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Bivariate BCI Algebras

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Abstract

In this paper, the concept of bivariate BCI algebras is introduced. Properties of ρ- variate, λ-variate and bivariate BCI algebras are investigated.

Keywords: Bivariate BCI Algebras, ρ- Variate BCI Algebras, λ-Variate BCI Algebras.
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1 Introduction

An algebra of type (2, 0) is a non-empty set, having a constant element, on which is defined a binary operation such that certain axioms are satisfied. BCI algebras and BCK algebras, introduced in [16] and [15], are common varieties of such algebras. There are several other varieties of algebras of type (2, 0). There are also several generalizations of BCI algebras. In [5], BCH algebras were studied. In [22], d algebras were studied. In [19], the notion of BE algebras was introduced. Ideals and upper sets in BE algebras were investigated in [1] and [2]. Pre- commutative algebras were studied in [20]. Fenyves algebras were studied in [17], [13] and [18]. Recently, it has been shown in [3] that algebras of type (2,0) have diverse applications in coding theory. Motivated by this, more research interest has been given to the study of algebras of type (2,0). In [21], Q algebras were introduced. Nayo algebras were studied in [7]. Obic algebras were introduced and properties of implicative obic algebras were investigated in [8]. In [9], torian algebras were studied. It was shown that the class of torian algebras is a wider class than the class of obic algebras.
In this paper, bivariate BCI algebras are introduced. Properties of ρ- variate, λ- variate and bivariate BCI algebras are investigated.

2 Preliminaries

In this section, some basic concepts necessary for proper understanding of this paper are discussed.

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Definition 2.1. *[16]. An algebra $(X; *, 0)$; where X is a non-empty set, $*$ a binary operation defined on X, and 0 a constant element of X is called a BCI algebra if the following hold for all $x, y, z \in X$:*

1. $((x * y) * (x * z)) * (z * y) = 0$
2. $(x * (x * y)) * y = 0$
3. $x * x = 0$
4. $x * y = 0,\ y * x = 0 \Rightarrow x = y$
5. $x * 0 = x$

Define a binary relation $\leq$ on a BCI algebra $(X; *, 0)$ by $x \leq y$ if and only if $x * y = 0$. Then $(X; \leq)$ is a partially ordered set.

Definition 2.2. *[16]. A BCI algebra $(X; *, 0)$ which satisfies $0 * x = 0$ for all $x \in X$ is called a BCK algebra.*

Proposition 2.3. *[23]. Let x, y, z be elements of a BCI algebra X. Then $x \leq y \Rightarrow z * y \leq z * x$.*

Definition 2.4. *Let X be a BCI algebra. We define $x * y^k = [(x * y) * y] * ....] * y$ (k times); where k is a natural number.*

The following Lemmas are straightforward from definition.

Lemma 2.5. *Let X be a BCI algebra. Then $x * (x * (x * y)) = x * y$ for all $x, y \in X$.*

Lemma 2.6. *Let $(X; *, 0)$ be a BCI algebra. Then $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$.*

We shall denote a BCI algebra by X unless there is the need to emphasize its binary operation and the constant element.

3 Main Results

In this section, we introduce ρ- variate, λ- variate and bivariate BCI algebras and some of their properties are investigated.

Definition 3.1. *Let X be a BCI algebra. An element $x \in X$ is called a ρ- variate element if $(y * z) * x = (y * x) * (z * x)$ for all $y, z \in X$.*

The collection of all ρ- variate elements of X is denoted by X^ρ. If $X^\rho = X$, then X is called a ρ- variate BCI algebra.

Definition 3.2. *Let X be a BCI algebra. An element $x \in X$ is called a λ- variate element if $x * (y * z) = (x * y) * (x * z)$ for all $y, z \in X$.*
The collection of all λ- variate elements of X is denoted by X^λ. Notice that $X^\lambda \neq X$ for any BCI algebra X.

Definition 3.3. *An element x in a BCI algebra X, is called a bivariate element if x is both λ- variate and ρ- variate.*

Proposition 3.4. *Let X be a BCI algebra. Then 0 is a bivariate element of X.*

Proof. Let $x, y \in X$. Then $(0*x)*(0*y) = (((x*y)*(x*y))*x)*(0*y) = (((x*y)*x)*(x*y))*(0*y) = ((0*y)*(x*y))*(0*y) = 0*(x*y)$. Thus, 0 is λ- variate. The fact that 0 is ρ- variate is obvious. □

Definition 3.5. *A BCI algebra X is called a bivariate BCI algebra if the following hold:*

1 . *X is a ρ- variate;*

2 . *$\{0\} \subset X^\lambda$.*

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Example 3.6. *Let $X = \{0, 1, 2, 3, 4\}$. Define a binary operation $*$ on X by the following table:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	3	2	0

Then $(X; *, 0)$ is a bivariate BCI algebra. Notice that $\{0\} \neq X^\lambda \cap X^\rho$ because $2 \in X^\lambda \cap X^\rho$.

The following Lemma is obvious from definition.

Lemma 3.7. *Let X be a ρ- variate BCI algebra. Then the following hold for all $x, y, z \in X$:*

1 . $y * x = (y * x) * (0 * x)$;

2 . $0 * x = 0$;

3 . $(x * z) * x = 0 * (z * x)$;

4 . $0 * z = 0 * (z * x)$;

5 . $(y * z) * z = y * z$;

6 . $(y * x) * z = y * z$;

7 . $((0 * x) * z) * x = ((0 * x) * x) * (z * x)$;

8 . $(y * x) = (y * x) * ((0 * x) * x)$;

9 . $(x * z) * x = (0 * x) * (z * x)$;

10 . $(0 * x) * z = (0 * x) * (z * x)$;

11 . $(x * z) * x = 0$.

Theorem 3.8. *Let X be a BCI algebra. Then X is ρ- variate if and only if $x * y = (x * y) * y$ for all $x, y \in X$.*

Proof. Suppose $x * y = (x * y) * y$ for all $x, y \in X$. Notice that $(x * z) * (y * z) = ((x * z) * z) * (y * z) \leq (x * z) * y$. So,

$$((x * z) * (y * z)) * ((x * z) * y) = 0 \tag{1}$$

By Lemma 3.1(11), we have $(y * z) \leq y$. By Proposition 2.1, we have $(x * z) * y \leq (x * z) * (y * z)$. Therefore,

$$((x * z) * y) * ((x * z) * (y * z)) = 0 \tag{2}$$

From expressions (1) and (2), we have $(x * z) * y = (x * z) * (y * z)$ or, equivalently, $(x * y) * z = (x * z) * (y * z)$ as required.
By Lemma 3.1(5), the converse holds. $\square$

Corollary 3.9. *Let X be a BCI algebra. Then X is ρ- variate if and only if $(x*y)*y = x*(x*(x*y))$ for all $x, y \in X$.*

Proof. This follows from Theorem 3.1 and the fact that $x * (x * (x * y)) = x * y$ for all $x, y \in X$. $\square$

Theorem 3.10. *Let X be a ρ- variate BCI algebra such that the following hold for all $x, y, z, p, v \in X$:*

1. $(x*(y*z))*(x*(y*p)) \leq (z*p)$;
2. $x \leq y \Rightarrow (z*y) \leq (z*x)$;
3. $(x*y) \leq v \Rightarrow (x*v) \leq (x*(x*y))$.

Then $(x*(x*y))*(y*x) = (x*x*(y*(y*x)))$ *for all* $x, y \in X$.

Proof. Notice that $[x*(x*y)]*[x*[x*[y*(y*x)]]] \leq [y*[y*(y*x)]] = y*x$. Hence, $[x*(x*y)]*(y*x) \leq [x*[x*[y*(y*x)]]]$. Now let $[x*[y*(y*x)] = v$. Then we have $(x*v) \leq [y*(y*x)]$. Notice that $[y*(y*x)] \leq y$. So, $(x*y) \leq [x*[y*(y*x)]]$; giving us $(x*y) \leq v$; so that $(x*v) \leq [x*(x*y)]$. Now notice also that $[y*(y*x)] = [y*(y*x)]*(y*x) \leq [x*(y*x)]$. Since $(x*v) \leq [y*(y*x)]$ and $[y*(y*x)] \leq [x*(y*x)]$, we have $(x*v) \leq [x*(y*x)]$.
Now, 'multiply' both sides of the last relation on the right by v to get $[(x*v)*v] \leq [x*(y*x)]*v$. That is, $[(x*v)*v] \leq (x*v)*(y*x)$; giving us $(x*v) \leq [(x*v)*(y*x)]$; leading to $(x*v) \leq [[x*(x*y)]*(y*x)]$. Substituting back for v, we have $[x*[x*[y*(y*x)]]] \leq [x*(x*y)]*(y*x)$. Since $[x*(x*y)]*(y*x) \leq [x*[x*[y*(y*x)]]]$ and $[x*[x*[y*(y*x)]]] \leq [x*(x*y)]*(y*x)$, we conclude that $[x*[x*[y*(y*x)]]] = [x*(x*y)]*(y*x)$ as required. □

Corollary 3.11. *Let* X *be a bivariate BCI algebra such that the following hold for all* $x, y, z, p, v\ in X^\lambda$*:*

1. $((x*y)*(x*z))*((x*y)*(x*p)) \leq (z*p)$;
2. $x \leq y \Rightarrow z*(y*x) = 0$;
3. $(x*y) \leq v \Rightarrow (x*v)*(0*(x*y)) = 0$.

Then $(x*(x*(0*(x*y)))) = (0*(x*y))*(y*x)$ *for all* $x, y \in X^\lambda$.

Proof. It follows from Theorem 3.2 and the definition of X^λ. □

Corollary 3.12. *Let* X *be a* ρ*- variate BCI algebra such that the following hold for all* $x, y, z, p, v \in X$*:*

1. $(x*(y*z))*(x*(y*p)) \leq (z*p)$;
2. $x \leq y \Rightarrow (z*y) \leq (z*x)$;
3. $(x*y) \leq v \Rightarrow (x*v) \leq (x*(x*y))$.

Then $(x*(y*x))*((x*y)*(y*x)) = (x*(x*(y*(y*x))))$ *for all* $x, y \in X$.

Proof. It follows from Theorem 3.2 and the definition of X^ρ. □

Theorem 3.13. *Let* X *be a* ρ*- variate BCI algebra such that the following hold for all* $x, y, z, p, v \in X$*:*

1. $(x*(y*z))*(x*(y*p)) \leq (z*p)$;
2. $x \leq y \Rightarrow (z*y) \leq (z*x)$;
3. $(x*y) \leq v \Rightarrow (x*v) \leq (x*(x*y))$.

Then $(x*y)*(x*(x*y)) = x*y$ *for all* $x, y \in X$.

Proof. By Theorem 3.2, for all $x, y \in X$, we have

$$[x*(x*y)]*(y*x) = [x*[x*[y*(y*x)]]] \tag{3}$$

Put $x*y$ for x, and put x for y in expression (1). Then the left hand side becomes $[(x*y)*[(x*y)*x]]*[x*(x*y)]$
$= [(x*y)*[(x*x)*y]]*[x*(x*y)] = [(x*y)*(0*y)]*[x*(x*y)]$

$= (x * y) * [x * (x * y)]$.
Also, the right hand side becomes $(x * y) * [(x * y) * [x * [x * (x * y)]]]$
$= (x * y) * [(x * y) * (x * y)] = x * y$.
Hence, equating the left and right hand sides, we have $(x * y) * [x * (x * y)] = x * y$ as required. □

Corollary 3.14. *Let X be a bivariate BCI algebra such that the following hold for all $x, y, z, p, v\ in X^{\lambda}$:*

1. $((x * y) * (x * z)) * ((x * y) * (x * p)) \leq (z * p)$;
2. $x \leq y \Rightarrow z * (y * x) = 0$;
3. $(x * y) \leq v \Rightarrow (x * v) * (0 * (x * y)) = 0$.

Then $(x * y) * (x * (x * y)) = x * y$ for all $x, y \in X^{\lambda}$.

Proof. It follows from Theorem 3.3 and the definition of X^{λ}. □

Corollary 3.15. *Let X be a ρ- variate BCI algebra such that the following hold for all $x, y, z, p, v \in X$:*

1. $(x * (y * z)) * (x * (y * p)) \leq (z * p)$;
2. $x \leq y \Rightarrow (z * y) \leq (z * x)$;
3. $(x * y) \leq v \Rightarrow (x * v) \leq (x * (x * y))$.

Then $(x * (x * (x * y))) * ((y * x) * (x * y)) = x * y$ for all $x, y \in X$.

Proof. It follows from Theorem 3.3 and the definition of X^{ρ}. □

Theorem 3.16. *Let X be a ρ- variate BCI algebra such that the following hold for all $x, y, z \in X$:*

1. $x \leq y \Rightarrow (x * z) \leq (y * z)$
2. $x * y^k = x * y^{k+1}$; *where $k \in \mathbb{N}$; the set of natural numbers.*
3. $x * y^k = x * y^l$ *for all* $l \geq k \in \mathbb{N}$
4. $(x * z^k) * (y * z^k \leq (x * y)$.

Then $(x * y) * z^k = (x * z^k) = (x * z^k) * (y * z^k)$ for all $x, y, z \in X$.

Proof. By hypothesis, we have $x * z^k = x * z^{2k}$. Since, $(x * z^k) * (y * z^k) \leq (x * y)$, we have $[(x * z^k) * (y * z^k)] * z^k \leq (x * y) * z^k$; which gives $[(x * z^k) * z^k] * (y * z^k) \leq (x * y) * z^k$; which results to $(x * z^{2k}) * (y * z^k) \leq (x * y) * z^k$. Since $x * z^k = x * z^{2k}$, we now have

$$(x * z^k) * (y * z^k) \leq (x * y) * z^k \tag{4}$$

Notice that $(y * z^k) * y = 0$. So, $(y * z^k) \leq y$. We therefore have $[(x * z^k) * y] \leq [(x * z^k) * (y * z^k)]$; which gives

$$[(x * y) * z^k] \leq [(x * z^k) * (y * z^k)] \tag{5}$$

By expressions (4) and (5), we have $(x * y) * z^k = (x * z^k) * (y * z^k)$ as required. □

Proposition 3.17. *Let X be a ρ- variate BCI algebra. If If $(x * y) * z^k = (x * z^k) * (y * z^k)$, then $x * z^k = x * z^{k+1}$ for all $x, y, z \in X; k \in \mathbb{N}$.*

Proof. By hypothesis, we have $(x * z) * z^k = (x * z^k) * (z * z^k)$;
which gives $x * z^{k+1} = x * z^k$ as required. □

Theorem 3.18. *Let X be a ρ- variate BCI algebra such that the following hold for all $x, y, z \in X$:*

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1. $x * y^k = x * y^{k+1}, k \in \mathbb{N}$;

2. $x * y^k = x * y^l$ *for all* $\geq k$;

3. $x \leq y \Rightarrow (x * z) * (y * z)$.

Then $[y * (y * x)^k] * (x * y)^k = [x * (x * y)^k] * (y * x)^k$ *for all* $x, y \in X$.

Proof. By hypothesis, we have

$$x * (x * y)^{k_1} = x * (x * y)^{k_1} \tag{6}$$

and

$$y * (y * x)^{k_2} = y * (y * x)^{k_2} \tag{7}$$

Let k be the maximum of k_1 and k_2. Then

$$x * (x * y)^k = x * (x * y)^{k+1} \tag{8}$$

and

$$y * (y * x)^k = y * (y * x)^{k+1} \tag{9}$$

Notice that $[x * (x * y)] * y = 0$. So, $x * (x * y) \leq y$; and from expression (6), we have

$$x * [(x * y)^k \leq y * (x * y)^k \tag{10}$$

Now, 'multiply' expression (8) on both sides on the right by $y * x$ (k times) to get

$$[x * (x * y)^k] * (y * x)^k \sim [y * (x * y)^k] * (y * x)^k \tag{11}$$

Now apply Lemma 2.2 to expression (9) to get

$$[x * (x * y)^k] * (y * x)^k \sim [y * (y *)^k] * (x * y)^k \tag{12}$$

Also notice that $[y * (y * x)] * x = 0$. So, $[y * (y * x)] \leq x$; and so from expression (7), we have

$$[y * (y * x)^k] \leq [x * (y * x)^k] \tag{13}$$

'Multiply' both sides of expression (11) on the right by $x * y$ (k times) to get

$$[y * (y * x)^k] * (x * y)^k \leq [x * (y * x)^k] * (x * y)^k \tag{14}$$

Now apply Lemma 2.2 to expression (12) to get

$$[y * (y * x)^k] * (x * y)^k \leq [x * (x * y)^k] * (y * x)^k \tag{15}$$

From expressions (12) and (15), we have $[y * (y * x)^k] * (x * y)^k$
$= [x * (x * y)^k] * (y * x)^k$ as required. □

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