THE DISTRIBUTIONAL PROPERTIES OF THE FAMILY OF LOGISTIC DISTRIBUTIONS*

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Abstract
The distributional properties of half logistic distribution and Type I generalized logistic distribution were studied, bringing out the L-moments (up to order four) of each of these. Skewness and Kurtosis were obtained.

Key words: Logistic distribution, L-moments

1. Introduction
Family of logistic distribution like standard logistic, half logistic, Type I generalized logistic, etc., were studied in Ph.D thesis of Olapade (2006); he obtained the properties of the family, using moment generating function and characteristic functions but did not consider the L-moment of the family of the distribution studied in this paper. Also, the L-skewness and L-kurtosis of the family of distribution are obtained. The standard probability density function of the logistic random variable x is given by:

\[ f_x(x) = \frac{e^x}{(1+e^x)^2}, -\infty < x < \infty \]  \hspace{1cm} (1.1)

The cumulative distribution function (c.d.f) is given as

\[ F_x(x) = \frac{1}{1+e^{-x}}, -\infty < x < \infty \]  \hspace{1cm} (1.2)

The quartile (inverse distribution) function is given as:

\[ x(F) = \ln\left(\frac{F}{1-F}\right), 0 \leq F \leq 1 \]

and the L-moment of a given distribution as proposed by Hosking (1990) can be expressed as:

\[ L_r = \int_0^1 x(F)P_{r-1}(F)dF, \]

where \( P_{r-1}(F) = \sum_{k=0}^{r-1} \frac{(-1)^{r-k-1}(r-1)(r+k-1)}{k} F^k \),

called Legendre polynomial of order \((r-1)\), Hosking (1990). The L-moment of the logistic distribution can be expressed as:

\[ L_r = \int_0^1 \ln\left(\frac{F}{1-F}\right)P_{r-1}(F)dF \]

\[ = \int_0^1 [\ln(F)]P_{r-1}(F)dF - \int_0^1 [\ln(1-F)]P_{r-1}(F)dF \]

\[ = c \int_0^1 F^k \ln(F)dF - \int_0^1 F^k \ln(1-F)dF \]

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The following identities of integrals will help us to simplify the above equations

(i) \[ \int_1^x \ln(x) \, dx = \frac{1}{(k+1)^2} \]

(ii) \[ \int_0^1 \ln(1-x) \, dx = -\sum_{n=1}^{\infty} \frac{1}{n(n+k+1)} \]

Therefore, if \( k = 0 \), \[ \lim_{j \to \infty} \sum_{n=0}^{\infty} \frac{1}{n(n+1)} = 1; \]
if \( k = 0 \); \[ \lim_{j \to \infty} \sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{3}{4} \]; if \( k = 2 \); \[ \lim_{j \to \infty} \sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \frac{11}{18} \];
the first four L-moment of the logistic distribution can be expressed as:
\[
\begin{align*}
L_1 &= 0 \\
L_2 &= 1 \\
L_3 &= 0 \\
L_4 &= 4.2
\end{align*}
\]
According to Hosking (1990), the L-skewness and L-kurtosis can be obtained by using
\[ \tau_r = \frac{L_r}{L_2}, \quad r > 2. \]
Therefore for the logistic distribution,
L-skewness = \( \tau_3 = 0 \) and L-kurtosis = \( \tau_4 = 4.2 \).
If the location (\( \delta \)) and scale (\( \omega \)) are included in the quartile function of logistic distribution as:
\[ x(F) = \delta + \omega \ln \left( \frac{F}{1-F} \right), 0 < F < 1 \]
The property of the half logistic distribution,
One of the probability distributions which is a member of the family of the logistic distribution is half logistic distribution.
Its probability density function can be expressed as: \( L_1 = 0 \).
\[ f_T(y) = \frac{2e^y}{(1 + e^y)^2}, \, 0 < y < \infty. \]

The cumulative distribution function is
\[ F_T(y) = \frac{e^y - 1}{1 + e^y}, \, 0 < y < \infty. \]

The inverse (quartile) distribution function of the half logistic distribution can be expressed as:
\[ y(F) = \ln \frac{1 + F}{1 - F}. \]

The L-moment of the half logistic distribution can be expressed as:
\[
L_r = \frac{1}{r} \int_0^1 \ln \left( \frac{1 + F}{1 - F} \right) P_{r-1}(F) dF
\]
\[
= \frac{1}{r} \int_0^1 \left[ \ln(1 + F) \right] P_{r-1}(F) dF - \frac{1}{r} \int_0^1 \left[ \ln(1 - F) \right] P_{r-1}(F) dF
\]
\[
= \left[ \int F^k \ln(1 + F) dF \right] - \frac{1}{r} \int F^k \ln(1 - F) dF \]

The identity of integrals below will give clues on solving the L-moment, of the half logistic distribution:

(i) \[ \int_0^1 x^k \ln(1 + x) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \]

(ii) \[ \int_0^1 x^k \ln(1 - x) dx = - \sum_{n=1}^{\infty} \frac{1}{n(n+k+1)} \]

Therefore, we have the equation \( L_r \) above to be

\[
L_r = \frac{c}{1 + k} \left( \sum_{n=1}^{\infty} \frac{1}{n(2n-1)} - \sum_{n=1}^{\infty} \frac{1}{n(2n+k)} \right)
\]

If \( k = 0 \), \( \lim_{j \to \infty} \sum_{n=1}^{j} \left( \frac{1}{(2n-1)} - \frac{1}{(2n)} \right) = \log_2 2; \)

If \( k = 1 \), \( \lim_{j \to \infty} \sum_{n=1}^{j} \left( \frac{1}{(2n-1)} - \frac{1}{(2n+1)} \right) = 1; \)

If \( k = 2 \), \( \lim_{j \to \infty} \sum_{n=1}^{j} \left( \frac{1}{(2n-1)} - \frac{1}{(2n+2)} \right) = \log_2 2 + \frac{1}{2}; \)

If \( k = 3 \), \( \lim_{j \to \infty} \sum_{n=1}^{j} \left( \frac{1}{(2n-1)} - \frac{1}{(2n+3)} \right) = \frac{4}{3}. \)

The L-moment of the half logistic distribution can be expressed as:

\[ L_1 = 2 \log_2 2, \]
\[ L_2 = -2 \log_2 2 + 2 = 2(1 - \log_2 2), \]
\[ L_3 = 2 \log_2 2 - 6 + 4 \left( \log_2 2 + \frac{1}{2} \right) = 6 \log_2 2 - 4, \]
\[ L_4 = 2 \log_2 2 + 12 - 2(5) \left( \log_2 2 + \frac{1}{2} \right) + \frac{4}{3} = 15 \frac{1}{3} - 22 \log_2 2 \]
The L-skewness and L-kurtosis of the half logistic distribution are

$$L_{\text{skewness}} = \tau_3 = \frac{6 \log_e 2 - 4}{2(1 - \log_e 2)}$$

and

$$L_{\text{kurtosis}} = \tau_4 = \frac{15}{3} - \frac{22 \log_e 2}{2(1 - \log_e 2)}.$$  

If the location ($\delta$) and scale ($\omega$) parameter are included in the quartile function, we have:

$$x(F) = \delta + \omega \ln \left( \frac{F}{1 - F} \right).$$

The L-moment of the half logistic can now be written as:

$$L_1 = \delta + \omega (2 \log_e 2),$$

$$L_2 = \omega (-2 \log_e 2 + 2) = 2 \omega (1 - \log_e 2),$$

$$L_3 = \omega \left( 2 \log_e 2 - 6 + 4 \left( \log_e 2 + \frac{1}{2} \right) \right) = \omega (6 \log_e 2 - 4),$$

$$L_4 = \omega \left( -2 \log_e 2 + 12 - 2 (15 \left( \log_e 2 + \frac{40}{3} \right)) \right) = \omega \left( 15 \frac{1}{3} - 22 \log_e 2 \right).$$

**The Distributional Property of the Type I generalized Logistic Distribution**

The probability density function of a random variable $X$ that has type I generalized logistic distribution is:

$$f_X(x) = \frac{b e^{-x}}{(1 + e^{-x})^{b+1}}, \quad -\infty < x < \infty, b > 0.$$  

The corresponding cumulative distribution function is:

$$F_X(x) = \frac{1}{(1 + e^{-x})^{b}}, \quad -\infty < x < \infty, b > 0.$$  

The quartile function of the type I distribution function is

$$x(F) = \ln \left( \frac{F^{\frac{1}{b}}}{1 - F^{\frac{1}{b}}} \right), 0 \leq F \leq 1.$$  

The L-moment of the type I generalized logistic distribution can be expressed as

$$\int_0^1 \ln \left( \frac{F^{\frac{1}{b}}}{1 - F^{\frac{1}{b}}} \right) P_{x^{-1}}(F) dF = c \left( \int_0^1 F^k \ln F^{\frac{1}{b}} dF - \left[ F^{k - 1} \ln F^{\frac{1}{b}} \right] \right).$$

The identity of integrals below will give clues of solving the above equation.

Let $x^b = u$, $dx = \frac{b du}{x^{b-1}}$, if $x = 0$, $u = 0$ and $x = 1$, $u = 1$.

Therefore, we have

$$\int_0^1 u^{b-1} \ln u du = \frac{-b}{(kb + b)^2}.$$
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Also, \( \int_0^1 x^b \ln \left( 1 - x^b \right) dx = b \int_0^1 u^{k+b-1} \ln(1-u)du = -b \sum_{n=0}^{\infty} \frac{1}{n bk + b + n} \)

\[
\sum_{n=0}^{\infty} \frac{1}{n bk + b + n} = \sum_{n=0}^{\infty} \left( \frac{1}{n bk + b} - \frac{1}{b(k + n)(n + bk + b)} \right)
\]

\[
= \frac{1}{(bk + b)} \sum_{n=0}^{\infty} \left( \frac{1}{n} - \frac{1}{(n + bk + b)} \right)
\]

If \( k = 0 \), \( \sum_{n=0}^{\infty} \left( \frac{1}{n} - \frac{1}{(n + b)} \right) = \sum_{n=0}^{\infty} \left( \frac{1}{n} \right) \)

If \( k = 1 \), \( b = \text{only value} \),

\[
\sum_{n=0}^{\infty} \left( \frac{1}{n} - \frac{1}{(n + 2b)} \right) = \sum_{n=0}^{\infty} \left( \frac{1}{n} \right)
\]

If \( k = 2 \), \( \sum_{n=0}^{\infty} \left( \frac{1}{n} - \frac{1}{(n + 3b)} \right) = \sum_{n=0}^{\infty} \left( \frac{1}{n} \right) \)

If \( k = 3 \), \( \sum_{n=0}^{\infty} \left( \frac{1}{n} - \frac{1}{(n + 4b)} \right) = \sum_{n=0}^{\infty} \left( \frac{1}{n} \right) \)

We can assume for any value of \( k \), \( \sum_{n=0}^{\infty} \left( \frac{1}{n} - \frac{1}{(n + bk + b)} \right) = \sum_{n=0}^{\infty} \left( \frac{1}{n} \right) \)

But \( k \) and \( b \) must be positive integers so the L-moment of the type I generalized logistic distribution can be expressed as

\[
c \left( \frac{1}{b(k+1)^2} + \frac{1}{b(k+1)} \sum_{n=0}^{\infty} \left( \frac{1}{n} \right) \right)
\]

The value of \( b \) is obtained, using the maximum likelihood method of estimation, and it is assumed to be approximated to the nearest integer.

If \( b = 1 \), the distribution gives the standard logistic distribution.

REFERENCES


