## NUMERICAL INTEGRATION OF NONLINEAR FITZHUGH-NAGUMO PARTIAL DIFFERENTIAL EQUATIONS USING SECOND DERIVATIVE TWO-STEP HYBRID ALGORITHM

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#### ABSTRACT

This manuscript presents a second derivative two-step hybrid block method derived through collocation techniques. The derived scheme and the sixth order compact difference schemes are used to efficiently solve the nonlinear FitzHugh-Nagumo Partial Differential Equations (PDE). The sixth order standard compact difference schemes are used to semi-discretize the nonlinear FitzHugh PDE to a first-order system of ordinary differential equations (ODE). The derived two-step hybrid block scheme profer approximate solution to the resulting system of ODEs. The analysis of the derived hybrid methods are shown. The numerical results reveal that the derived block scheme is efficient and effective for solving FitzHugh-Nagumo PDE.

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Keywords and Phrases: Collocation technique, Compact Difference Scheme, FitzHugh-Nagumo Equation, Hybrid Block Method, Nonlinear PDEs.

#### **INTRODUCTION**

We are interested in solving one-dimensional nonlinear FitzHugh-Nagumo PDE of the form

$$V_t = V_{xx} + V(1 - V)(V - \mu)$$
(1*a*)

with the initial condition

$$V(x,0) = f(x), \qquad a \le x \le b, \tag{1b}$$

and two boundary conditions as

where the variables t, x and V denote time, space and the solution of the problem respectively. Also  $\mu \in (0, 1)$  represent the overall dynamics of the equation parameter. The FitzHugh-Nagumo PDE (1) was named after two mathematicians Richard FitzHugh and John Nagumo who independently proposed it in the 1960s. It describes the behaviour of a neuron after stimulation by an external input current (Ara, 2019). The PDE had been used to study different phenomena in the fields of neurophysiology, population growth models, genetics, biology and many other fields of sciences (Ramos et al., 2022). In the past years, various numerical methods have been applied to solve (1). Among these are Haar wavelet method (Hariharan and Kannan, 2010), Jacobi-Gaus-Lobatto collocation

 $V(a,t) = f_1(t) = V_1(t), V(b,t) = f_2(t) = V_{N+1}(t), t \ge 0$ (1c)method (Bhrawy, 2013), Galerkin finite element method (Ali et al., 2020), the qhomotopy analysis approach (Kumar et al., 2018), Compact and Finite Difference Schemes (Agbavon and Appadu, 2020), block hybrid method (Mehta et al., 2023; Ramos et al., 2022), to mention but a few. The Compact Difference Schemes are preferred over the standard Finite Difference Schemes to optimize accuracy in approximating the spatial derivatives in (1). Block methods profer approximate solutions to first-order system of ODEs. These block methods are self starting schemes introduced by Milne (Milne, 1953) and also provide approximate solution for more than one point at a time saving computational time. Some researchers (Akinnukawe et al., 2016; Akinnukawe and Odekunle, 2023) had used

block methods for solving first and higher orders ODEs.

This manuscript presents the combination of the sixth order compact difference schemes (Li

and Chen, 2008) and the derived second derivative two-step hybrid block method to numerically solve the nonlinear FitzHugh-Nagumo PDE.

#### **Development of SDTHBM**

The PDE (1) can be converted to first-order system of ODE:

$$V' = g(t, V), \quad V(t_0) = V_0, \quad t_0 \le t \le t_m,$$
 (2)

To derive SDTHBM, we assume  $t_m = t_0 + mh$  with uniform step size  $h = t_m - t_{m-1}$ . The true solution of problem (1) can be approximated by the polynomial in (3)

$$V(t) = \sum_{r=0}^{r} a_r t^r,$$
 (3)

with first and second derivatives as

$$V'(t) = \sum_{\substack{r=1\\ \frac{7}{7}}}^{\prime} ra_r t^{r-1},$$
(4a)

$$V''(t) = \sum_{r=2}^{n} r(r-1)a_r t^{r-2},$$
(4b)

where  $a_r \in R$ , r = 0(1)7 are unknown parameters that should be determined. Since there are eight parameters to be determined and we need eight equations to find the eight  $V_{n+i} = V(t$ 

unknown hence equation (3) is a seven degree polynomial with  $a_r$ , r = 0(1)7. Interpolating equation (3) and collocating equation (4) at given grid points gives

$$t_{n+j}), \qquad j=0, \tag{5a}$$

$$V'_{n+j} = V'(t_{n+j}) = g_{n+j}, \qquad j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$$

$$V''_{n+j} = V''(t_{n+j}) = g'_{n+j}, \qquad j = 0, 2$$
(5b)
(5c)

$$V''_{n+j} = V''(t_{n+j}) = g'_{n+j}, \quad j =$$

where  $V_{n+j}$ ,  $g_{n+j}$  and  $g'_{n+j}$  are approximations for  $V(t_{n+j})$ ,  $V'(t_{n+j})$  and

 $V''(t_{n+i})$  respectively. In matrix form, the system of eight equations in equation (5) will be

$$\begin{bmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 & t_n^6 & t_n^7 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 & 6t_n^5 & 7t_n^6 \\ 0 & 1 & 2t_{n+\frac{1}{2}} & 3t_{n+\frac{1}{2}}^2 & 4t_{n+\frac{1}{2}}^3 & 5t_{n+\frac{1}{2}}^4 & 6t_{n+\frac{1}{2}}^5 & 7t_{n+\frac{1}{2}}^6 \\ 0 & 1 & 2t_{n+1} & 3t_{n+1}^2 & 4t_{n+1}^3 & 5t_{n+1}^4 & 6t_{n+1}^5 & 7t_{n+1}^6 \\ 0 & 1 & 2t_{n+\frac{3}{2}} & 3t_{n+\frac{3}{2}}^2 & 4t_{n+\frac{3}{2}}^3 & 5t_{n+\frac{3}{2}}^4 & 6t_{n+\frac{3}{2}}^5 & 7t_{n+\frac{3}{2}}^6 \\ 0 & 1 & 2t_{n+2} & 3t_{n+\frac{3}{2}}^2 & 4t_{n+2}^3 & 5t_{n+2}^4 & 6t_{n+2}^5 & 7t_{n+2}^6 \\ 0 & 1 & 2t_{n+2} & 3t_{n+2}^2 & 4t_{n+2}^3 & 5t_{n+2}^4 & 6t_{n+2}^5 & 7t_{n+2}^6 \\ 0 & 0 & 2 & 6t_n & 12t_n^2 & 20t_n^3 & 30t_n^4 & 42t_n^5 \\ 0 & 0 & 2 & 6t_{n+2} & 12t_{n+2}^2 & 20t_{n+2}^3 & 30t_{n+2}^4 & 42t_{n+2}^5 \end{bmatrix} = \begin{bmatrix} V_n \\ a_0 \\ a_1 \\ a_2 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix}$$

Equation (5) is solved simultaneously to obtain the coefficients  $a_r, r = 0(1)7$ . Then the resulting coefficients  $a_r, r = 0(1)7$ are substituted into equation (3) and its derivatives.

Evaluation is done at the points  $t_{n+\frac{1}{2}}, t_{n+1}$ ,  $t_{n+\frac{3}{2}}$  and  $t_{n+2}$  to give the following second derivative two-step block hybrid schemes

$$\begin{split} V_{n+\frac{1}{2}} &= V_{n} + \frac{h}{241920} [62923 \ g_{n} + 69728 \ g_{n+\frac{1}{2}} - 17712 \ g_{n+1} + 9248 g_{n+\frac{3}{2}} - 3227 g_{n+2} \\ &+ \frac{h^{2}}{16128} [337g'_{n} + 41g'_{n+2}] & (6a) \\ V_{n+1} &= V_{n} + \frac{h}{15120} \Big[ 3381 \ g_{n} + 8576 \ g_{n+\frac{1}{2}} + 3456 \ g_{n+1} - 384g_{n+\frac{3}{2}} + 91g_{n+2} \Big] \\ &+ \frac{h^{2}}{1008} [15g'_{n} - g'_{n+2}] & (6b) \\ V_{n+\frac{3}{2}} &= V_{n} + \frac{h}{8960} \Big[ 2177 \ g_{n} + 4512 \ g_{n+\frac{1}{2}} + 4752 \ g_{n+1} + 2272g_{n+\frac{3}{2}} - 273g_{n+2} \Big] \\ &+ \frac{h^{2}}{1792} [33g'_{n} + 9g'_{n+2}] & (6c) \\ V_{n+2} &= V_{n} + \frac{h}{945} \Big[ 217 \ g_{n} + 512 \ g_{n+\frac{1}{2}} + 432 \ g_{n+1} + 512g_{n+\frac{3}{2}} + 217g_{n+2} \Big] \\ &+ \frac{h^{2}}{63} [g'_{n} - g'_{n+2}] & (6d) \end{split}$$

Equations (6a) - (6d) form the Second Derivative Two-step Hybrid Block Method (SDTHBM) developed to efficiently solve the nonlinear FitzHugh-Nagumo PDEs (1).

# Theoretical Analysis of SDTHBM Zero-stability of the method

The schemes in (6a) - (6d) is presented in matrix form as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_{n+\frac{1}{2}} \\ V_{n+1} \\ V_{n+\frac{3}{2}} \\ V_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_{n-\frac{3}{2}} \\ V_{n-1} \\ V_{n-\frac{1}{2}} \\ V_{n} \end{bmatrix} + h \begin{bmatrix} \frac{69728}{241920} & -\frac{17712}{241920} & \frac{9248}{241920} & -\frac{3227}{241920} \\ \frac{8576}{15120} & \frac{3456}{15120} & -\frac{384}{15120} \\ \frac{4512}{8960} & \frac{4752}{8960} & \frac{2272}{8960} \\ \frac{4512}{8960} & \frac{4752}{8960} & \frac{2272}{8960} \\ \frac{512}{945} & \frac{432}{945} & \frac{512}{945} & \frac{217}{945} \end{bmatrix} \begin{bmatrix} g_{n+\frac{3}{2}} \\ g_{n+2} \\ g_{n+2} \end{bmatrix} \\ + h \begin{bmatrix} 0 & 0 & 0 & \frac{62923}{241920} \\ 0 & 0 & 0 & \frac{3381}{15120} \\ 0 & 0 & 0 & \frac{2177}{8960} \\ 0 & 0 & 0 & \frac{2177}{8960} \\ g_{n-\frac{1}{2}} \\ g_{n} \end{bmatrix} + h^2 \begin{bmatrix} \frac{41}{16128} & 0 & 0 & \frac{337}{16128} \\ -\frac{1}{1008} & 0 & 0 & \frac{15}{1008} \\ \frac{9}{1792} & 0 & 0 & \frac{33}{1792} \\ -\frac{1}{63} & 0 & 0 & \frac{1}{63} \end{bmatrix} \begin{bmatrix} g'_{n+2} \\ g'_{n+1} \\ g'_{n} \end{bmatrix}$$

$$(7)$$

Equation (7) is written as  $E^{(1)}V_{n+i} = E^{(0)}V_{n-i} + hB^{(1)}G_{n+i} + hB^{(0)}G_{n-i} + h^2B^{(2)}G'_{n+i}$ which is already normalized. The first characteristic polynomial of (7) is defined as:

$$S(\emptyset) = det[\emptyset E^{(1)} - E^{(0)}]$$

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$$S(\phi) = \phi \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \phi & 0 & 0 & -1 \\ 0 & \phi & 0 & -1 \\ 0 & 0 & \phi & -1 \\ 0 & 0 & 0 & \phi -1 \end{bmatrix} = 0$$

$$S(\emptyset) = \emptyset^3(\emptyset - 1) = 0,$$
  
since  $\emptyset_1 = \emptyset_2 = \emptyset_3 = 0, \emptyset_4 = 1$ , then the hybrid schemes (6) are zero-stable.

**Error Constant and Order of the SDTHBM** The error constants and order of the Method (6) are obtained by evaluating their local truncation error (LTE) as seen in (Akinfenwa et al., 2020).

Let  $V(t_n)$  be a sufficiently differentiable function and recall that  $V'(t_n) = g(t_n)$ ,  $V''(t_n) = g'(t_n)$ . We consider the Taylor's series expansion of  $V(t_n + \frac{1}{2}h)$ ,  $V(t_n + h)$ ,  $V(t_n + \frac{3}{2}h)$ ,  $V(t_n + 2h)$ ,  $g(t_n + \frac{1}{2}h)$ ,  $g(t_n + h)$ ,  $g(t_n + \frac{3}{2}h)$ ,  $g(t_n + 2h)$ ,  $g'(t_n)$ ,  $g'(t_n + 2h)$ . The local truncation error of each of the schemes (6a) -(6d) is obtained by performing the Taylor's series expansion of each of its component. The LTE of the scheme (6a) is:

(8)

$$\begin{aligned} \text{LTE} &= V\left(t_n + \frac{1}{2}h\right) - (V(t_n) + h\frac{62923g(t_n) + 69728g\left(t_n + \frac{1}{2}h\right) - 17712g(t_n + h)}{241920} \\ &+ \frac{9248g\left(t_n + \frac{3}{2}h\right) - 3227g(t_n + 2h)}{241920} + h^2\frac{337g'^{(t_n)} + 41g'^{(t_n + 2h)}}{16128}) = C_8h^8V^8(t_n) + O(h^9) \end{aligned}$$

The same process is repeated for equations (6b) - (6d) to obtain the LTE of the individual scheme. After the expansion and simplification of equation (6), the order and error constant of the schemes (6a) - (6d) are obtained and tabulated as follows:

Schemes	Order (p)	Error Constant
		$(\mathcal{C}_{p+1})$
$V_{n+1}$	7	3
<i>n</i> + <u>2</u>		1146880
$V_{n+1}$	8	1
		6350400
$V_{m+3}$	7	3
$n+\frac{1}{2}$		1146880
$V_{n+2}$	8	1
		3175200

Table 1: The Order and Error Constants of SDTHBM

## Consistency and Convergence of SDTHBM

A numerical scheme is consistent if the scheme have order  $p \ge 1$  (Henrici, 1962). Hence the SDTHBM is consistent since it is of order at least seven as shown in Table 1. Also SDTHBM is convergent since it is zero-stable and consistent (Fatunla, 1988).

$$A\begin{bmatrix} V_{n+\frac{1}{2}} \\ V_{n+1} \\ V_{n+\frac{3}{2}} \\ V_{n+2} \end{bmatrix} = B\begin{bmatrix} V_{n-\frac{3}{2}} \\ V_{n-1} \\ V_{n-\frac{1}{2}} \\ V_{n} \end{bmatrix}$$

#### **Stability Analysis**

We applied the SDTHBM (6) to the Dahlquist test equation:

$$V'(t) = \lambda V(t), \quad \operatorname{Re}(\lambda) < 0,$$
 (9a)

to obtain the region of absolute stability for the method where  $z = h \lambda$ , we have

(9b)

$$A = \begin{bmatrix} 1 - \frac{69728z}{241920} & \frac{17712z}{241920} & -\frac{9248z}{241920} & \frac{3227z}{241920} - \frac{41z^2}{16128} \\ -\frac{8576z}{15120} & 1 - \frac{3456z}{15120} & \frac{384z}{15120} & -\frac{91z}{15120} + \frac{z^2}{1008} \\ -\frac{4512z}{8960} & -\frac{4752z}{8960} & 1 - \frac{2272z}{8960} & \frac{273z}{8960} - \frac{9z^2}{1792} \\ -\frac{512z}{945} & -\frac{432z}{945} & -\frac{512z}{945} & 1 - \frac{217z}{945} + \frac{z^2}{63} \end{bmatrix}$$
$$B = \begin{bmatrix} 0 & 0 & 0 & 1 + \frac{62923z}{241920} + \frac{337z^2}{16128} \\ 0 & 0 & 0 & 1 + \frac{3381z}{15120} + \frac{15z^2}{1008} \\ 0 & 0 & 0 & 1 + \frac{2177z}{8960} + \frac{33z^2}{1792} \\ 0 & 0 & 0 & 1 + \frac{2177z}{945} + \frac{z^2}{63} \end{bmatrix}$$
Clearly, (9b) can be written as:

$$\begin{bmatrix} V_{n+\frac{1}{2}} \\ V_{n+1} \\ V_{n+\frac{3}{2}} \\ V_{n+2} \end{bmatrix} = W(z) \begin{bmatrix} V_{n-\frac{3}{2}} \\ V_{n-1} \\ V_{n-1} \\ V_{n-\frac{1}{2}} \\ V_{n} \end{bmatrix}$$

where  $W(z) = A^{-1}B$  denotes the stability matrix. Then the eigenvalues of the stability

matrix are obtained as 0, 0, 0, Q(z) where Q(z) is the stability function.

$$Q(z) = \frac{-z^5 - 14z^4 - 95z^3 - 375z^2 - 840z - 840}{z^5 - 14z^4 + 95z^3 - 375z^2 + 840z - 840}$$

Finally, let us consider a set

 $s = \{z \in C : |Q(z) < 1|\}.$ 

The set s is the absolute stability region of the SDTHBM. This region is shown in Figure 1 and it is A-stable in nature. Note that the region

of absolute stability of SDTHBM is the shaded region of the graph in Figure 1.



Figure 1: A graph showing the stability region of the proposed method

**Discretization of the space variable** The sixth order compact difference scheme is used to semi-discretize the spatial derivatives in (1) (Li and Chen, 2008). To do this, we partitioned the space interval [a,b] into N+1 equal parts as

$$\Delta x = x_2 - x_1 = x_3 - x_2 = \dots = x_{N+1} - x_N.$$
The  $V_x$  at the interior node satisfies the following relation:  

$$\frac{1}{3}V'_{j-1} + V'_j + \frac{1}{3}V'_{j+1} = \frac{1}{\Delta x} \left[ \frac{1}{36} (V_{j+2} - V_{j-2}) + \frac{7}{9} (V_{j+1} - V_{j-1}) \right], \quad j = 3, 4, \dots, N-1, \quad (10)$$
and the following relations holds at the boundary points 1, 2, N, N + 1.  
for  $j = 1$ ,  
 $V'_1 + 6V'_2 = \frac{1}{\Delta x} \left[ -\frac{69}{20}V_1 - \frac{17}{10}V_2 + \frac{15}{2}V_3 - \frac{10}{3}V_4 + \frac{5}{4}V_5 - \frac{3}{10}V_6 + \frac{1}{30}V_7 \right], \quad (11)$   
for  $j = 2$ ,  
 $\frac{5}{32}V'_1 + V'_2 + \frac{5}{32}V'_3 = \frac{1}{\Delta x} \left[ -\frac{209}{384}V_1 - \frac{49}{120}V_2 + \frac{475}{384}V_3 - \frac{5}{12}V_4 + \frac{65}{384}V_5 - \frac{1}{24}V_6 + \frac{3}{640}V_7 \right], \quad (12)$   
for  $j = N$ ,  
 $\frac{5}{32}V'_{N-1} + V'_N + \frac{5}{32}V'_{N+1}$   
 $= \frac{1}{\Delta x} \left[ \frac{209}{384}V_{N+1} + \frac{49}{120}V_N - \frac{475}{384}V_{N-1} + \frac{5}{12}V_{N-2} - \frac{65}{384}V_{N-3} + \frac{1}{24}V_{N-4} - \frac{3}{640}V_{N-5} \right], \quad (13)$ 

for j = N + 1,  

$$6V'_{N} + V'_{N+1} = \frac{1}{\Delta x} \left[ \frac{69}{20} V_{N+1} + \frac{17}{10} V_{N} - \frac{15}{2} V_{N-1} + \frac{10}{3} V_{N-2} - \frac{5}{4} V_{N-3} + \frac{3}{10} V_{N-4} - \frac{1}{30} V_{N-5} \right], \quad (14)$$

For any value of N, equations (10) - (14) can be expressed in matrix form as:

$$R_1 V' = R_2 V \tag{15}$$

$V' = \left[ egin{array}{c} V'_1 \ V'_2 \ dots \ V \end{array}  ight], V = \left[ egin{array}{c} V_1 \ V_2 \ dots \ dots \ dots \end{array}  ight],$	
$\begin{bmatrix} V'_{N} \\ V'_{N} \end{bmatrix} = \begin{bmatrix} V_{N} \\ V_{N} \end{bmatrix}$	
$\begin{bmatrix} 1 & 6 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}$	
$\begin{bmatrix} \frac{1}{5} & 0 & \frac{5}{5} & 0 & 0 & 0 & 0 & 0 \\ \frac{5}{5} & 1 & \frac{5}{5} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}$	
$\begin{vmatrix} 32 & 32 \\ 1 & 1 \end{vmatrix}$	
$\begin{bmatrix} 0 & \frac{1}{3} & 1 & \frac{1}{3} & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}$	
$\begin{bmatrix} 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}$	
$R_1 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 1 & \frac{1}{2} & 0 & 0 & 0 \end{vmatrix}$	
$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \end{bmatrix}$	
$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{3} & 1 & \frac{1}{3} & 0 \end{bmatrix}$	
$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{2} & 1 & \frac{5}{2} \end{bmatrix}$	
$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 32 & 32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 1 \end{bmatrix}$	
$\begin{bmatrix} 69 & 17 & 15 & 10 \end{bmatrix}$	٦
$\left  \begin{array}{ccc} -\frac{37}{20} & -\frac{17}{10} & \frac{12}{2} & -\frac{13}{3} & \cdots & 0 & 0 & 0 \end{array} \right $	0
$\left  -\frac{209}{204} - \frac{49}{120} - \frac{475}{204} - \frac{5}{12} - \frac{5}{12}$	0
$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	
$\begin{bmatrix} -\frac{1}{36} & -\frac{1}{9} & 0 & \frac{1}{9} & \cdots & 0 & 0 & 0 \\ 1 & 7 & 9 & \cdots & 0 & 0 & 0 \end{bmatrix}$	0
$1 \qquad 0 \qquad -\frac{1}{36}  -\frac{7}{9} \qquad 0  \cdots  0 \qquad 0 \qquad 0$	0
$R_2 = \frac{1}{\Delta x} \begin{vmatrix} \vdots & \vdots$	:,
$0 0 0 0 0 \cdots 0 \frac{7}{9} \frac{1}{36}$	0
$0 0 0 0 0 \cdots -\frac{7}{2} 0 \frac{7}{2}$	$\frac{1}{2\epsilon}$
	36 209
	201
12  384  120	384

From equation (15), the approximate value of  $V_x$  can be obtained by  $V_x = V' = R_1^{-1}R_2V$ . Note that  $R_1$  and  $R_2$  matrices are not used in the computations below because of the absence of  $V_x$  in the FitzHugh PDE.

Following the above procedure as  $inV_x$ , the approximate value of the second order spatial derivative  $V_{xx}$  is obtained as:

$$V_{xx} = V'' = R_3^{-1} R_4 V \tag{16}$$

where

	1	$\frac{126}{11}$	0	0	0		0	0	0	0	0			
	$\frac{11}{122}$	11	$\frac{11}{100}$	0	0		0	0	0	0	0			
	$\begin{array}{c c} 128 \\ 0 \end{array}$	2	128	2	0		0	0	0	0	0			
		11	2	11 1	2		0	0	0	0	0			
		0	11	2	11		0	0	0	0	0			
P —		0	0	11	1	•••	0	0	0 ·	0 ·	0			
$\Lambda_3 -$		:	: 0	:	: 0		: 1	: 	: 0	: 0	:	,		
		0	0	0	0		2	11	2	0	0			
		0	0	0	0	•••	11	1 2	11	2	0			
	0	0	0	0	0	•••	0	$\frac{2}{11}$	1	$\frac{2}{11}$	0			
	0	0	0	0	0		0	0	$\frac{11}{128}$	1	$\frac{11}{128}$			
	0	0	0	0	0		0	0	0	$\frac{126}{11}$	1			
		[13097	7	2943	57	73	167		0	0	_	0	0	7
		990 585		$\begin{array}{c} 110\\ 141 \end{array}$	4 4	.4 59	99 9		0	0		0	0	
		$\overline{512}$	_	64	5	12	$\frac{32}{12}$	•••	0	0		0	0	
		$\frac{3}{44}$		$\frac{12}{11}$		$\frac{31}{22}$	$\frac{12}{11}$	•••	0	0		0	0	
	1	0		$\frac{3}{44}$	$\frac{1}{1}$	$\frac{2}{1}$	$-\frac{51}{22}$		0	0		0	0	
$R_{4} =$	$\frac{1}{\left(\Delta x\right)^2}$	:		:	1	:	:		: 51	:		:	:	,
		0		0	(	C	0	•••	$-\frac{31}{22}$	$\frac{12}{11}$		$\frac{3}{44}$	0	
		0		0	(	C	0		$\frac{12}{11}$	$-\frac{5}{22}$	L	$\frac{12}{11}$	$\frac{3}{44}$	
		0		0	(	C	0		$\frac{9}{32}$	$\frac{459}{512}$	)	$\frac{141}{64}$	$\frac{585}{512}$	
		0		0	(	Э	0		$\frac{167}{99}$	$\frac{512}{573}$	<u> </u>	$\frac{2943}{110}$	<u>13097</u>	7
	Γ	$V''_1$		$\begin{bmatrix} V_1 \end{bmatrix}$	]				77	44		110	990	L
	<b>T</b> 7 11	$V''_2$	<b>T</b> 7	$V_2$ .										
	$\mathbf{v}$ =	: $V''_{x'}$	, v =	$\begin{vmatrix} : \\ V_{\star} \end{vmatrix}$	,									
				$\begin{bmatrix} V \\ V_{N+1} \end{bmatrix}$										

Finally, substituting  $R_3^{-1}R_4V$  for  $V_{xx}$  in equation (1) we have:  $V_t = (R_3^{-1}R_4 - \mu I)V + (1 + \mu)V^2 - V^3$ 

$$\begin{bmatrix} \frac{dV_{1}}{dt} \\ \frac{dV_{2}}{dt} \\ \vdots \\ \frac{dV_{N}}{dt} \\ \frac{dV_{N+1}}{dt} \end{bmatrix} = (R_{3}^{-1}R_{4} - \mu I) \begin{bmatrix} V_{1} \\ V_{2} \\ \vdots \\ V_{N} \\ V_{N+1} \end{bmatrix} + (1 + \mu) \begin{bmatrix} V_{1}^{2} \\ V_{2}^{2} \\ \vdots \\ V_{N}^{2} \\ V_{N+1}^{2} \end{bmatrix} - \begin{bmatrix} V_{1}^{3} \\ V_{2}^{3} \\ \vdots \\ V_{N}^{3} \\ V_{N+1}^{3} \end{bmatrix}$$

which is now the resulting nonlinear system of ODE to be solved numerically using the new derived block method (SDTHBM) in equations (6a) - (6d).

#### 1. Numerical Results

Three special cases of the FitzHugh-Nagumo partial differential equation are solved to show the accuracy of SDTHBM. The numerical results obtained from SDTHBM are compared with numerical results from existing methods and exact solution of the PDE. Mathematica 13.3 software is used for the computation of the work. The calculations of the maximum absolute error  $M_{\infty}$  and the root mean square error  $M_{rms}$  are done using the formulae below.

$$M_{\infty} = max |\varepsilon_j|$$
$$M_{rms} = \left(\sum_{j=1}^{N+1} \frac{\varepsilon_j^2}{N+1}\right)^{\frac{1}{2}}$$
$$\varepsilon_j = v(x_j, t) - V(x_j, t)$$

where

- $v(x_i, t)$  denotes the theoretical solution at point  $(x_i, t)$ .
- $V(x_j, t)$  denotes the approximate solution at point  $(x_j, t)$ .
- $\varepsilon_i$  represents the error at point *j* and j = 1, 2, ..., N + 1.

#### **Experiment 1**

Consider,

$$V_t = V_{xx} + V(1-V)(V-\mu)$$

with  $\mu = 0.75$  and initial condition:

$$V(x,0) = \frac{1}{2} + \frac{1}{2} tanh\left(\frac{1}{2\sqrt{2}}x\right), \quad x \in [-10, \ 10]$$

boundary conditions:

$$V(-10, t) = \frac{1}{2} + \frac{1}{2} tanh\left(\frac{1}{2\sqrt{2}}\left(-10 - \frac{2\mu - 1}{\sqrt{2}}t\right)\right),$$
  
$$V(10, t) = \frac{1}{2} + \frac{1}{2} tanh\left(\frac{1}{2\sqrt{2}}\left(10 - \frac{2\mu - 1}{\sqrt{2}}t\right)\right),$$

and exact solution:

V(x, t) = 
$$\frac{1}{2} + \frac{1}{2} tanh\left(\frac{1}{2\sqrt{2}}\left(x - \frac{2\mu - 1}{\sqrt{2}}t\right)\right)$$
,

	$M_{\infty}$ SDTHBM	$M_{\infty}$ (method in
		Akkoyunlu, 2019)
12	$3.9505 \times 10^{-5}$	$3.9857 \times 10^{-4}$
24	$2.4452 \times 10^{-6}$	$2.3475 \times 10^{-5}$
48	$6.4200 \times 10^{-8}$	$8.3749 \times 10^{-6}$
64	$3.6500 \times 10^{-8}$	$5.9363 \times 10^{-6}$
Number of	1	20
iterations		

**Table 2**:  $M_{\infty}$  for Experiment 1 at t = 0.2

Table 3:  $\underline{M}_{\infty}$  for Experiment 1 at  $\mu = 0.75$  and N = 100

t	$M_{\infty}$ (method in	$M_{\infty}(method$ in	$M_{\infty}$ SDTHBM
	Ahmad et al., 2019)	Ramos et al., 2023)	
0.2	$1.8896 \times 10^{-5}$	$1.8876 \times 10^{-5}$	$1.6664 \times 10^{-8}$
0.5	$4.1554 \times 10^{-5}$	$4.1519 \times 10^{-5}$	$3.6950 \times 10^{-7}$
1	$6.9891 \times 10^{-5}$	$6.9734 \times 10^{-5}$	$4.0990 \times 10^{-6}$
1.5	$9.1687 \times 10^{-5}$	$9.1180 \times 10^{-5}$	$1.6744 \times 10^{-5}$
2	$1.0969 \times 10^{-4}$	$1.0854 \times 10^{-4}$	$4.4916 \times 10^{-5}$
3	$1.3942 \times 10^{-4}$	1.3651× 10 <sup>-4</sup>	$1.7390 \times 10^{-4}$

Table 4:  $M_{rms}$  for Experiment 1 at  $\mu = 0.75$  and N = 100

	1 /		
t	M <sub>rms</sub> (method in	M <sub>rms</sub> (method	M <sub>rms</sub> SDTHBM
	Ahmad et al., 2019)	in Ramos et al., 2023)	
0.2	$2.1960 \times 10^{-7}$	$7.4559 \times 10^{-6}$	$2.4966 \times 10^{-9}$
0.5	$1.5696 \times 10^{-6}$	1.6411 ×	$5.7614 \times 10^{-8}$
		10 <sup>-5</sup>	
1.0	$7.1449 \times 10^{-6}$	2.7433 ×	$6.5297 \times 10^{-7}$
		10 <sup>-5</sup>	
1.5	$1.7262 \times 10^{-5}$	3.5345 ×	$2.6934 \times 10^{-6}$
		10 <sup>-5</sup>	
2.0	$3.1857 \times 10^{-5}$	4.1285 ×	$7.3000 \times 10^{-6}$
		10 <sup>-5</sup>	
3.0	$7.2878 \times 10^{-5}$	$4.9731 \times 10^{-5}$	$2.9095 \times 10^{-5}$

## **Experiment 2**

Consider,

$$V_t = V_{xx} + V(1 - V)(V - \mu)$$

with  $\mu = 0.75$  and initial condition (Inan *et al.*, 2020)

$$V(x, 0) = \frac{1}{1 + \exp\left(\frac{-x}{\sqrt{2}}\right)}, \quad 0 \le x \le 1,$$

and exact solution:

$$V(x,t) = \frac{1}{1 + \exp\left(\frac{-w}{\sqrt{2}}\right)}, \quad t > 0,$$

where w = x + ct and  $c = \sqrt{2} \left(\frac{1}{2} - \mu\right)$ .

**Table 5**: Absolute errors for Experiment 2 at t = 0.04 and N = 10.

x	SDTHBM	ANM (Inan et al., 2020)
0.2	$2.0331 \times 10^{-8}$	$2.00 \times 10^{-7}$
0.4	$2.7564 \times 10^{-9}$	$5.00 \times 10^{-7}$
0.6	$4.1075 \times 10^{-9}$	$7.00 \times 10^{-7}$
0.8	$3.2904 \times 10^{-8}$	$6.00 \times 10^{-7}$
Number of	1	8
iterations		

#### **Experiment 3**

Consider stiff case of the FitzHugh-Nagumo equation (Agbavon and Appadu, 2020),

$$V_t = V_{xx} + \tau (1 - V)(V - \mu)$$

where  $\tau > 0$  represents the natural growth rate. The initial condition is:

$$V(x,0) = \frac{1}{2} - \frac{1}{2} tanh\left(\frac{\sqrt{\tau}}{2\sqrt{2}}x\right), \quad -10 < x < 10,$$

The exact solution of the problem is:

$$V(x,t) = \frac{1}{2} - \frac{1}{2} tanh\left(\frac{\sqrt{\tau}}{2\sqrt{2}}(x-ct)\right), \quad -10 < x < 10,$$
  
where  $c = \sqrt{\frac{\tau}{2}}(2\mu - 1).$ 

Table 6: Ma	, for	Experin	nent 3 a	t time <b>t</b>	= 0.5	for $\mu$ :	= 0.2 :	and $N =$	100
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τ	Scheme 4 (Agbavon	SDTHBM
	and Appadu, 2020)	
0.5	$4.132 \times 10^{-6}$	$2.1768 \times 10^{-7}$
1	$2.4227 \times 10^{-5}$	$3.7778 \times 10^{-7}$
2	$1.5759  imes 10^{-4}$	$5.4279 \times 10^{-7}$
Δt	0.00125	0.1

Table 7:  $M_{\infty}$  for Experiment 3 at time t = 0.5 for  $\mu$  = 0.2, large values of  $\tau$  various values of N

τ	Ν	SDTHBM
5	25	$8.8723 \times 10^{-4}$
	48	$1.3793 \times 10^{-5}$
	100	$1.5527 \times 10^{-7}$
10	25	6.9994 × 10 <sup>-3</sup>
	48	$1.3884 \times 10^{-4}$
	100	$1.3537 \times 10^{-6}$

#### **RESULT DISCUSSION**

In Tables 2 and 3, the  $M_{\infty}$  errors obtained using SDTHBM were smaller when compared with other existing methods and the convergent rate was faster after one iteration unlike the other method that converged after 20 iterations. Table 4 also showed better accuracy for the derived method when compared with two existing methods solving same problem with same parameters. The absolute error obtained using SDTHBM was better than that of ANM in (Inan et al., 2020) after solving Experiment problem 2 as shown in Table 5. Even at a higher time step size, SDTHBM solved the stiff case of the nonlinear FitzHugh-Nagumo PDE better than the scheme 4 of (Agbavon and Appadu, 2020) as shown in Tables 6 and 7. The numerical results shown in Tables 2 - 7 confirm that SDTHBM can efficiently and effectively proffer numerical solution to nonlinear differential FitzHugh-Nagumo partial equation.

## CONCLUSION

Second derivative two-step hybrid Algorithm is developed and coupled with the standard sixth order compact difference schemes to numerically solve time dependent nonlinear FitzHugh-Nagumo partial differential equations. The FitzHugh-Nagumo PDE is first semi-discretized into first-order system of ODEs using the sixth order compact difference schemes and then the resulting ODEs are numerically integrated using the derived block Method, SDTHBM. The analysis of SDTHBM is shown to be zero stable, consistent, convergent and A-stable in nature. SDTHBM is applied to three special cases of the FitzHugh-Nagumo equation and it has proven effective in the numerical integration of FitzHugh-Nagumo nonlinear partial differential equation.

## **CONFLICT OF INTEREST**

No competing conflict of interest exists between the authors.

## **AUTHORS' CONTRIBUTION**

A. B. I.: Conceptualization, Introduction, Development of the Method, Methodology,

Implementation and Numerical Results, Discussion of Results, Conclusion, proofreading of the manuscript.

A. E. M.: Introduction, Methodology, Analysis of the derived method, Discussion of Results, Conclusion.

#### REFERENCES

Agbavon, K. M. and Appadu, A. R., 2020. Construction and analysis of some nonstandard finite difference methods for the FitzHugh-Nagumo equation. Numerical Methods for Partial Differential Equations, 36(5), 1145-1169.

doi: 10.1002/num.22468.

- Ahmad, I., Ahsan, M. and Din, Z. U., Masood
  A. and Kuman, P., 2019. An efficient local formulation for time-dependent PDEs. Mathematics 7(3), 216.
  doi: 10.3390/math7030216.
- Akinfenwa, O. A., Abdulganiy, R. I., Akinnukawe, B. I. and Okunuga, S. A.,2020. Seventh order hybrid block method for solution of first order stiff systems of initial value problems. Journal of Egyptian Mathematical Society 28, 34.

doi: 10.1186/s42787-020-00095-3.

- Akinnukawe, B. I., Akinfenwa, O. A. and Okunuga, S. A., 2016. L-stable Block Backward Differentiation Formula for Parabolic Partial Differential Equations, Ains Shams Engineering Journal, Vol.7(2), 867 - 872. doi: 10.1016/j.asej.2015.12.012.
- Akinnukawe, B. I. and Odekunle M. R., 2023. Block Bi-Bias collocation method for direct approximation of Fourth-order IVP. Journal of the Nigerian Mathematical Society, 42(1), 1-18. <u>https://ojs.ictp.it/jnms/index.php/jn</u> ms/article/view/888.
- Akkoyunlu, C., 2019. Compact finite differences method for FitzHugh-Nagumo equation. Universal Journal of Mathematics and Applications, 2(4), 180-187. doi: 10.32323/ujma.561873.

Ali, H., Kamrujjaman, M., and Islam, M., 2020. Numerical computation of FitzHugh-Nagumo equation: a novel Galerkin finite element approach. International Journal of Mathematical Research 9(1), 20-27.

doi: 10.18488/journal.24.2020.91.20.27.

- Ara, I., 2019. Parameters estimation of FitzHugh-Nagumo model. Biomedical Research, 30(5), 713-715.
- Bhrawy, A. 2013. A Jacobi-Gaus-Lobatto for solving generalized FitzHugh-Nagumo equation with time-dependent coefficients. Applied Mathematics and Computation, 222, 255-264. doi: 10.1016/j.amc.2013.07.056.
- Fatunla, S. O., 1988. Numerical Methods for Initial Value Problems in Ordinary Differential Equations. Academic Press Inc. Harcourt Brace, Jovanovich Publishers, New York.
- Hariharan, G. and Kannan, K., 2010. Haar wavelet method for solving FitzHugh-Nagumo equation. International Journal of Mathematical and Statistical Sciences, 2(2), 59 - 63.
- Henrici, P., 1962. Discrete Variable Methods in Ordinary Differential Equations. New York, Wiley 571.91, 419.
- Inan, B., Ali, K. K., Saha, A. and Ak, T., 2020. Analytical and numerical solutions of the FitzHugh-Nagumo equation and their multistability behaviour. Numerical Methods of Partial Differential Equations, 37(1), 7 - 23. doi: 10.1002/num.22516.

- Kumar, D., Singh, J. and Badeanu, D., 2018. A new numerical algorithm for fractional FitzHugh-Nagumo equation arising in transmission of nerve impulses. Nonlinear Dynamics, 91(1), 307-317. https://link.springer.com/article/10.1 007/s11071-017-3870-x.
- Li, J. and Chen, Y., 2008. Computational partial differential equations using MATLAB. A Chapman and Hall Book, London.
- Mehta, A., Singh, G., and Ramos, H., 2023. Numerical solution of time dependent nonlinear partial differential equations using a novel block method coupled with compact finite difference schemes. Computational and Applied Mathematics, 42(4), 201. doi: 10.1007/s40314-023-02345-3.
- Milne, W. E., 1953. Numerical solution of Differential Equations, John Wiley and sons, New York.
- Ramos, H., Kaur, A. and Kanwar, V., 2022. Using a cubic B-spline method in conjunction with a one step optimized hybrid block approach to solve nonlinear partial differential equation. Computational and Applied Mathematics, 41, 34. doi: 10.1007/s40314-021-01729-7.