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INTEGRAL REPRESENTATIONS OF THE GENERATING FUNCTION OF THE RIEMANN ZETA FUNCTION OF INTEGER ARGUMENTS

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ABSTRACT

In this article we give new integral representations for the ordinary generating functions of $\zeta(2n)$, $n\zeta(2n+1)$ and $\zeta(2n+1)$ for $n \in \mathbb{Z}^+, n \geq 1$; where $\zeta(j)$ is the Riemann zeta function. We also give closed form expressions for the generating functions.

Keywords: Riemann zeta function, generating function, residue, integer argument, integral representation.

INTRODUCTION

The Riemann zeta function of a positive integer argument defined by

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}, \quad n \in \mathbb{Z}^+, n > 1, \tag{1.1}$$

occurs in various areas of statistical physics, for example in Bose-Einstein systems, black body radiation and in the Sommerfeld expansion (Ashcroft and Mermin, 1976; Gross and Witten, 1986; Pathria, 1996), usually in integral form

$$\zeta(n) = \int_0^{\infty} \frac{x^{n-1}}{(n-1)!} \frac{dx}{e^x - 1}. \tag{1.2}$$

The Rieman zeta function also appears in the calculations of Feynman diagrams (Kreimer, 2000) as well as in string theory (Gross and Witten, 1986).

In the study of Bose-Einstein distribution one often encounters integrals of the form

$$g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^{\infty} \frac{x^{\nu-1}}{z^{-1}e^x - 1} dx,$$

$$0 \leq z < 1, \nu > 0; \quad z = 1, \nu > 1, \tag{1.3}$$

where $\Gamma(y) = \int_0^{\infty} e^{-t} t^{y-1} \mathrm{d}t$

is the Gamma function defined for $\Re(y)>0$ and extended to the rest of the complex plane, excluding the non-positive integers, by analytic continuation.

For small values of z, the integral in (1.3) may be expanded in powers of z to obtain

$$g_\nu(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^\nu} = z + \frac{z^2}{2^\nu} + \frac{z^3}{3^\nu} + \cdots$$

Thus for $\nu > 1$, $g_\nu(z)$ approaches the Riemann zeta function, $\zeta(\nu)$, defined in (1.1).

Planck's law gives the intensity of light emitted by a blackbody as

$$I(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}, \tag{1.4}$$

where h is Planck constant, c is the speed of light, ν is the radiation frequency, k is Boltzmann constant and T is temperature.

The Stefan-Boltzmann law gives the emitted power per unit area, B, of a blackbody as

$$B = \pi \int_0^{\infty} I(\nu, T) d\nu;$$

that is

$$B = \frac{2\pi h}{c^2} \int_0^{\infty} \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu.$$

Substituting $y = h\nu/kT$ gives

$$B = \frac{2\pi h}{c^2} \left(\frac{kT}{h}\right)^4 \int_0^{\infty} \frac{y^3}{e^y - 1} dy.$$

Thus, using (1.1), the emitted power per unit area of an emitting body is given by

$B = \sigma T^4$,

where

$$\sigma = \frac{2\pi^5 k^4}{15c^2h^3}$$

is the Stefan-Boltzmann constant.

A Sommerfeld (1929) expansion is an approximation procedure applicable to certain integrals which occur in various areas of condensed matter physics and statistical physics such as the thermal conduction of electrons. The integrals represent statistical averages based on the Fermi-Dirac distribution. Fermi-Dirac integrals (Morales, 2011, Kim *et al.*, 2019) are defined by

$$F_{j/2}(\eta)=\int_0^\infty \frac{\xi^{j/2}}{1+e^{\xi-\eta}}d\xi,$$

where $\eta=\mu/kT$, μ is the chemical potential and j is a positive integer.

For electrons (j = 1) the Fermi-Dirac integral is expressed in terms of the Riemann zeta function by (Morales, 2011)

$$\int_0^\infty \frac{\xi^{1/2}}{1+e^{\xi-\eta}}d\xi=\frac{2}{3}\eta^{3/2}+2\sum_{n=1}^{\infty}(1-2^{1-2n})\zeta(2n)\frac{(4n-5)!!}{2^{2n-1}\eta^{(4n-3)/2}}.$$

Let B_j be the Bernoulli numbers defined by the generating function

$$\frac{z}{e^z-1}=\sum_{j=0}^{\infty}B_j\frac{z^j}{j!},\quad z<2\pi. \qquad (1.5)$$

The first few Bernoulli numbers are

$$B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_3=0, B_4=-\frac{1}{30}, B_5=0, B_6=\frac{1}{42}, B_7=0,\ldots \qquad (1.6)$$

For positive even arguments, the numbers $\zeta(2n)$ are directly related to the Bernoulli numbers, B_{2n}:

$$\zeta(2n)=(-1)^{n+1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n}. \qquad (1.7)$$

No analogous closed evaluation is known for the Riemann zeta function at odd integer arguments.

Integral representations for the Riemann zeta function of even and odd integers are

$$\zeta(2n)=\frac{(-1)^{n+1}2^{2n-3}\pi^{2n}}{(2n-2)!(2^{2n}-1)}\int_0^1 E_{2(n-1)}(x)dx$$

and

$$\zeta(2n+1)=\frac{(-1)^n 2^{2n-1}\pi^{2n+1}}{(2n)!(2^{2n+1}-1)}\int_0^1 E_{2n}(x)\cot\left(\frac{\pi x}{2}\right)dx;$$

where $E_j(x)$ are the Euler polynomials defined through the generating function

$$\frac{2e^{xt}}{e^t+1}=\sum_{j=0}^{\infty}E_j(x)\frac{t^j}{j!}.$$

Many useful and important properties of the Riemann zeta function can be accessed through its generating function.

The generating function, $u(a)$, of the Riemann zeta function of even integers is well known (Apostol, 1973; Borwein *et al.*, 2001):

$$u(a)=\frac{1-\pi a\cot\pi a}{2a^2}=\sum_{n=0}^{\infty}\zeta(2n+2)a^{2n}. \qquad (1.8)$$

No such simple expression is known for the generating function of the ζ function of an odd integer argument (Cvijović and Klinowski, 2002; Sondow and Weisstein, 2002). We remark that the Riemann zeta function with odd arguments is especially important in calculating the first few values of the emptiness-formation probability in the anti-ferromagnetic *XXX* spin chain. Boss and Korepin (2001) obtained the third and fourth, $P(3)$ and $P(4)$, such probabilities in terms of $\zeta(3)$ and $\zeta(5)$ and gave an asymptotic of $P(n)$ when n approaches infinity.

In this article we will derive the generating function, $w(a)$, of the Riemann zeta function of odd integer argument multiplied by an integer. Specifically, we shall show that

$$\begin{aligned} w(a) &= \int_0^\infty \frac{\sinh ax}{2a}\frac{x}{e^x-1}dx \\ &= \frac{1+\pi^2 a^2 cosec^2\pi a-2a^2\psi'(a)}{4a^3} \\ &= \sum_{n=0}^{\infty}(n+1)\zeta(2n+3)a^{2n}, \end{aligned}$$

where ψ' is the trigamma function defined by

$$\psi'(z)=\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\Gamma(z)=\sum_{n=1}^{\infty}\frac{1}{(z-1+n)^2}.$$

We will also derive an integral representation for the Riemann zeta function of an odd integer argument, namely,

$$v(a)=\int_0^\infty \frac{\cosh ax-1}{a}\frac{dx}{e^x-1}=\sum_{n=1}^{\infty}\zeta(2n+1)a^{2n-1}.$$

Integral representation of the generating function of Riemann zeta function of an even integer argument

Theorem 1 *The function* $u(a)$ *defined by the integral*

$$u(a)=\int_0^\infty \frac{\sinh ax}{a}\frac{1}{e^x-1}dx \qquad (2.1)$$

is a generating function of the Riemann zeta function of a positive even integer argument.

Proof. Since

$$\frac{\sinh ax}{a}\equiv\frac{e^{ax}-e^{-ax}}{2a},$$

we have, after using the Taylor series expansion of the exponential function,

$$\frac{\sinh ax}{a}=\sum_{n=0}^\infty\frac{x^{2n+1}}{(2n+1)!}a^{2n}. \qquad (2.2)$$

Thus

$$\int_0^\infty\frac{\sinh ax}{a}\frac{dx}{e^x-1}=\int_0^\infty\sum_{n=0}^\infty\frac{x^{2n+1}}{(2n+1)!}a^{2n}\frac{dx}{e^x-1}$$
$$=\sum_{n=0}^\infty a^{2n}\int_0^\infty\frac{x^{2n+1}}{(2n+1)!}\frac{dx}{e^x-1}=\sum_{n=0}^\infty a^{2n}\zeta(2n+2),\quad by\ .$$

Therefore, $u(a)$ is a generating function of the Riemann Riemann zeta function of even integers. Note that the uniform convergence of the series provides the justification for the interchange of summation and integration.

Next we put $u(a)$ in closed form.

Writing

$$u(a)=\int_0^\infty\frac{\sinh ax}{a}\frac{1}{e^x-1}\mathrm{d}x=\int_0^\infty\frac{\sinh ax}{a}\frac{e^{-x}}{1-e^{-x}}\mathrm{d}x,$$

we have

$$u(a)=\int_0^\infty\left(\sum_{k=1}^\infty e^{-kx}\right)\frac{\sinh ax}{a}\mathrm{d}x$$
$$=\sum_{k=1}^\infty\left(\int_0^\infty e^{-kx}\sinh ax\mathrm{d}x/a\right)$$
$$=\sum_{k=1}^\infty\frac{1}{k^2-a^2}.$$

Let

$$f(k)=\frac{(-1)^k}{k^2-a^2}=\frac{e^{-ik\pi}}{k^2-a^2}$$

Note that $f(k)$ has simple poles at $k=\pm a$. The above sum is evaluated by the method of residues as

$$u(a)=\sum_{k=1}^\infty(-1)^k f(k)=\frac{1}{2}\left(\sum_{k=-\infty}^\infty(-1)^k f(k)-f(0)\right)$$
$$=-\frac{\pi}{2}\sum_{k=\pm a}\frac{\mathrm{R}es(f,\pm a)}{\sin\pi(\pm a)}+\frac{1}{2a^2}$$
$$=-\frac{\pi}{2a}\cot\pi a+\frac{1}{2a^2}$$
$$=\frac{1-\pi a\cot\pi a}{2a^2},$$

which is the well-known closed form generating function of the ζ function of an even integer argument. Next we obtain the generating function of the Riemann zeta function of an odd integer argument multiplied by a positive integer.

Generating function of the Riemann zeta function of an odd integer argument multiplied by a positive integer

Theorem 2 *The function* $w(a)$ *defined by the integral*

$$w(a)=\int_0^\infty\frac{\sinh ax}{2a}\frac{x}{e^x-1}\mathrm{d}x \qquad (3.1)$$

is a generating function of the Riemann ζ function of an odd positive integer argument multiplied by a positive integer.

Proof. Using (2.2), we have

$$\int_0^\infty\frac{\sinh ax}{2a}\frac{xdx}{e^x-1}=\int_0^\infty\sum_{n=0}^\infty\frac{x^{2n+2}}{(2n+1)!2}a^{2n}\frac{dx}{e^x-1}$$
$$=\sum_{n=0}^\infty a^{2n}\int_0^\infty\frac{x^{2n+1}(n+1)}{(2n+2)!}\frac{dx}{e^x-1}=\sum_{n=0}^\infty a^{2n}(n+1)\zeta(2n+3). \qquad (3.2)$$

Therefore, the function $w(a)$ defined by the integral (3.1) is a generating function of the Riemann zeta function of an odd integer argument multiplied by a positive integer.

In order to express $w(a)$ in closed form we write

$$w(a)=\int_0^\infty\left(\sum_{k=1}^\infty e^{-kx}\right)\frac{x\sinh ax}{2a}\mathrm{d}x$$
$$=\sum_{k=1}^\infty\frac{1}{2a}\frac{\mathrm{d}}{\mathrm{d}a}\left(\int_0^\infty e^{-kx}\cosh ax\mathrm{d}x\right)$$
$$=\sum_{k=1}^\infty\frac{1}{2a}\frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{k}{a^2-k^2}\right)$$
$$=\sum_{k=1}^\infty\frac{k}{(a^2-k^2)^2}. \qquad (3.3)$$

Consider the identity

$$\sum_{k=1}^{\infty}\frac{k}{(a^2-k^2)^2}=-\frac{1}{4a}\left(\sum_{k=1}^{\infty}\frac{1}{(k+a)^2}-\sum_{k=1}^{\infty}\frac{1}{(k-a)^2}\right).$$

Expressing each of the sums on the right hand side as a trigamma function, we have

$$\sum_{k=1}^{\infty}\frac{k}{(a^2-k^2)^2}=-\frac{1}{4a}\left(\psi'(a+1)-\psi'(-a+1)\right)$$

But

$$\psi'(a+1)=\psi'(a)-1/a^2$$

and

$$\psi'(-a+1)=-\psi'(a)+\pi^2\csc^2\pi a.$$

Hence

$$w(a)=\sum_{n=1}^{\infty}\frac{n}{(a^2-n^2)^2}=-\frac{1}{4a}\frac{2a^2\psi'(a)-\pi^2a^2\csc^2\pi a-1}{a^2}.$$

Generating function of the Riemann zeta function of an odd integer argument

Theorem 3 *The function* $v(a)$ *defined by the integral*

$$v(a)=\int_0^{\infty}\frac{\cosh ax-1}{a}\frac{dx}{e^x-1} \qquad (4.1)$$

is a generating function of the Riemann zeta function of an odd integer argument.

Proof. From (3.1) and (3.2) we have

$$aw(a)=\int_0^{\infty}\frac{\sinh ax}{2}\frac{xdx}{e^x-1}=\sum_{n=0}^{\infty}a^{2n+1}(n+1)\zeta(2n+3),$$

from which, after integrating both sides of $(aw)a$ with respect to a, we find

$$\int_0^{\infty}(\cosh ax-1)\frac{dx}{e^x-1}=\sum_{n=0}^{\infty}\zeta(2n+3)a^{2n+2}=\sum_{n=1}^{\infty}\zeta(2n+1)a^{2n}, \qquad (4.2)$$

which shows that the $v(a)$ given in (4.1) is the generating function of $\zeta(2n+1)$.

Before ending this section, we give a closed form for the generating function of the Riemann zeta function of an odd integer argument.

It is known that (Srivastava and Choi, 2012)

$$\sum_{n=1}^{\infty}\zeta(2n+1)\frac{a^{2n+1}}{2n+1}=\frac{1}{2}\log\left(\frac{\Gamma(1-a)}{\Gamma(1+a)}\right)-\gamma a, \quad |a|<1,$$

which, upon differentiation with respect to a gives

$$\sum_{n=1}^{\infty}\zeta(2n+1)a^{2n}=-\frac{1}{2}\psi(1-a)-\frac{1}{2}\psi(1+a)-\gamma; \qquad (4.3)$$

where $\gamma\approx 0.5772157$ is the Euler-Mascheroni constant and $\psi(z)$ is the digamma function defined by

$$\psi(z)=\frac{d}{dz}\log\Gamma(z)$$
$$=-\gamma+\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+z}\right), \quad z\neq 0,-1,-2,\ldots$$

Thus, from (4.2), (4.3) and (4.1), it is established that

$$v(a)=\sum_{n=1}^{\infty}\zeta(2n+1)a^{2n-1}=-\frac{1}{2}\frac{\psi(1-a)+\psi(1+a)}{a}-\frac{\gamma}{a}.$$

CONCLUSION

As noted in the introduction, valuable information about any mathematical quantity is stored in the generating function of that quantity or object.

In this paper, we have derived new integral representations for the generating functions of the Riemann zeta function of integer arguments.

For even integer arguments, we found the following representation

$$u(a)=\int_0^{\infty}\frac{\sinh ax}{a}\frac{dx}{e^x-1}=\sum_{n=0}^{\infty}a^{2n}\zeta(2n+2),$$

while for odd arguments, we obtained

$$w(a)=\int_0^{\infty}\frac{\sinh ax}{2a}\frac{xdx}{e^x-1}=\sum_{n=0}^{\infty}a^{2n}(n+1)\zeta(2n+3)$$

and

$$v(a)=\int_0^{\infty}\frac{\cosh ax-1}{a}\frac{dx}{e^x-1}=\sum_{n=1}^{\infty}\zeta(2n+1)a^{2n-1}.$$

We also showed that

$$u(a)=\frac{1-\pi a\cot\pi a}{2a^2},$$

$$w(a)=-\frac{1}{4a}\frac{2a^2\psi'(a)-\pi^2a^2\csc^2\pi a-1}{a^2},$$

and

$$v(a)=-\frac{1}{2}\frac{\psi(1-a)+\psi(1+a)}{a}-\frac{\gamma}{a}.$$

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