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Block Method Coupled with the Compact Difference Schemes for the Numerical Solution of Nonlinear Burgers' Partial Differential Equations

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Abstract

In this paper, a novel block method is proposed to solve the nonlinear time dependent Burgers' equation. The Burgers' PDE is semi discretized in spatial direction by using the standard fourth-order compact difference schemes to yield system of nonlinear ordinary differential equations (ODE) in time.The resulting system of first-order ODE from the Burgers' equation is approximated by a new derived Block method. The new two-step hybrid methods are developed through the Interpolation and Collocation techniques. The derived methods are applied as a block method for the numerical solution of the nonlinear Burgers' Partial Differential Equations (PDE) which is of physical relevance. The proposed block scheme has been proven to be zero-stable, consistent and convergent, also saving computational time while maintaining good accuracy. The efficiency of the derived method is demonstrated using three test problems.

Keywords: Block Method, Burgers' Equation, Collocation Technique, Compact Difference Scheme, Nonlinear PDEs.
MSC2010: 65M06, 65N35, 35F50.

1 Introduction

Nonlinear partial differential equations (PDEs) are used to model phenomena in a wide range of scientific and technical fields, including heat transport, fluid dynamics, quantum physics. A very important phenomena to research in fluid dynamics 'turbulence' is modeled by a nonlinear partial differential equation called the Burgers' equation. It describes the behaviour of one dimensional viscous fluid undergoing both advection and diffusion. The one dimensional nonlinear time dependent Burgers' equation is Mathematically expressed in the form:

$$\mu_t + \mu\mu_x = \nu\mu_{xx} \tag{1}$$

along with the initial condition

$$\mu(x, 0) = g(x), \quad a \leq x \leq b, \tag{2}$$

and two boundary conditions as

$$\mu(a, t) = g_1(t) = \mu_1(t), \quad \mu(b, t) = g_2(t) = \mu_{N+1}(t), \quad t \geq 0. \tag{3}$$

where μ, x and t represent the solution of the problem, the space and the time variables respectively. Also ν denotes the kinematic viscosity of the fluid. Burgers' equation has both linear and nonlinear components. the nonlinear component $\mu\mu_x$ represents advection or transport of velocity, while the linear component or viscosity component $\nu\mu xx$ represents diffusion or smoothing of velocity due to viscosity. Many numerical methods have been used to profer solution to the Burgers' equation. Methods based on finite difference were proposed in Evans and Abdullah [1]; Kutluay et al. [2]; Kadalbajoo and Awasthi [3]; Yang et al. [4]; Mukundan and Awasthi [5]; those based on finite element were introduced in Varoglu and LiamFinn [6]; Tadmor [7]; Kutluay et al. [8], a method based on a blend of both the predictor-corrector technique and finite difference was introduced in Zhang and Wang [9] and methods based on quasi-interpolation and quasi-linearisation were introduced in Wang et al. [10] and Jwari [11] respectively. But most of these methods are single-step methods and hence using them involves iterating through many steps. Also, a very small step size has to be used to get good accuracy. The most recent advancement in the numerical treatment of the Burgers' equation is the derivation of block methods for the solution of the resulting nonlinear system of ordinary differential equations after its semi-discretisation. Ramos et al. [12] derived a block hybrid method and used it to solve the Burgers' equation and few other time dependent PDEs. Similarly, Mehta et al. [13] used a three step block method coupled with the fourth order compact finite difference scheme to solve the Burgers equation. These methods gave result with very high order accuracy even with a relatively large step size and approximate solution at more than one grid point at a time.
In this paper, various form of the Burgers' equation were solved. The fourth-order compact difference scheme is used for spatial discretisation of Burgers' equation and a novel optimised second derivative two-step block hybrid method for numerical solution of the resulting system of ODEs is developed. The two-step block hybrid method is derived using polynomial interpolation and optimised for accuracy by introducing hybrid points and some points on the first derivative of its basis function. The stability and convergence analysis are carried out using boundary locus plot.

2 Derivation of the Second Derivative Two step Hybrid Block Method

Each component of equations (1) - (3) can be expressed as an Ordinary Differential Equation (ODE) of the form:

$$\mu' = f(t, \mu), \quad \mu(t_0) = \mu_0, \quad t_0 \leq t \leq t_m \tag{4}$$

Now we can discretize the time interval $[t_0, t_m]$ into m steps of equal distance $h = \frac{t_m - t_0}{m}$, and the grid points would be :

$$t_0 < t_1 < t_2 \cdots < t_m,$$

let

$$\mu(t) = \sum_{r=0}^{8} b_r t^r \tag{5}$$

be a eight degree polynomial approximation of the exact solution of equation (4) with corresponding derivatives as

$$\mu'(t) = \sum_{r=1}^{8} r b_r t^{r-1} \tag{6}$$

$$\mu''(t) = \sum_{r=2}^{8} r(r-1) b_r t^{r-2} \tag{7}$$

where $b_r, r = 0(1)8$ are nine unknown coefficients to be determined. Interpolating equation (5) and collocating equations (6) and (7) at given grid points give

$$\mu_{n+j} = \mu(t_{n+j}), \;\; j = 0, \tag{8}$$

$$\mu'_{n+j} = \mu'(t_{n+j}) = f_{n+j}, \;\; j = 0, w_1, 1, w_2, 2, \tag{9}$$

$$\mu''_{n+j} = \mu''(t_{n+j}) = g_{n+j}, \;\; j = 0, 1, 2, \tag{10}$$

where μ_{n+j} , f_{n+j} and g_{n+j} are approximations for $\mu(t_{n+j})$, $\mu'(t_{n+j})$ and $\mu''(t_{n+j})$ respectively and w_1, w_2 are hybrid points. The system of nine equations in equation $(8) - (10)$ in matrix form is given as

$$\begin{bmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 & t_n^6 & t_n^7 & t_n^8 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 & 6t_n^5 & 7t_n^6 & 8t_n^7 \\ 0 & 1 & 2t_{n+\frac{1}{3}} & 3t_{n+w_1}^2 & 4t_{n+w_1}^3 & 5t_{n+w_1}^4 & 6t_{n+w_1}^5 & 7t_{n+w_1}^6 & 8t_{n+w_1}^7 \\ 0 & 1 & 2t_{n+1} & 3t_{n+1}^2 & 4t_{n+1}^3 & 5t_{n+1}^4 & 6t_{n+1}^5 & 7t_{n+1}^6 & 8t_{n+1}^7 \\ 0 & 1 & 2t_{n+w_2} & 3t_{n+w_2}^2 & 4t_{n+w_2}^3 & 5t_{n+w_2}^4 & 6t_{n+w_2}^5 & 7t_{n+w_2}^6 & 8t_{n+w_2}^7 \\ 0 & 1 & 2t_{n+2} & 3t_{n+2}^2 & 4t_{n+2}^3 & 5t_{n+2}^4 & 6t_{n+2}^5 & 7t_{n+2}^6 & 8t_{n+2}^7 \\ 0 & 0 & 2 & 6t_n & 12t_n^2 & 20t_n^3 & 30t_n^4 & 42t_n^5 & 56t_n^6 \\ 0 & 0 & 2 & 6t_{n+1} & 12t_{n+1}^2 & 20t_{n+1}^3 & 30t_{n+1}^4 & 42t_{n+1}^5 & 56t_{n+1}^6 \\ 0 & 0 & 2 & 6t_{n+2} & 12t_{n+2}^2 & 20t_{n+2}^3 & 30t_{n+2}^4 & 42t_{n+2}^5 & 56t_{n+2}^6 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \\ b_8 \end{bmatrix} = \begin{bmatrix} \mu_n \\ f_n \\ f_{n+w_1} \\ f_{n+1} \\ f_{n+w_2} \\ f_{n+2} \\ g_n \\ g_{n+1} \\ g_{n+2} \end{bmatrix} \tag{11}$$

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Solving (11) simultaneously gives the corresponding coefficients of $b_r, r = 0(1)8$. Substituting the resulting coefficients $b_r, r = 0(1)8$ into equation (5) and its derivatives yields a continuous implicit scheme of the form,

$$\alpha_z \mu_{n+z} = \alpha_0 \mu_n + h \sum_{j=0}^{2} \beta_j f_{n+j} + h \sum_{j=1}^{2} \beta_{w_j} f_{n+w_j} + h^2 \sum_{j=0}^{2} \rho_j g_{n+j}, \quad z = w_1, 1, w_2, 2. \tag{12}$$

To obtain the approximate values of w_1 and w_2 hybrid points, first derive the schemes $\mu_{n+z}, z = w_1, 1, w_2, 2$ in equation (12), then find the error constant of any of the schemes, I specifically used μ_{n+2}. In this paper, the error constant for the scheme μ_{n+2} is $c_9 \neq 0$. Set the error constant of the scheme to be zero and ensure that the hybrid points satisfy the interval $0 < w_1 < 1 < w_2 < 2$ (see [12]), we have

$$3w_1w_2 - 3w_1 - 3w_2 + 4 = 0, \tag{13}$$

$$0 < w_1 < 1 < w_2 < 2, \tag{14}$$

Use (13) and (14) to scan for w_1 and w_2 such that the scheme attains order nine yields the solution as $w_1 = \frac{1}{3}$ and $w_2 = \frac{3}{2}$ as one of the possible solutions.
The discrete hybrid schemes derived by evaluating (12) at grid and non-grid points $t_{n+\frac{1}{3}}$,t_{n+1},$t_{n+\frac{3}{2}}$ and t_{n+2} are

$$\begin{aligned}\mu_{n+\frac{1}{3}} = \mu_n + \frac{\Delta t}{1928934000}[346046225 f_n + 331940673 f_{n+\frac{1}{3}} - 15017625 f_{n+1} - 256064 f_{n+\frac{3}{2}} + 5615127 f_{n+2}] \\ + \frac{(\Delta t)^2}{1928934000}[19248600 g_n + 20377350 g_{n+1} - 912870 g_{n+2}]\end{aligned} \tag{15}$$

$$\begin{aligned}\mu_{n+1} = \mu_n + \frac{\Delta t}{2646000}[286825 f_n + 1318761 f_{n+\frac{1}{3}} + 921375 f_{n+1} + 147200 f_{n+\frac{3}{2}} - 28161 f_{n+2}] \\ \frac{(\Delta t)^2}{2646000}[4200 g_n - 211050 g_{n+1} + 4410 g_{n+2}]\end{aligned} \tag{16}$$

$$\begin{aligned}\mu_{n+\frac{3}{2}} = \mu_n + \frac{\Delta t}{25088000}[2426900 f_n + 13167927 f_{n+\frac{1}{3}} + 16466625 f_{n+1} + 6150400 f_{n+\frac{3}{2}} - 579852 f_{n+2}] \\ + \frac{(\Delta t)^2}{25088000}[525 g_n - 784350 g_{n+1} + 85995 g_{n+2}]\end{aligned} \tag{17}$$

$$\begin{aligned}\mu_{n+2} = \mu_n + \frac{\Delta t}{165375}[19775 f_n + 78732 f_{n+\frac{1}{3}} + 94500 f_{n+1} + 102400 f_{n+\frac{3}{2}} + 35343 f_{n+2}] \\ + \frac{(\Delta t)^2}{165375}[525 g_n - 12600 g_{n+1} - 2205 g_{n+2}]\end{aligned} \tag{18}$$

Equations (15) – (18) form the proposed Two-step Hybrid Block Method (TSHBM) developed for the approximation of the resulting system of ODEs obtained from the discretized nonlinear time dependent Burgers' PDE.

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3 Analysis of TSHBM

3.1 Order and Error Constant of TSHBM

The order and error constants of the Block Method (15) - (18) is obtained by evaluating their local truncation error as shown in Akinnukawe [14,15] and Modebei et al. [16] . Suppose that $\mu(t_n)$ is a continuously differentiable function and recall that $\mu'(t_n) = f(t_n)$, $\mu''(t_n) = g(t_n)$. The local truncation error of the block scheme (15) - (18) is as follows:

$$L[\mu(t_n);h] = \sum_{j=0}^{2}\alpha_j\mu(x_n+jh) - h\sum_{j=0}^{2}\beta_j\mu'(t_n+jh) - h\sum_{j=1}^{2}\beta_{w_j}\mu'(t_n+(w_j)h) - h^2\sum_{j=0}^{2}\rho_j\mu''(x_n+jh) \tag{19}$$

Assuming that $\mu(t_n)$ is sufficiently differentiable, then using Taylor series expansion on $\mu(t_n + jh)$,$\mu'(t_n + jh)$ and $\mu''(t_n + jh)$ about t_n, we have

$$\mu(t_n + jh) = \sum_{m=0}^{\infty}\frac{(jh)^m}{m!}\mu^{(m)}(t_n),$$

$$\mu'(t_n + jh) = \sum_{m=1}^{\infty}\frac{(jh)^m}{m!}\rho^{(m+1)}(t_n),$$

$$\mu''(t_n + jh) = \sum_{m=2}^{\infty}\frac{(jh)^m}{m!}\mu^{(m+2)}(t_n).$$

Substituting $\mu(t_n + jh)$, $\mu'(t_n + jh)$ and $\mu''(t_n + jh)$ in equation (19) to obtain

$$L[\mu(t_n);h] = C_0\mu(t_n)+C_1h\mu'(t_n)+C_2h^2\mu''(t_n)+C_3h^3\mu'''(t_n)+...+C_{m+2}h^{m+2}\mu^{(m+2)}(t_n)+... \tag{20}$$

where $C_m, m = 0, 1, 2, ...$ are constants given as:

$$C_0 = \sum_{j=0}^{2}\alpha_j + \sum_{j=1}^{2}\alpha_{w_j},$$

$$C_1 = \left[\sum_{j=0}^{2} j\alpha_j + \sum_{j=1}^{2} w_j\alpha_{w_j}\right] - \beta_j,$$

$$\vdots$$

$$C_{m+1} = \frac{1}{(m+1)!}\left[\sum_{j=0}^{2} j^{m+1}\alpha_j + \sum_{j=1}^{2}(w_j)^{m+1}\alpha_{w_j}\right] - \frac{1}{(m)!}\left[\sum_{j=0}^{2} j^m\beta_j + \sum_{j=1}^{2}(w_j)^m\beta_{w_j}\right] - \frac{1}{(m-1)!}\left[\sum_{j=0}^{2} j^{m-1}\rho_j\right]. \tag{21}$$

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Table 1: Order and Error Constants of TSHBM

S/N	Scheme	Order(m)	Error Constant (C_{m+1})
1	$\mu_{n+\frac{1}{3}}$	8	$\frac{1625}{22856214528}$
2	μ_{n+1}	8	$-\frac{1}{5806080}$
3	$\mu_{n+\frac{3}{2}}$	8	$-\frac{1}{3670016}$
4	μ_{n+2}	9	$\frac{1}{42865200}$

3.2 Zero-stability of the method

The block method is said to be zero-stable if the roots Z_u, $u = 1, 2, \ldots, 4$ of the first characteristic polynomial $\gamma(Z)$ satisfy $|Z_u| \leq 1, u = 1, \ldots, 4$ multiplicity not exceeding the order of the differential equation [15]. Since the partial differential equation has been reduced to a system of first-order ODE, the order of the ODE is one. The derived method written in a standard form

$$\Rightarrow A^{(0)}\mu_{n+i} = A^{(1)}\mu_{n-i} + \Delta t B^{(0)} F_{n+i} + \Delta t B^{(1)} F_{n-i} + (\Delta t)^2 B^{(2)} G_{n+i}$$

The first characteristic polynomial of the block scheme (15) - (18) is defined as:

$$P(\lambda) = det[\lambda A^{(0)} - A^{(1)}] = \left| \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & \lambda - 1 \end{vmatrix} = 0, \tag{22}$$

$$P(\lambda) = \lambda^3(\lambda - 1) = 0$$

since $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 = 1$, the block method is zero-stable.

3.3 Consistency

According to Fatunla [17] and Henrici [18], the necessary and sufficient condition for a numerical scheme to be consistent is for the method to have order of at least one ($m \geq 1$). The derived method is of at least order 8 since the least order of the block method is of order 8 as shown in Table 1.

3.4 Convergence of the Method

A numerical method converges if it is consistent and zero-stable [18]. This implies that TSHBM converges since the method is of order $m = 8 > 1$ and it satisfies the conditions for zero-stability.

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3.5 Linear Stability Analysis

To carry out the linear stability analysis of TSHBM, we consider the Dahlquist test equation given by:

$$u'(t) = \lambda u(t), \quad Re(\lambda) < 0, \tag{23}$$

The general exact solution of equation (23) is given by $u(t) = exp(\lambda t)$. This solution will vanish as t tends to infinity. A numerical method is said to be linearly stable if the solution obtained by using it has similar qualitative behavior as the exact solution ([12]). To determine the region for which this is true for the proposed block hybrid method, TSHBM, we apply it to ODE (23), and this gives the following result.

$$A \begin{bmatrix} \mu_{n+\frac{1}{3}} \\ \mu_{n+1} \\ \mu_{n+\frac{3}{2}} \\ \mu_{n+2} \end{bmatrix} = B \begin{bmatrix} \mu_{n-\frac{3}{2}} \\ u\mu_{n-1} \\ \mu_{n-\frac{1}{3}} \\ \mu_{n} \end{bmatrix} \tag{24}$$

where

$$A = \begin{bmatrix} 1-\frac{331940673z}{1928934000} & \frac{15017625z}{1928934000}-\frac{20377350z^2}{1928934000} & \frac{25606400z}{1928934000} & -\frac{5615127z}{1928934000}+\frac{912870z^2}{1928934000} \\ -\frac{1318761z}{2646000} & 1-\frac{921375z}{2646000}+\frac{211050z^2}{2646000} & -\frac{147200z}{2646000} & \frac{28161z}{2646000}-\frac{4410z^2}{2646000} \\ -\frac{13167927z}{25088000} & -\frac{16466625z}{25088000}+\frac{784350z^2}{25088000} & 1-\frac{6150400z}{25088000} & \frac{579852z}{25088000}-\frac{85995z^2}{25088000} \\ -\frac{78732z}{165375} & -\frac{94500z}{165375}+\frac{12600z^2}{165375} & -\frac{102400z}{165375} & 1-\frac{35343z}{165375}+\frac{2205z^2}{165375} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 1+\frac{346046225z}{1928934000}+\frac{19248600z^2}{1928934000} \\ 0 & 0 & 0 & 1+\frac{286825z}{2646000}+\frac{4200z^2}{2646000} \\ 0 & 0 & 0 & 1+\frac{2426900z}{25088000}+\frac{525z^2}{25088000} \\ 0 & 0 & 0 & 1+\frac{19775z}{165375}+\frac{525z^2}{165375} \end{bmatrix}$$

and

$$z = \lambda \Delta t$$

. Clearly, (24) can be written as:

$$\begin{bmatrix} \mu_{n+\frac{1}{3}} \\ \mu_{n+1} \\ \mu_{n+\frac{3}{2}} \\ \mu_{n+2} \end{bmatrix} = M(z) \begin{bmatrix} \mu_{n-\frac{3}{2}} \\ \mu_{n-1} \\ \mu_{n-\frac{1}{3}} \\ \mu_{n} \end{bmatrix},$$

Where the stability matrix is $M(z) = A^{-1}B$. Next, we obtained the eigenvalues of the stability matrix to be $(0, 0, 0, P(z))$, where

$$P(z) = \frac{5z^6 + 84z^5 + 735z^4 + 4065z^3 + 14490z^2 + 30870z + 30240}{3(z^6 - 20z^5 + 195z^4 - 1165z^3 + 4410z^2 - 9870z + 10080)}$$

is the stability function.

Finally, consider a set

$$S = \{z \in \mathbb{C} : |P(z)| < 1\}$$

which is the stability region of the proposed block hybrid method. Figure 1 shows that the stability region, which proves that TSHBM is A-stable.

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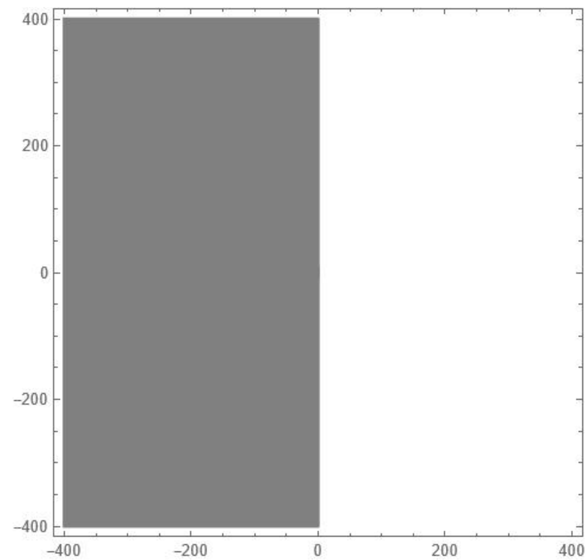

Figure 1: A graph showing the stability region of the proposed method

4 Discretization of the Space Variable

To obtain the semi-discretization of equations (1) - (3), We approximate the spatial derivatives using the standard fourth-order compact difference scheme (Mehta et al. [13]. To do this, we first partition the space interval $[a, b]$ into N equal parts as shown below:

$$a = x_1 < x_2 < \cdots < x_N < x_{N+1} = b$$

with equal distance between each consecutive points(i.e. $\Delta x = x_2 - x_1 = x_3 - x_2 = \cdots = x_{N+1} - x_N$). From Li and Chen [18], the first order derivative of $u(x,t)$ with respect to the space variable x at the interior node satisfies the following relation:

$$\frac{1}{4}\mu'_{j-1} + \mu'_j + \frac{1}{4}\mu'_{j+1} = \frac{3}{4\Delta x}\left[\mu_{j+1} - \mu_{j-1}\right], \quad j = 2, 3, 4, ..., N, \tag{25}$$

and at the boundary points the following relations:
for $j = 1$

$$\mu'_1 + 3\mu'_2 = \frac{1}{\Delta x}\left(-\frac{17}{6}\mu_1 + \frac{3}{2}\mu_2 + \frac{3}{2}\mu_3 - \frac{1}{6}\mu_4\right), \tag{26}$$

for $j = N + 1$

$$3\mu'_N + \mu'_{N+1} = \frac{1}{\Delta x}\left(-\frac{17}{6}\mu_{N+1} + \frac{3}{2}\mu_N + \frac{3}{2}\mu_{N-1} - \frac{1}{6}\mu_{N-2}\right), \tag{27}$$

The above formulae have fourth-order accuracy. For any value of N, equation (25)-(27) can be expressed in matrix compact form as:

$$A_1 U' = A_2 U \tag{28}$$

where

$$A_1 = \begin{bmatrix} 1 & 3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & \frac{1}{4} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 1 & \frac{1}{4} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 1 & \frac{1}{4} & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{1}{4} & 1 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{4} & 1 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{4} & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 3 & 1 \end{bmatrix},$$

$$A_2 = \frac{1}{2\Delta x} \begin{bmatrix} -\frac{17}{3} & 3 & 3 & -\frac{1}{3} & \cdots & 0 & 0 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{3}{2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & 0 & \frac{3}{2} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{3}{2} & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{3} & -3 & -3 & \frac{17}{3} \end{bmatrix},$$

$$U' = \begin{bmatrix} \mu_1' \\ \mu_2' \\ \vdots \\ \mu_N' \\ \mu_{N+1}' \end{bmatrix} \text{ and } U = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \\ \mu_{N+1} \end{bmatrix}$$

From equation(28), we can obtain the approximate value of μ_x by carrying out the matrix multiplication $A_1^{-1}A_2U$ (i.e. $\mu_x = U' = A_1^{-1}A_2U$).
Similarly, from Li and Chen [19] and by following the same procedure as in the first-order spatial discretization above, we obtain the approximate value of the second-order spatial derivative to be:

$$\mu_{xx} = U'' = A_3^{-1}A_4U \tag{29}$$

Where,

$$A_3 = \begin{bmatrix} 1 & 11 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{10} & 1 & \frac{1}{10} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{10} & 1 & \frac{1}{10} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{10} & 1 & \frac{1}{10} & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{1}{10} & 1 & \frac{1}{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{10} & 1 & \frac{1}{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{10} & 1 & \frac{1}{10} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 11 & 1 \end{bmatrix}$$

,

$$A_4 = \frac{1}{\Delta x} \begin{bmatrix} 13 & -27 & 15 & -1 & \cdots & 0 & 0 & 0 & 0 \\ \frac{6}{5} & -\frac{12}{5} & \frac{6}{5} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5} & -\frac{12}{5} & \frac{6}{5} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{6}{5} & -\frac{12}{5} & \frac{6}{5} \\ 0 & 0 & 0 & 0 & \cdots & -1 & 15 & -27 & 13 \end{bmatrix},$$

$$U'' = \begin{bmatrix} \mu_1'' \\ \mu_2'' \\ \vdots \\ \mu_N'' \\ \mu_{N+1}'' \end{bmatrix} \text{ and } U = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \\ \mu_{N+1} \end{bmatrix}.$$

Substituting the transformation $\mu_x = \mu' = A_1^{-1}A_2\mu$ and $\mu_{xx} = \mu'' = A_3^{-1}A_4\mu$ into the Burgers' equation yields

$$\mu_t = v(A_3^{-1}A_4)\mu - \mu \circ (A_1^{-1}A_2)\mu$$

$$\begin{bmatrix} \mu_1' \\ \mu_2' \\ \vdots \\ \mu_N' \\ \mu_{N+1}' \end{bmatrix} = v(A_3^{-1}A_4) \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \\ \mu_{N+1} \end{bmatrix} - \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \\ \mu_{N+1} \end{bmatrix} \circ (A_1^{-1}A_2) \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \\ \mu_{N+1} \end{bmatrix} \tag{30}$$

The symbol $\circ$ stands for the element-wise product of two matrices of the same dimension. Equation (30) can be further expressed as

$$\mu' = A\mu + C$$

where $A = v(A_3^{-1}A_4)$ and C is the remaining nonlinear components.

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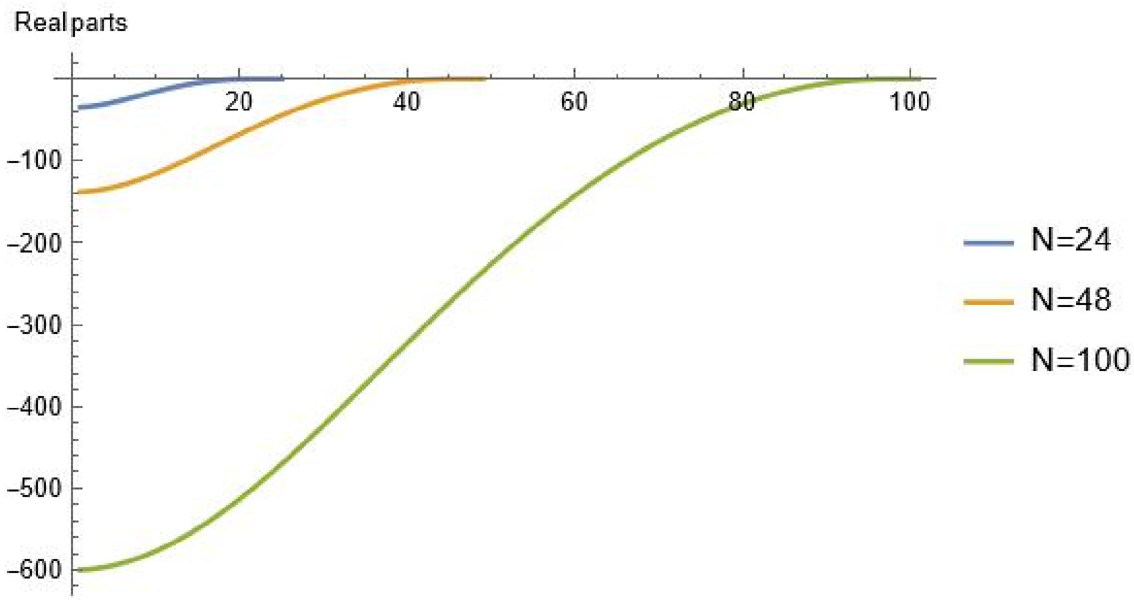

Figure 2: A graph showing the real parts of all the eigenvalues of matrix A for some values of N

5 Stability of the Differential System

To investigate the stability of the differential system (30), we follow the same procedure as in (Mehta et al. [13]). The first step is to linearize it by assuming a constant value for $\mu(x,t)$ in the nonlinear terms(i.e. $\mu(x,t) = \mu_{i,j}$ in the nonlinear terms). After doing this, if the resulting linear system is stable, then the initial nonlinear system is stable. More explicitly, system (30) is stable if all the eigenvalues of matrix A has either zero or negative real parts. The graph showing the real parts of all the eigenvalues of matrix A for some values of N is given in Figure 2. And from the graph, they all fall on the negative plane. Hence, the differential system (30) is stable.

6 Numerical Experiments

In this section, the performance of the derived method is shown by solving three special cases of the Burgers' equation. The numerical results from the developed method is compared with some existing methods. Figures 3 and 4 show the physical behaviour of the numerical and exact solutions of examples 2 and 3 respectively. All computations were done using Wolfram MATHEMATICA 13.3 software.

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6.1 Example 1

Consider the one-dimensional Burgers' equation [5]

$$\frac{\partial \mu}{\partial t} + \mu \frac{\partial \mu}{\partial x} = v \frac{\partial^2 \mu}{\partial x^2}$$

with initial condition:

$$\mu(x, 0) = \sin(\pi x), 0 \leq x \leq 1$$

and boundary condition:

$$\mu(0, t) = \mu(1, t) = 0$$

The exact solution of the problem is given by

$$\mu(x, t) = 2\pi v \frac{\sum_{n=1}^{\infty} C_n \exp{(-n^2 \pi^2 v t)} n \sin{(n\pi x)}}{C_0 + \sum_{n=1}^{\infty} C_n \exp{(-n^2 \pi^2 v t)} \cos{(n\pi x)}}$$

where

$$C_0 = \int_0^1 \exp\{-\frac{1}{2\pi v}[1 - \cos{(\pi x)}]\} dx$$

and

$$C_n = 2 \int_0^1 \exp\{-\frac{1}{2\pi v}[1 - \cos{(\pi x)}]\} \cos{(n\pi x)} dx$$

Table 2: Results for Example 1 with values of $v = 0.02$, $N = 20$, $N = 40$ at $t = 0.01$

x	Mukundan and Awasthi [5] $\Delta t = 0.001$		TSHBM ($\Delta t = 0.001$)		Exact
	N=20	N=40	N=20	N=40	
0.1	0.29951	0.29949	0.29948	0.29948	0.29948
0.2	0.57212	0.57208	0.57205	0.57205	0.57205
0.3	0.79267	0.79262	0.79260	0.79260	0.79260
0.4	0.93974	0.93970	0.93968	0.93968	0.93968
0.5	0.99756	0.99755	0.99754	0.99754	0.99754
0.6	0.95799	0.95801	0.95802	0.95802	0.95801
0.7	0.82222	0.82226	0.82228	0.82228	0.82228
0.8	0.60170	0.60175	0.60175	0.60175	0.60175
0.9	0.31782	0.31785	0.31785	0.31785	0.31785
no. of iterations	10		5		

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6.2 Example 2

Consider the Burgers' equation [5]

$$\frac{\partial \mu}{\partial t} + \mu \frac{\partial \mu}{\partial x} = v \frac{\partial^2 \mu}{\partial x^2}$$

with initial condition:

$$\mu(x, 0) = 4x(1 - x), 0 \leq x \leq 1$$

and boundary condition:

$$\mu(0, t) = \mu(1, t) = 0$$

The exact solution of the problem is given by :

$$\mu(x, t) = 2\pi v \frac{\sum_{n=1}^{\infty} D_n \exp{(-n^2\pi^2 vt)} n \sin{(n\pi x)}}{D_0 + \sum_{n=1}^{\infty} D_n \exp{(-n^2\pi^2 vt)} \cos{(n\pi x)}}$$

where D_0 and D_n are Fourier coefficients given by:

$$D_0 = \int_0^1 \exp\{-\frac{1}{3v}[x^2(3 - 2x)]\}dx$$

and

$$D_n = 2\int_0^1 \exp\{-\frac{1}{3v}[x^2(3 - 2x)]\} \cos{(n\pi x)}dx$$

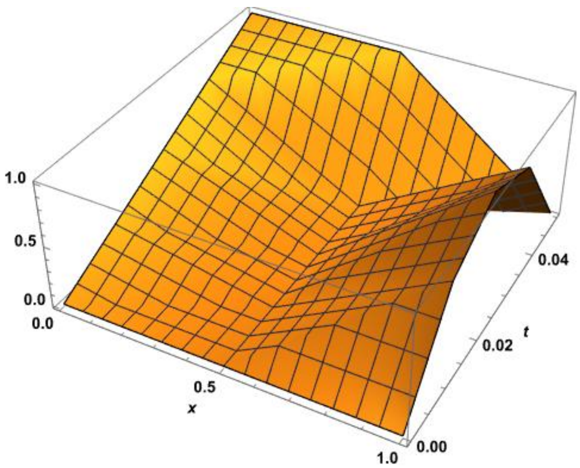
Table 3: Results for Example 2 with values of $v = 0.02$, $N = 80$

x	t	Mukundan and Awasthi [5] $\Delta t = 0.001$	TSHBM ($\Delta t = 0.001$)	Exact
0.25	0.01	0.73356	0.73355	0.73355
	0.02	0.71743	0.71741	0.71741
	0.03	0.70165	0.70162	0.70162
0.50	0.01	0.99800	0.99800	0.99800
	0.02	0.99523	0.99523	0.99523
	0.03	0.99170	0.99169	0.99169
0.75	0.01	0.76341	0.76340	0.76340
	0.02	0.77679	0.77678	0.77678
	0.03	0.79008	0.79008	0.79008
no. of iterations		30	15	

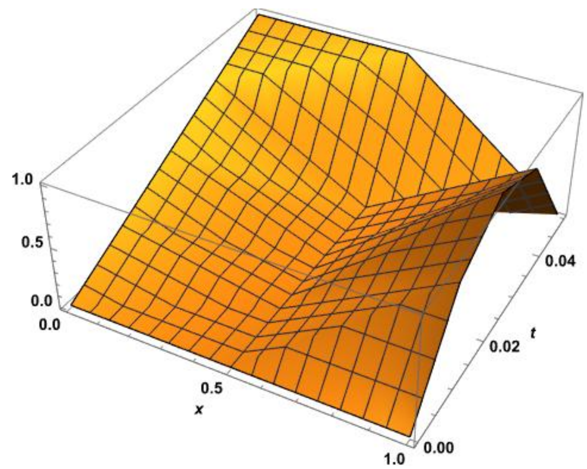
6.3 Example 3

Consider the Burgers' equation [5]

$$\frac{\partial \mu}{\partial t} + \mu \frac{\partial \mu}{\partial x} = v \frac{\partial^2 \mu}{\partial x^2}$$



(a) A 3D graph showing the physical behavior of the numerical solution of Example 2 using the proposed method for $N = 80$, $v = 0.02$ across different times



(b) A 3D graph showing the physical behavior of the exact solution of Example 2 across different times

Figure 3

with initial condition:

$$\mu(x,0) = \frac{2v\pi \sin(\pi x)}{b + \cos(\pi x)}, 0 \leq x \leq 1$$

and boundary condition:

$$\mu(0,t) = \mu(1,t) = 0$$

The exact solution of the problem is given by

$$\mu(x,t) = \frac{2v\pi \exp(-\pi^2 vt) \sin(\pi x)}{b + \exp(-\pi^2 vt) \cos(\pi x)}$$

where $b > 1$ is a parameter.

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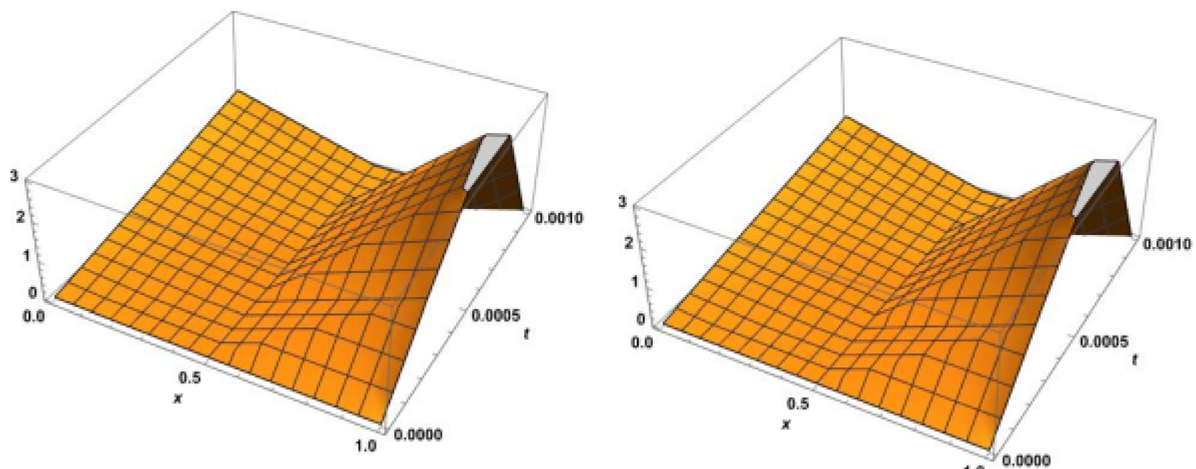
Table 4: Results for Example 3 with values of $v = 1$, $b = 2$, $N = 40$ at $t = 0.001$

x	Mukundan and Awasthi [5] $\Delta t = 0.0001$	TSHBM ($\Delta t = 0.0001$)	Exact
0.1	0.653539	0.653544	0.653544
0.2	1.305530	1.305534	1.305534
0.3	1.949364	1.949364	1.949364
0.4	2.565921	2.565925	2.565925
0.5	3.110744	3.110739	3.110739
0.6	3.492901	3.492866	3.492866
0.7	3.549703	3.549595	3.549595
0.8	3.050339	3.050135	3.050134
0.9	1.816859	1.816662	1.816660
no. of iterations	10	5	



(a) A 3D graph showing the physical behavior of the numerical solution of Example 3 using the proposed method for $N = 40$, $v = 0.01$ across different times

(b) A 3D graph showing the physical behavior of the exact solution of Example 3 across different times

Figure 4

7 Result Discussion

Tables 2 - 4 show the numerical results obtained by using the derived method (TSHBM) and the method in Mukundan and Awasthi [5] to solve the nonlinear time dependent Burgers' PDE in Examples 1 - 3. TSHBM had better results at all points considered when compared to the exact solutions of same problems. Also, the number of iterations undergone before obtaining the numerical results using TSHBM is less than that of the method in Mukundan and Awasthi [5]. This shows that the proposed method saves computational time. Finally, the pictorial representation of the physical behavior of the solution obtained by using the proposed method to solve Example 2 and Example 3 is presented on Figures 3 and figure 4 respectively. This presentation is done in comparison with the physical behavior of the exact solution. From the graph, it is clear that TSHBM gives result that simulates the exact solution. The numerical results from TSHBM proves that the block scheme can effectively be used for the numerical integration of the nonlinear time dependent Burgers' PDE.

8 Conclusion

A new two-step second derivative hybrid block method is developed for the numerical solution of nonlinear Burgers' equations. The Burgers' PDEs is first semi-discretized in spatial direction to system of nonlinear first-order ODEs using the standard fourth-order compact differences schemes. Then the derived block method is applied to the resulting system of nonlinear first-order ODEs from the Burgers' PDEs. Two hybrid points are introduced in the development of the block scheme such that the hybrid points lies in the interval $0 \leq w_1 \leq 1 \leq w_2 \leq 2$. The characteristics of the derived method, TSHBM, are analyzed and shown to be zero-stable, consistent and convergent. The application of TSHBM on three test problems shows that the method can effectively integrate nonlinear Burgers' partial differential equation.

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