric.

Example 2.16. Consider $X = \mathbf{R}^2$. Define a mapping $d : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$ by $d(x, y) = |\, x_1 - y_1|\,^2$ where $x = (x_1, x_1)$, $y = (y_1, y_1)$. Then (X, d) is a b-metric space but not a metric space. Indeed, for $x, y, z \in \mathbf{R}^2$. Set $a = x - z$, $b = z - y$ so $x - y = a + b$. Using the inequality

$$(u + v)^2 \leq (2 \max\{u, v\})^2 \leq 2^2(u^2 + v^2),$$

we have,

$$|\, x - y|\,^2 = |\, a + b|\,^2 \leq (|\, a|\, + |\, b|\,)^2 = 4(|\, x - z|\,^2 + |\, x - y|\,^2)$$

for any $a, b > 0$. That is,

$$d(x, y) \leq s[d(x, z) + d(z, y)]$$

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with $s>1$. On the other hand, by inequality, $(u+v)^2 > u^2+v^2$ for all $a,b>0$, we get that

$$\begin{aligned} |\ x-y|\ ^2 &= |\ a+b|\ ^2 = (a+b)^2 > a^2+b^2 \\ &= (x-z)^2+(z-y)^2 = |\ x-z|\ ^2 + |\ x-y|\ ^2 \end{aligned}$$

for all $x>z>y$. Therefore (iii) in (2.4) is not satisfied when $s=1$, and hence (X,d) is not a metric space.

Now we state and prove the following lemma to justify the main result.
Lemma 2.17. Let A and B be nonempty subsets of a metric space (X,d) and (F,G) generalised Hardy- Rogers cyclic (ω)-contraction. If for $(x_0,y_0)\in A\times A$, we define

$$\begin{cases} x_{2n+1} = F(x_{2n},y_{2n}) \\ y_{2n+1} = F(y_{2n},x_{2n}) \end{cases} \text{ and }$$
$$\begin{cases} x_{2n+2} = G(x_{2n+1},y_{2n+1}) \\ y_{2n+2} = G(y_{2n+1},x_{2n+1}) \end{cases}$$

where $n\in\mathbf{N}\cup\{0\}$, then $d(x_{2n},x_{2n+1})\to d(A,B)$ and $d(y_{2n},y_{2n+1})\to d(A,B)$.

Proof. For each $n\in\mathbf{N}$, we have

$$\begin{aligned} & d(x_{2n},x_{2n+1}) \\ = \;& d(F(x_{2n},y_{2n}),G(x_{2n-1},y_{2n-1}) \\ \le \;& \omega(d(x_{2n},x_{2n-1}),d(y_{2n},y_{2n-1}))[d(x_{2n},x_{2n-1}) \\ & +d(y_{2n},y_{2n-1}))+d(x_{2n},F(x_{2n},y_{2n})) \\ & +d(x_{2n-1},G(x_{2n-1},y_{2n-1}))+d(x_{2n},G(x_{2n-1},y_{2n-1})) \\ & +d(x_{2n-1},F(x_{2n},y_{2n}))]+[1-6\omega(d(x_{2n},x_{2n-1}),d(y_{2n},y_{2n-1}))]d(A,B). \end{aligned}$$

and

$$\begin{aligned} d(y_{2n},y_{2n+1}) &= d(F(y_{2n},z_{2n}),G(y_{2n-1},z_{2n-1})) \\ &\le \omega(d(y_{2n},y_{2n-1}),d(z_{2n},z_{2n-1}))[d(y_{2n},y_{2n-1}) \\ &\quad +d(z_{2n},z_{2n-1}))+d(y_{2n},F(y_{2n},z_{2n})) \\ &\quad +d(y_{2n-1},G(y_{2n-1},z_{2n-1}))+d(y_{2n},G(y_{2n-1},z_{2n-1})) \\ &\quad +d(y_{2n-1},F(y_{2n},z_{2n}))]+[1-6\omega(d(y_{2n},y_{2n-1}),d(z_{2n},z_{2n-1}))]d(A,B). \end{aligned}$$

If we put $d_{2n} = d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})$, $\omega(d(x_{2n},x_{2n-1}),d(y_{2n},y_{2n-1})) =$

$\omega(x_{2n}, y_{2n})$ and $\omega(x_n, y_n) + \omega(y_n, z_n) = \eta_n$, then taking the sum of above inequalities above, we obtain that

$$\begin{aligned} d_{2n} &\leq (\omega(x_{2n}, y_{2n}) + \omega(y_{2n}, z_{2n}))[d_{2n-1} + d_{2n-1} + d_{2n} + d_{2n-1} \\ &\quad +[2 - 6(\omega(x_{2n}, y_{2n}) + \omega(y_{2n}, z_{2n}))]d(A, B) \\ &= 3\eta_{2n}d_{2n-1} + \eta_{2n}d_{2n} + 2(1 - 3\eta_{2n})d(A, B). \end{aligned}$$

and

$$\begin{aligned} d_{2n-1} &\leq (\omega(x_{2n}, y_{2n}) + \omega(y_{2n}, z_{2n}))[d_{2n-2} + d_{2n-2} + d_{2n-1} + d_{2n-2}] \\ &\quad +[2 - 6(\omega(x_{2n}, y_{2n}) + \omega(y_{2n}, z_{2n}))]d(A, B) \\ &= 3\eta_{2n}d_{2n-2} + \eta_{2n}d_{2n-1} + 2(1 - 3\eta_{2n})d(A, B) \\ &= \frac{3}{1-\eta_{2n}}\eta_{2n}d_{2n-2} + \frac{2}{1-\eta_{2n}}(1 - 3\eta_{2n})d(A, B). \end{aligned}$$

Now

$$\begin{aligned} d_{2n} &\leq 3\eta_{2n}(\frac{3}{1-\eta_{2n}}\eta_{2n}d_{2n-2} + \frac{2}{1-\eta_{2n}}(1 - 3\eta_{2n})d(A, B)) + \eta_{2n}d_{2n} \\ &\quad +2(1 - 3\eta_{2n})d(A, B) \\ &\leq 3\eta_{2n}(\frac{3}{(1-\eta_{2n})^2}\eta_{2n}d_{2n-2} + \frac{2}{(1-\eta_{2n})^2}(1 - 3\eta_{2n})d(A, B)) \\ &\quad +\frac{2}{1-\eta_{2n}}(1 - 3\eta_{2n})d(A, B) \\ &\leq 3\eta_{2n}[\eta_{2n}d_{2n-2} + 2(1 - 3\eta_{2n})d(A, B) + 6(1 - 3\eta_{2n})d(A, B)] \\ &\leq 3\eta_{2n}^2 d_{2n-2} + 6(1 - 3\eta_{2n})(1 + 3\eta_{2n})d(A, B) \\ &\leq \eta_{2n}^2 d_{2n-2} + 2(1 - \eta_{2n}^2)d(A, B) \end{aligned}$$

That is,

$$d_{2n} \leq \eta_{2n}^2 d_{2n-2} + 2(1 - \eta_{2n}^2)d(A, B).$$

As $\{\eta_n\}$ is a sequence in $[0, \frac{1}{3})$,

$$d_{2n} \leq (\frac{1}{3})^2 d_{2n-2} + 2(1 - (\frac{1}{3})^2)d(A, B).$$

Continuing this way, we have

$$d_{2n} \leq (\frac{1}{3})^n d_0 + 2(1-(\frac{1}{3})^n)d(A, B).$$

Note that, $d(A, B) \leq d(x_{2n}, x_{n+1})$ and $d(A, B) \leq d(y_{2n}, y_{n+1})$ $\forall n$. Indeed, $x_{2n}, y_{2n} \in A$ and $x_{2n+1}, y_{2n+1} \in B$. Thus, we have $2d(A, B) \leq d_{2n}$ and hence

$$(\frac{1}{3})^n d_0 + 2(1-(\frac{1}{3})^n)d(A, B) \geq d_0 \geq 2d(A, B).$$

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On taking limit as $n \to \infty$, we obtain that $\lim_{n\to\infty} d_{2n} = 2d(A,B)$, that is, $\lim_{n\to\infty}[d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1})] = 2d(A,B)$.

Since, $d(A,B) \leq d(x_{2n}, x_{2n+1})$ and $d(A,B) \leq d(y_{2n}, y_{2n+1}))$ and hence

$$d(x_{2n}, x_{2n+1}) \to d(A,B) \text{ and } d(y_{2n}, y_{2n+1}) \to d(A,B).$$

3 Coupled Best Proximity Point Results

We start with the following result.

Theorem 3.1. Let A and B be nonempty closed subsets of a complete b-metric space X, (F,G) a generalized Hardy-Rogers type cyclic ω-contraction pair. Suppose that (A,B) and (B,A) satisfy the property UC. If for $(x_0, x'_0) \in A \times A$, we define

$$\begin{aligned} x_{2n+1} &= F(x_{2n}, x'_{2n})\ , \quad x'_{2n+1} = F(x'_{2n}, x_{2n}), \\ x_{2n+2} &= G(x_{2n+1}, x'_{2n+1}) \text{ and } \ x'_{2n+2} = G(x'_{2n+1}, x_{2n+1}) \end{aligned}$$

for all $n \in N \cup \{0\}$. Then $x_{2n} \to x$, $x'_{2n} \to x'$, $x_{2n-1} \to y$ and $x'_{2n-1} \to y'$ as $n \to \infty$. Moreover, $(x,x') \in A \times A$ and $(y,y') \in B \times B$ are coupled best proximity points of F and G, respectively.

Proof. If $d(A,B) = 0$, then we obtain the coupled fixed points of F and G in the same way as in [2] and the result follows. Assume that $d(A,B) > 0$. Now by (2.3), we have

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &= d(F(x_{2n}, x'_{2n}), G(x_{2n-1}, x'_{2n-1})) \\ &\leq \omega(d(x_{2n}, x_{2n-1}), d(x'_{2n}, x'_{2n-1}))[d(x_{2n}, x_{2n-1}) \\ &\quad + d(x'_{2n}, x'_{2n-1}) + d(x_{2n}, F(x_{2n}, x'_{2n})) \\ &\quad + d(x_{2n-1}, G(x_{2n-1}, x'_{2n-1})) \\ &\quad + d(x_{2n}, G(x_{2n-1}, x'_{2n-1})) + d(x_{2n-1}, F(x_{2n}, x'_{2n}))] \\ &\quad + [1 - 6\omega(d(x_{2n}, x_{2n-1}), d(x'_{2n}, x'_{2n-1}))]d(A,B) \qquad (3.1) \end{aligned}$$

for each $n \in N$. Since (A,B) has property UC, $d(x_{2n}, x_{2n+2}) \to 0$ and $d(x'_{2n}, x'_{2n+2}) \to 0$. Also, (B,A) satisfies property UC, $d(x_{2n-1}, x_{2n+1}) \to 0$ and $d(x'_{2n-1}, x'_{2n+1}) \to 0$. Note that

$$\omega(d(x_{2n}, x_{2n-1}), d(x'_{2n}, x'_{2n-1})) < \frac{1}{6},\ d(x_{2n}, x_{2n-1}) \geq d(A,B),$$

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and

$$
\begin{aligned}
& \omega(d(x_{2n},x_{2n-1}),d(x'_{2n},x'_{2n-1}))[d(x_{2n},x_{2n-1})+d(x'_{2n},x'_{2n-1}) \\
& +d(x_{2n},F(x_{2n},x'_{2n}))+d(x_{2n-1},G(x_{2n-1},x'_{2n-1}))+d(x_{2n},G(x_{2n-1},x'_{2n-1})) \\
& +d(x_{2n-1},F(x_{2n},x'_{2n}))]+[1-6\omega(d(x_{2n},x_{2n-1}),d(x'_{2n},x'_{2n-1}))]d(A,B) \\
< \; & \frac{1}{6}[2d(x_{2n},x_{2n-1})+d(x'_{2n},x'_{2n-1})+d(x_{2n},x_{2n+1})]. \qquad (3.2)
\end{aligned}
$$

From (3.1) and (3.2), we have

$$d(x_{2n+1},x_{2n}) \leq \frac{1}{6}[2d(x_{2n},x_{2n-1})+d(x'_{2n},x'_{2n-1})+d(x_{2n},x_{2n+1})].$$

Thus,

$$
\begin{aligned}
d(x_{2n+1},x_{2n}) & \leq \frac{1}{5}[2d(x_{2n},x_{2n-1})+d(x'_{2n},x'_{2n-1})] \\
& < d(x_{2n},x_{2n-1})+d(x'_{2n},x'_{2n-1}) \qquad (3.3)
\end{aligned}
$$

for each $n \in N$. We conclude that

$$d(x_{n+1},x_n) \leq d(x_{n-1},x_n)+d(x'_{n-1},x'_n)$$

for each $n \in N$. Thus $\{d(x_{n+1},x_n)\}$ is a non increasing sequence of positive real numbers. Consequently, there exist some real number $t_0 \geq d(A,B)$ such that $\{d(x_{n+1},x_n)\}$ converges to t_0. Now we claim that $t_0 = d(A,B)$. On the contrary that $t_0 > d(A,B)$. Since $\limsup_{(s,s')\to(t_0,t'_0)} \omega(s,s') < \frac{1}{6}$ and $\omega(t_0,t'_0) < \frac{1}{6}$, there exists $t \in (0,\frac{1}{6})$ and $\epsilon > 0$ such that $\omega(s,s') \leq t$ for all $(s,s') \in [t_0,t_o+\epsilon]\times[t'_0,t'_0+\epsilon']$. If $N_0 \in N$ is such that $t_0 \leq d(x_n,x_{n+1}) \leq t_0+\epsilon$ for all $n \geq N_0$. Then, we have $\omega(d(x_n,x_{n+1}),d(x'_n,x'_{n+1})) \leq t$ for $n \geq N_0$. From (3.1), we get that

$$
\begin{aligned}
& d(x_{2n+1},x_{2n}) \\
\leq \; & \omega(d(x_{2n},x_{2n-1}),d(x'_{2n},x'_{2n-1}))[2d(x_{2n},x_{2n-1})+d(x'_{2n},x'_{2n-1}) \\
& +d(x_{2n},x_{2n+1})]+d(A,B)-6\omega(d(x_{2n},x_{2n-1}),d(x'_{2n},x'_{2n-1}))d(A,B) \\
\leq \; & t[2d(x_{2n},x_{2n-1})+d(x'_{2n},x'_{2n-1})+d(x_{2n},x_{2n+1})-6d(A,B)]+d(A,B).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
d(x_{2n+1},x_{2n}) & \leq \frac{t}{1-t}[2d(x_{2n},x_{2n-1})+d(x'_{2n},x'_{2n-1})]+\frac{(1-6t)}{1-t}d(A,B) \\
& \leq \frac{t}{2}d(x_{2n},x_{2n-1})+d(x'_{2n},x'_{2n-1})+(1-t)d(A,B)
\end{aligned}
$$

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for each $n \geq N_0$. Thus

$$d(x_n, x_{n+1}) \leq \frac{t}{2} d(x_n, x_{n-1}) + d(x'_n, x'_{n-1}) + (1-t) d(A, B), \quad n \geq 2N_0. \quad (3.4)$$

On taking limit as $n \to \infty$ on both sides of (3.4), we have

$$t_0 = \lim_{n \to \infty} d(x_n, x_{n+1}) \leq d(A, B),$$

a contradiction. Then it follows that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = d(A, B).$$

That is, $\lim\limits_{n \to \infty} d(x_n, F(x_n, x'_n)) = d(A, B)$. Similarly, $\lim\limits_{n \to \infty} d(y_n, G(y_n, y'_n)) = d(A, B)$. We now show that for every $\epsilon > 0$, there exists N such that

$$d(x_{2m}, x_{2n+1}) \leq \epsilon + d(A, B) \ \text{ for all } m > n \geq N. \quad (3.5)$$

Assume on contrary that there exists $\epsilon_1 > 0$ such that for all $k \in N$, there exists $m_k > n_k \geq k$ such that

$$d(x_{2mk}, x_{2nk+1}) > \epsilon_1 + d(A, B) \quad (3.6)$$

and

$$d(x_{2(mk-1)}, x_{2nk+1}) < \epsilon_1 + d(A, B). \quad (3.7)$$

It follows from (3.6) and (3.7) that

$$\begin{aligned} d(A, B) + \epsilon_1 &\leq d(x_{2mk}, x_{2nk+1}) \\ &\leq s[d(x_{2mk}, x_{2(mk-1)}) + d(x_{2(mk-1)}, x_{2nk+1})] \\ &< s[d(x_{2mk}, x_{2(mk-1)}) + d(A, B) + \epsilon_1]. \end{aligned}$$

On taking limit as $k \to \infty$, we have

$$\lim_{k \to \infty} d(x_{2mk}, x_{2nk+1}) = s(\epsilon_1 + d(A, B)), s \geq 1. \quad (3.8)$$

Let $r_0 = d(A, B) + \epsilon_1$. As $\limsup_{(s,s') \to (r_0, r'_0)} \omega(s, s') < \frac{1}{6}$ and $\omega(r_0, r'_0) < \frac{1}{6}$, there exists $t' \in (0, \frac{1}{6})$ and $\epsilon > 0$ such that $\omega(s, s') \leq t'$
for all $(s, s') \in [r_0, r_0 + \epsilon] \times [r'_0, r'_o + \epsilon']$. Taking $N_0 \in N$ such that $r_0 \leq d(x_{2mk}, x_{2nk+1}) \leq r_0 + \epsilon$ for all $k \geq K_0$. Thus

$$\omega(d(x_{2mk}, x_{2nk+1}), d(x'_{2mk}, x'_{2nk+1})) \leq t'$$

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for $k \geq K_0$. From (3.1) we get that

$$d(x_{2mk+1}, x_{2nk+2}) \leq t'(d(x_{2mk}, x_{2nk+1}) + d(x'_{2mk}, x'_{2nk+1}) + (1-t')d(A,B), \tag{3.9}$$

$k \geq K_0$.
By (3.3) and (3.9), we have
$d(x_{2mk}, x_{2nk+1}) \leq s[d(x_{2mk}, x_{2mk+2}) + d(x_{2mk+2}, x_{2nk+3}) + d(x_{2nk+3}, x_{2nk+1})]$
$\leq s[d(x_{2mk}, x_{2mk+2}) + d(x_{2mk+1}, x_{2nk+2}) + d(x_{2nk+3}, x_{2nk+1})]$
$\leq s[d(x_{2mk}, x_{2mk+2}) + d(x_{2nk+3}, x_{2nk+1}) + t'(d(x_{2mk}, x_{2nk+1})$

$$+d(x'_{2mk}, x'_{2nk+1}) + (1-t')d(A,B)], \tag{3.10}$$

for all $k \geq K_0$. It follows from (3.8) and (3.10) that

$$s(d(A,B)+\epsilon_1) < s^2t'(d(A,B)+\epsilon_1)+sd(A,B)-s^2t'd(A,B) = s(d(A,B)+t'\epsilon_1),$$

a contradiction. So (3.5) holds.

Now we show that $\{x_{2n}\}, \{x'_{2n}\}, \{x_{2n+1}\}$ and $\{x'_{2n+1}\}$ are Cauchy sequences. To prove $\{x_{2n}\}$ is a Cauchy sequence; Assume on contrary that there exists $\epsilon_2 > 0$, such that for each $k \geq 1$, there exists $j_k > l_k \geq k$ such that

$$d(x_{2jk}, x_{2lk}) \geq \epsilon_2. \tag{3.11}$$

Using Lemma (2.17) and (3.9), we have

$$\begin{aligned} d(x_{2mk}, x_{2nk+1}) &\leq s[d(x_{2mk}, x_{2nk}) + d(x_{2nk}, x_{2nk+1})] \text{ and} \\ s(d(A,B)+\epsilon) &\leq sd(A,B), \end{aligned}$$

a contradiction. Hence $\{x_{2n}\}$ is a Cauchy sequence. Similarly, $\{x'_{2n}\}, \{x_{2n+1}\}$ and $\{x'_{2n+1}\}$ are Cauchy sequences.
Since (X,d) is a complete b-metric space, there exists $x \in X$ such that $x_{2n} \to x$. Similarly, there exists $x' \in X$ such that $x'_{2n} \to x'$ and $y, y' \in X$ such that $x_{2n+1} \to y$ and $x'_{2n+1} \to y'$ respectively.
Now we show that the coupled best proximity point $(x,x') \in A \times A$ is unique. If there is another fixed point $a \in A$, then $G(a,a') = a$ such that $x \neq a$.

$$\begin{aligned} d(x,a) &= d(F(x,x'), G(a,a')) \\ &\leq \omega(d(x,a), d(x',a')))[d(x,a) + d(x',a') \\ &\quad +d(x, F(x,x')) + d(a, G(a,a')) \\ &\quad +d(x, G(a,a')) + d(a, F(x,x')) \\ &\quad +[1-6\omega(d(x,a), d(x',a'))]d(A,B) \\ &\leq \omega(d(x,a), d(x',a')))[d(x,a) + d(x',a') + d(x,a) + d(a,x) \\ &\quad +[1-6\omega(d(x,a), d(x',a'))]d(A,B). \end{aligned}$$

Since $\omega(d(x,a),d(x',a')) \leq t < \frac{1}{6}$, take $x = x'$ and $a = a'$ to obtain that

$$\begin{aligned} d(x,a) &\leq t(4d(x,a)) + (1-6t)d(A,B) \text{ and} \\ d(x,a) &\leq \frac{1-6t}{1-4t}d(A,B) < d(A,B). \end{aligned}$$

Hence, $x = a$. Similarly, $x' = a'$ and $(x,x') \in A \times A$ is a unique coupled best proximity point of F. Similarly, $(y,y') \in B \times B$ is a unique coupled best proximity point of G.

The following example shows that in the statement of Theorem 3.1, the condition $\lim sup_{(s,s')\to(t,t')}\omega(s,s') < \frac{1}{6}$ is necessary.

Example 3.2. Let $F : A \times A \to B$ and $G : B \times B \to A$ be defined by $F(x,x') = \frac{3x+x'}{8}$ and $G(y,y') = \frac{3y+y'}{8}$ for each $x \in (0,\infty)$, where $\omega : [d(A,B),\infty) \times [d(A,B),\infty) \to [0,\frac{1}{6})$ satisfies $\lim \sup\limits_{(s,s')\to(t,t')} \omega(s,s') < \frac{1}{6}$ for each $(t,t') \in [d(A,B),\infty) \times [d(A,B),\infty)$. Note that

$$\begin{aligned} d(F(x,x'),G(y,y')) &= \mid \frac{3x+x'}{8} - \frac{(3y+y')}{8} \mid \\ &= \mid \frac{x+x+x+x'-y-y-y-y'}{8} \mid \\ &\leq \frac{1}{8} \mid x-y \mid + \mid x-y \mid + \mid x-y \mid + \mid x'-y' \mid . \end{aligned}$$

Taking $x = G(y,y')$ and $y = F(x,x')$, we have

$$\begin{aligned} d(F(x,x'),G(y,y')) &= \mid \frac{3x+x'}{8} - \frac{(3y+y')}{8} \mid \\ &= \mid \frac{x+x+x+x'-y-y-y-y'}{8} \mid \\ &\leq \frac{1}{8} \mid x-y \mid + \mid x-y \mid + \mid x'-y' \mid + \mid x - F(x,x') \mid \\ &\quad + \mid y - G(y,y') \mid + \mid x - G(y,y') \mid \\ &\quad + \mid y - F(x,x') \mid + (1 - \frac{1}{8}(8))d(A,B) . \end{aligned}$$

If$\frac{1}{8} \leq \omega(d(x,y),d(x',y')) < \frac{1}{6}$, then

$$\begin{aligned} d(F(x,x'),G(y,y')) \quad &\leq \quad \omega(d(x,y),d(x',y'))[\mid x-y \mid + \mid x'-y' \mid \\ &\quad + \mid x - F(x,x') \mid + \mid y - G(y,y)' \mid + \mid x - G(y,y') \mid \\ &\quad + \mid y - F(x,x') \mid] + [1 - 6\omega(d(x,y),d(x',y'))]d(A,B) \end{aligned}$$

shows that (F,G) is a generalised Hardy-Rogers type cyclic ω-contraction. If

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the condition $\lim \sup_{(s,s')\to(t,t')} \omega(s,s') < \frac{1}{6}$ is missing, then

$$\begin{aligned} d(F(x,x'),G(y,y')) &\leq \frac{1}{8}[| x-y| + | x'-y'| + | x-F(x,x')| \\ &+| y-G(y,y')| + | x-G(y,y')| \\ &+| y-F(x,x')|] \end{aligned}$$

shows that (F,G) is not a generalised Hardy-Rogers type cyclic ω-contraction. Also, (F,G) has no coupled best proximity point.

We now give the definition of cyclic Ciric type quasi contraction mapping.
Definition 3.3. Let A and B be nonempty subset of a metric space (X,d), $F : A\times A \to B$ and $G : B\times B \to A$. An ordered pair (F,G) is said to be a cyclic Ciric type quasi contraction pair if there exists a mapping $\omega : [d(A,B),\infty)\times[d(A,B),\infty) \to [0,1)$ with $\lim_{(s,s')\to(t,t')} \sup \omega(s,s') < 1$ such that

$$\begin{aligned} & d(F(x,x'),G(y,y')) \\ \leq\ & \omega(d(x,y),d(x',y'))(M(x,y)) + (1-\omega(d(x,y),d(x',y')))d(A,B) \end{aligned}$$

holds for all $(x,x')\in A\times A$ and $(y,y')\in B\times B$, where

$$\begin{aligned} M(x,y) &= \max\{d(x,y),d(x',y'),d(x,F(x,x')),d(y,G(y,y')), \\ & d(y,F(x,x')),d(x,G(y,y'))\}. \end{aligned}$$

Theorem 3.4. Let A and B be nonempty closed subsets of a complete $b-$metric space X, (F,G) a generalized Ciric quasi cyclic ω-contraction pair. Suppose that (A,B) and (B,A) satisfy the property UC. If for $(x_0,x_0')\in A\times A$, we define

$$\begin{aligned} x_{2n+1} &= F(x_{2n},x'_{2n})\ ,\ \ x'_{2n+1} = F(x'_{2n},x_{2n}), \\ x_{2n+2} &= G(x_{2n+1},x'_{2n+1}) \text{ and } x'_{2n+2} = G(x'_{2n+1},x_{2n+1}) \end{aligned}$$

for all $n\in N\cup\{0\}$. Then $x_{2n}\to x$, $x'_{2n}\to x'$, $x_{2n-1}\to y$ and $x'_{2n-1}\to y'$ as $n\to\infty$. Moreover, $(x,x')\in A\times A$ and $(y,y')\in B\times B$ are coupled best proximity points of F and G, respectively.
Proof. If $d(A,B)=0$, then we obtain the coupled fixed points of F and G in the same way as in [21]. Assume that $d(A,B)>0$. Now
$d(x_{2n+1},x_{2n}) = d(F(x_{2n},x'_{2n}),G(x_{2n-1},x'_{2n-1}))$
$\leq \omega(d(x_{2n},x_{2n-1}),d(x'_{2n},x'_{2n-1}))[d(x_{2n},x_{2n-1}),d(x'_{2n},x'_{2n-1}),$
$d(x_{2n},F(x_{2n},x'_{2n})),d(x_{2n-1},G(x_{2n-1},x'_{2n-1})),$

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$d(x_{2n}, G(x_{2n-1}, x'_{2n-1})), d(x_{2n-1}, F(x_{2n}, x'_{2n}))]$

$$+(1 - \omega(d(x_{2n}, x_{2n-1}), d(x'_{2n}, x'_{2n-1})))d(A, B)) \tag{3.12}$$

for each $n \in N$. Since (A, B) has a property UC, $d(x_{2n}, x_{2n+2}) \to 0$ and $d(x'_{2n}, x'_{2n+2}) \to 0$. Also, (B, A) has a property UC, we have $d(x_{2n-1}, x_{2n+1}) \to 0$ and $d(x'_{2n-1}, x'_{2n+1}) \to 0$. Using $\omega(d(x_{2n}, x_{2n-1}), d(x'_{2n}, x'_{2n-1})) < 1$, and $d(x_{2n}, x_{2n-1}) \geq d(A, B)$, we have
$\omega(d(x_{2n}, x_{2n-1}), d(x'_{2n}, x'_{2n-1})) \max[d(x_{2n}, x_{2n-1}), d(x'_{2n}, x'_{2n-1}),$
$d(x_{2n}, F(x_{2n}, x'_{2n})), d(x_{2n-1}, G(x_{2n-1}, x'_{2n-1})), d(x_{2n}, G(x_{2n-1}, x'_{2n-1})),$
$d(x_{2n-1}, F(x_{2n}, x'_{2n}))] + (1 - \omega(d(x_{2n}, x_{2n-1}), d(x'_{2n}, x'_{2n-1})))d(A, B)$

$$\leq \frac{1}{6} \max[d(x_{2n}, x_{2n-1}), d(x'_{2n}, x'_{2n-1}) \leq d(x_{2n}, x_{2n-1}) \tag{3.13}$$

for each $n \in N$. From (3.12) and (3.13), we have

$$d(x_{2n+1}, x_{2n}) \leq d(x_{2n}, x_{2n-1})$$

for each $n \in N$. Also, we conclude that $d(x_{n+1}, x_n) \leq d(x_{n-1}, x_n)$ for each $n \in N$. Thus $\{d(x_{n+1}, x_n)\}$ is a non increasing sequence of positive real numbers. Consequently, there exist some real number $t_0 \geq d(A, B)$ such that $\{d(x_{n+1}, x_n)\}$ converges to t_0 . On the contrary that $t_0 > d(A, B)$. Since $\limsup_{(s,s')\to(t_0,t'_0)} \omega(s, s') < 1$ and $\omega(t_0, t'_0) < 1$, there exists $t \in (0, 1)$ and $\epsilon > 0$ such that $\omega(s, s') \leq t$ for all $(s, s') \in [t_0, t_o+\epsilon]\times[t'_0, t'_o+\epsilon']$. If $N_0 \in N$ is such that $t_0 \leq d(x_n, x_{n+1}) \leq t_0 + \epsilon$ for all $n \geq N_0$. Then $\omega(d(x_n, x_{n+1}), d(x'_n, x'_{n+1})) \leq t$ for $n \geq N_0$. Thus

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &\leq \omega(d(x_{2n}, x_{2n-1}), d(x'_{2n}, x'_{2n-1})) \max[d(x_{2n}, x_{2n-1}), d(x'_{2n}, x'_{2n-1})] \\ &\quad +(1 - \omega(d(x_{2n}, x_{2n-1}), d(x'_{2n}, x'_{2n-1}))d(A, B) \\ &\leq t[d(x_{2n}, x_{2n-1}) - d(A, B)] + d(A, B) \end{aligned}$$

On taking limit as $n \to \infty$ on both sides of above inequality, we have $t_0 = \lim_{n\to\infty} d(x_n, x_{n+1}) \leq d(A, B)$, a contradiction. Then it follows that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B).$$

That is, $\lim_{n\to\infty} d(x_n, F(x_n, x'_n)) = d(A, B)$. Similarly, $\lim_{n\to\infty} d(y_n, G(y_n, y'_n)) = d(A, B)$. Following arguments similar to those in the proof of Theorem (3.1), we obtain that $\{x_{2n}\}, \{x'_{2n}\}, \{x_{2n+1}\}$ and $\{x'_{2n+1}\}$ are Cauchy sequences.
Since (X, d) is a complete b-metric space, there exists $x \in X$ such that $x_{2n} \to x$. Similarly, there exists $x' \in X$ such that $x'_{2n} \to x'$ and $y, y' \in X$ such that $x_{2n+1} \to y$ and $x'_{2n+1} \to y'$ respectively.

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Now we show that the coupled best proximity point $(x,x') \in A \times A$ is unique. If there is another fixed point $a \in A$, then $G(a,a') = a$ such that $x \neq a$. Now

$$\begin{aligned} d(x,a) &= d(F(x,x'),G(a,a')) \\ &\leq \omega(d(x,a),d(x',a')))\max[d(x,a),d(x',a'),d(x,F(x,x')),d(a,G(a,a')), \\ &\quad d(x,G(a,a'))+d(a,F(x,x'))+[1-6\omega(d(x,a),d(x',a'))]d(A,B) \\ &\leq \omega(d(x,a),d(x',a')))[d(x,a)+d(x',a')+[1-6\omega(d(x,a),d(x',a'))]d(A,B). \end{aligned}$$

Since $\omega(d(x,a),d(x',a')) \leq t < \frac{1}{6}$, take $x = x'$ and $a = a'$ to obtain that

$$\begin{aligned} d(x,a) &\leq t(d(x,a)) + (1-6t)d(A,B) \text{ and} \\ d(x,a) &\leq \frac{1-6t}{1-t}d(A,B) < d(A,B). \end{aligned}$$

Hence, $x = a$ and $x' = a'$. Therefore $(x,x') \in A \times A$ is a unique coupled best proximity point of F. Similarly, $(y,y') \in B \times B$ is a unique coupled best proximity point of G.

Corollary 3.5([32]). Let A and B be nonempty closed subsets of a complete metric space (X,d). If mappings $F : A \times A \to B$, and $G : B \times B \to A$ satisfy

$$d(F(x,x'),G(y,y')) \leq \frac{k}{2}[d(x,y)+d(x',y')] + (1-k)d(A,B)$$

for all $(x,x') \in A \times A$ and $(y,y') \in B \times B$. Then $(p,q) \in A \times A$ and $(p',q') \in B \times B$ are unique coupled best proximity points of F and G, respectively.
Proof. Take, $F = G$, $\omega(d(x,y),d(x',y')) = k$ where

$$d(x,F(x,x')) = d(y,G(y,y')) \text{ and } F(x,x') = G(y,y').$$

Since b-metric space is the generalisation of a metric space. The result follows from Theorem 3.1.

Corollary 3.6([29]). Let A and B be nonempty closed subsets of a complete metric space (X,d), and $T : A \cup B \to A \cup B$. If T is any one of the following types:
(1) Kannan type cyclic contraction
(2) Chatterjea type cyclic contraction
(3) Reich type cyclic contraction and
(4) Ciric type cyclic contraction, then T has a unique best proximity point.

Corollary 3.7. Let (X,d) be a complete metric space and $T : X \to X$. If for any $x,y \in X$, we have

$$d(Tx,Ty) \leq \alpha(d(x,y))d(x,y)$$

where $\alpha : [0,\infty) \to [0,1)$ satisfies $\limsup_{s \to t^+} \alpha(s) < 1$ for each $t \in (0,\infty)$. Then T has a fixed point.

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Corollary 3.8([4]). Let A and B be nonempty closed subsets of a complete metric space (X, d), and $T : A \cup B \to A \cup B$ a generalised contraction map, that is, for any $x, y \in X$, we have

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y) + (1 - \alpha)d(A, B).$$

Then T has a best proximity point.
Proof. The proof follows from Theorem 3.1 if we take
d(x$'$, $y') = d(x, F(x, x')) = d(y, G(y, y')) = d(x, G(y, y')) = d(y, F(x, x'))$
$= 0$.

Corollary 3.9([34]). Let (X,d) be a complete metric space and T:X$\to$ X. If for any $x, y \in X$, with $x \neq y$, we have

$$d(Tx, Ty) < \max[d(x, y), \frac{d(x, T(x)) + d(y, T(y))}{2}, \frac{d(x, T(y), d(y, T(x))}{2}].$$

Then T has a unique fixed point.

4 Application

We give the following application of the main result:
Let A, B be nonempty subsets of X, $X = L_2$ $([0, 1])$, a space of integrable functions, where

$$\|f\| = [\int f^2(t)dt]^{\frac{1}{2}}, \text{ and } d(f, g) = \|f - g\|$$

for $f, g \in X$. Let

$$A = \{f \in X : -2 \leq f(t) \leq -1\}, \text{ and } B = \{g \in X : 2 \leq g(t) \leq 3\}.$$

Clearly, $d(A, B) = \inf_{f\in A, g\in B}[\int_0^1 (f - g)^2(t)dt]^{\frac{1}{2}} = 3$. Let $\sqrt{3} \leq s \leq 12\sqrt{3} + \frac{1}{3}$ and $\omega(t, t') = \frac{t\sqrt{3}}{18}$, where $(t, t') \in (d(A, B), \infty) \times (d(A, B), \infty)$. Define $F : A \times A \to B$ and $G : B \times B \to A$ by

$$F(f, u) = \frac{t}{18}[-2sf(t) + f(t) - su(t)] + 1 \text{ and } \tag{4.1}$$

$$G(g, v) = \frac{t}{18}[-g(t) - sv(t) - sG(g, v) + G(g, v)] - 1 \tag{4.2}$$

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Note that

$$\begin{aligned} d(F(f,u),G(g,v)) &= \|F(f,u)-G(g,v)\| \\ &= \|\frac{t}{18}[-2sf(t)+f(t)-su(t)]-\frac{t}{18}[-sg(t)-sv(t) \\ &\quad -sG(g,v)+G(g,v)]+2\| \\ &= \|\frac{t}{18}[sf(t)-sg(t)+su(t)-sv(t) \\ &\quad +sf(t)-sG(g,v)+f(t)-G(g,v)]+2\| \\ &\leq \frac{1}{18}\|t\|[s\|f-g\|+s\|u-v\|+s\|f-G(g,v)\|+s\|f-F(f,u)\| \\ &\quad +s\|F(f,u)-g\|+s\|g-G(g,v)\|]+2 \\ &\leq \frac{s}{18}\|t\|[\|f-g\|+\|u-v\|+\|f-G(g,v)\|+\|f-F(f,u)\| \\ &\quad +\|F(f,u)-g\|+\|g-G(g,v)\|]+2. \end{aligned}$$

Thus,

$$\begin{aligned} d(F(f,u),G(g,v)) &\leq \frac{s}{18}[d(f,g)+d(u,v)+d(f,G(g,v)) \\ &\quad +d(f,F(f,u)+d(F(f,u),g)+d(g,G(g,v))][\int_0^1(t^2dt)^{\frac{1}{2}}]+2. \end{aligned}$$

Also, $\omega(d(f,g),d(u,v)=\frac{1}{18}$, where $\|t\|=\frac{1}{\sqrt{3}}$ and $s=\sqrt{3}$, we have

$$\begin{aligned} d(F(f,\alpha),G(g,u)) &\leq \omega(d(f,g),d(u,v))[d(f,g)+d(u,v)+d(f,G(g,v))+d(f,F(f,u) \\ &\quad +d(F(f,u),g)+d(g,G(g,v))]+(1-6\omega(d(f,g),d(u,v)))d(A,B). \end{aligned}$$

Hence, (F,G) is a Hardy-Rogers type cyclic ω-contraction.
Now, we show that an initial value problem: $\frac{dy}{dt}=\frac{t}{18}[-2sy+y-sz)]+1$, $y(0)=0$ has a unique solution valid for all $x\in[0,1]$. That is, there is one and only function $f:[0,1]\to R$ with the property that $f'(t)=\frac{t}{18}[-2sf(t)+f(t)-su(t))]+1$ for all $t\in[0,1]$ and $f(0)=0$. Define $F: L_p\times L_p\to R$ by $F(f,u)=2$, for $s=6\sqrt{3}+\frac{1}{3}$, $f=-1$ and $u=-1$ in (4.1). Clearly, F has a unique coupled best proximity point $(-1,-1)$. So $d(f,F(f,u))=d(u,F(u,f))=d(A,B)$ solve the initial value problem under consideration if and only if $(-1,-1)$ is the coupled best proximity pair of F. Similarly, $(-1,-1)$ is the coupled best proximity pair of G. Since, (F,G) is a Hardy-Rogers cyclic ω-contraction, (F,G) has only one coupled best proximity point which is $(-1,-1)$.

Open Question.

Some authors have shown that results on coupled fixed point results can be retrieved from known results on fixed point and vice-versa (see[31]). Can we deduce results on coupled best proximity point from known results on best proximity points and vice-versa?

Competing financial Interest.

The authors declare that they have no competing financial interest.

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