Controllability and Null Controllability of Linear Systems

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ABSTRACT: This paper establishes sufficient conditions for the controllability and null controllability of linear systems. The aim is to use the variation of constant formula to deduce our controllability grammians, by exploiting the properties of the grammian and the asymptotic stability of the free system, we achieve our results @JASEM

Differential equations, in general, are an important tool for harnessing into single system and analyzing the inter-relationship between different components which otherwise may continue to remain independent on each other. It is known in Sebakhy and Bayoumi (1973) that, in the study of economics, biology and physiological systems as well as electromagnetic systems composed of such subsystems interconnected by hydraulic, mechanical and various other linkages, one encounters phenomena which cannot be readily modeled unless relations involving time delays are admitted. Models for such systems can be controlled. A delayed control on such systems will affect the

\[ \dot{x}(t) = \sum_{i=0}^{n} A_i(t)x(t-h_i) + \sum_{i=0}^{n} B_i(t)u(t-h_i) \] (1)

and gave sufficient conditions for the relative controllability of (1). Our interest, is to integrate the concept of null controllability into a generalized system with delay in state and control given by

\[ \dot{x}(t) = L(t,x_t) + C(t)u(t-h) \] (2)

We shall give sufficient conditions for the null controllability with constraint of (2) when relative controllability is assumed. Our results complement and extend known results.

BASIC NOTATIONS AND PRELIMINARIES

Let \( n \) and \( m \) be positive integers, \( E \) the real line \((-,\infty)\). We denote by \( E^n \) the space of real \( n \)-tuples with the Euclidean norm denoted by \( |\cdot| \). If \( J \) is any interval of \( E \) the usual Lebesgue space of square integrable (equivalent class of) functions from \( J \) to \( E^n \) will be denoted by \( L_2(J,E^n) \). \( L_1([t_0,t_1],E^n) \) denotes the space of integrable functions from \([t_0,t_1]\) to \( E^n \). Let \( h > 0 \) be given, for functions \( x:[t_0-h,t_1] \to E^n \), \( t \in [t_0,t_1] \), we use \( x_t \) to denote the functions on \([-h,0]\) defined by \( x_t(s) = x(t+s) \) for \( s \in [-h,0] \).

Consider the system

\[ \dot{x}(t) = L(t,x_t) + C(t)u(t-h) \] (3)

where

\[ L(t,\phi) = \int_{-h}^{0} d\eta(t,s)\phi \] (4)

satisfied almost everywhere on \([t_0,t_1]\). The integral is in the Lebesgue- Stieltjes sense with respect to \( \eta \). \( L(t,\phi) \) is continuous in \( t \), linear in \( \phi \). \( \eta(t,s) \) is an \( n \times n \) matrix function measurable in \( t \) and of

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bounded variation in $\phi$ on $[-h,0]$ for each $t \in [t_0, t_1]$. $C(t)$ is an $n \times m$ matrix assumed to be bounded and measurable on $[t_0, t_1]$. The control function $u(t) \subset E^n$, is assumed to be measurable and bounded on every finite interval. Throughout the sequel, the controls of interest are, $B = L_2([t_0, t_1], E^n), U \subseteq L_2([t_0, t_1], E^n)$ a closed and bounded subset of $B$ with zero in the interior relative to $B$.

If $X$ and $Y$ are linear spaces is a mapping, we shall use the symbols $D(T), R(T)$ and $N(T)$ to denote the domain, range and null spaces of $T$ respectively.

**Definition 1** - The complete state of system (3) at time $t$ is given by $z(t) = \{x(t), x_t, u_t\}$

**Definition 2** - System (3) is relatively controllable on $[t_0, t_1]$, if for every $z(t_0)$ and every vector $x_1 \in E^n$, there exist a control $u \in B$, such that the corresponding trajectory of system (3) satisfies $x(t) = x_1$. If system (3) is relatively controllable on each interval $[t_0, t_1], t_1 > t_0$, we say, system (3) is relatively controllable.

**Definition 3** - System (3) is said to be null controllable at $t = t_1$, if for any initial state $\{x_0, x_0, u_0\}$ on $[t_0, t_1]$, there exists an admissible control $u(t) \in U$ defined on $[t_0, t_1 - h]$ such that the response is brought to the origin of $E^n$ at $t = t_1$, using the control effort $u(t) = \begin{cases} u(t), & \text{on } [t_0, t_1 - h] \\ 0, & \text{on } [t_1 - h, t_1] \end{cases}$ see Sebakh and Bayouni (1973). It is null controllable with constraints at $t = t_1$, if for any initial state $\{x_0, x_0, u_0\}$ on $[t_0, t_1 - h, t_0]$, there exists an admissible control $u(t) \in U$, defined on $[t_0, t_1 - h]$ such that the response $x(t)$ of system (3) satisfies $x(t) = 0$, using the control effort $u(t) = \begin{cases} u(t) \in U, & \text{on } [t_0, t_1 - h] \\ 0, & \text{on } [t_1 - h, t_1] \end{cases}$

**Definition 4** - The domain $D$ of null controllability of system (3) is the set of all initial points $x_0 \in E^n$ for which the solution $x(t)$ of system (3) with $x(t_0) = x_0$ satisfies $x(t_1) = 0 \in E^n$ at some $t_1$ using $u \in U$.

**Definition 5** - An operator $T : X \rightarrow Y$, where $X$ and $Y$ are linear spaces, is said to be closed if for any sequence $u_n \in D(T)$ such that $u_n \rightarrow u$ and $Tu_n \rightarrow v$, $u$ belongs to $D(T)$ and $Tu = v$.

The variation of parameter of system (3) for $t > t_0 + h$ imply the existence of a unique absolutely continuous solution $x(t)$ of system (3), with initial complete state $z(t_0)$ of the form

$$x(t) = X(t, t_0)x(t_0) + \int_{t_0}^{t} X(t, s)C(s)u(s + h)ds$$  (5)

where $X(t, s)$ satisfies the equation

$$\frac{\partial X(t, s)}{\partial t} = L(t, X_t(t), s), \ t > s$$

almost everywhere

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\[ X(t, s) = \begin{cases} 0, & s - h < t < s \\ 1, & t = s \ (I = \text{Identity matrix}) \end{cases} \]

\( X(t, s) \) is called the fundamental matrix solution of the system

\[ x(t) = I(t, x_0) \quad (6) \]

We now obtain a more convenient form of the solution (3) by expressing (5) as

\[ x(t) = X(t, t_0) \left[ x(t_0) + \int_{t_0}^{t} X(t, s + h) C(s + h) u(s) ds \right] \]

\[ + \int_{t_0}^{t-h} X(t, s + h) C(s + h) u(s) ds \quad (7) \]

The solution \( x(t) \) of system (3) at \( t = t_1 \) (Klamka, 1980) becomes

\[ x(t_1) = X(t_1, t_0) \left[ x(t_0) + \int_{t_0}^{t_1} X(t, s + h) C(s + h) u(s) ds \right] \]

\[ + \int_{t_0}^{t_1-h} X(t, s + h) C(s + h) u(s) ds \quad (8) \]

We now define the \( n \times n \) controllability matrix of system (3), given by

\[ W(t_0, t_1) = \int_{t_0}^{t_1} \begin{bmatrix} X(t, s + h) C(s + h) \end{bmatrix}^T \left[ X(t, s + h) C(s + h) \right] \quad (9) \]

Where \( T \) denotes transpose

**Definition 6** - The reachable set of system (3) at time \( t_1 \) using \( L_2 \) controls is the subset of \( E^n \) given by

\[ P(t_1, t_0) = \left\{ \int_{t_0}^{t_1} X(t, s + h) C(s + h) u(s) ds : u \in L_2 \right\} \]

and the constraint reachable set of system (3) is given by

\[ R(t_1, t_0) = \left\{ \int_{t_0}^{t_1} X(t, s + h) C(s + h) u(s) ds : u \in U \right\} \]

The constraint reachable set with unspecified end time is given by

\[ R(t_1) = \bigcup_{t_0} R(t_1, t_0) \]

**Definition 7** - System (3) is said to be proper in \( E^n \) on \([t_0, t_1]\) if

\[ c^T \left[ X(t_1, s + h) C(s + h) \right] = 0 \]

almost everywhere \( t \in [t_0, t_1] \), \( c \in E^n \) implies \( c = 0 \). If system (3) is proper on \([t_0, t_0 + \delta]\) for each \( \delta > 0 \), we say system (3) is proper at time \( t_0 \). If system (3) is proper on each interval \([t_0, t_1]\), \( t_1 > t_0 > 0 \), we say the system is proper in \( E^n \).

**NULL CONTROLLABILITY WITH CONSTRAINED CONTROLS**

**Lemma 1** - The following are equivalent

(i) \( W(t_0, t_1) \) is non-singular for each \( t \)

(ii) System (3) is proper in \( E^n \) for each interval \([t_0, t_1]\)

(iii) System (3) is relatively controllable on each interval \([t_0, t_1]\)

**Proof** - \( W(t_0, t_1) \) is non-singular implies \( W(t_0, t_1) \) is positive definite, that is

\[ c^T \left[ X(t_1, s + h) C(s + h) \right] = 0 \]

almost everywhere on \([t_0, t_1]\) implies \( c = 0 \). Therefore, (i) implies (ii).

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To prove the equivalence of (ii) and (iii) let \( c \in E^n \), and assume that
\[
c^T \left[ X(t_1, s + h) C(s + h) \right] = 0
\]
almost everywhere \( t \in [t_0, t_1] \) for each \( t_1 \), then
\[
\int_{t_0}^{t_1} c^T X(t_1, s + h) C(s + h) u(s) ds = c^T \int_{t_0}^{t_1} X(t_1, s + h) C(s + h) u(s) ds = 0
\]
for \( u \in L_2 \). It follows from this that \( c \) is orthogonal to the set \( P(t_1, t_0) \). We assume that system (3) is relatively controllable, then \( P(t_1, t_0) = E^n \), so that \( c = 0 \), meaning that (iii) implies (ii). Conversely, assume that system (3) is not controllable, so that \( P(t_1, t_0) \subseteq E^n \), for \( t_1 > t_0 \). Then, there exists \( c \neq 0 \), \( c \in E^n \), such that \( c^T P(t_1, t_0) = 0 \).

It follows that for all admissible control \( u \in L_2 \)
\[
0 = c^T \int_{t_0}^{t_1} [X(t_1, s + h) C(s + h)] u(s) ds = c^T \int_{t_0}^{t_1} c^T \left[ X(t_1, s + h) C(s + h) \right] u(s) ds
\]
Hence \( c^T \left[ X(t_1, s + h) C(s + h) \right] \equiv 0 \), almost everywhere \( t \in [t_0, t_1] \), \( c \neq 0 \).

To show that (i) implies (ii)
We define the operator \( K : L_2([t_0, t_1], E^n) \rightarrow E^n \) by \( K(u) = \int_{t_0}^{t_1} X(t_1, s + h) C(s + h) u(s) ds \)

If we assume that \( W(t_0, t_1) \) is singular. Then the symmetric operator \( KK^T = W(t_0, t_1) \) is positive definite. But this holds, if and only if \( \text{rank } W(t_0, t_1) = n \).

**Theorem 1 -** System (3) is proper on \([t_0, t_1]\) if and only if \( 0 \in \text{int } R(t_1, t_0) \).

**Proof -** If \( R(t_1, t_0) \) is a closed and convex subset of \( E^n \) (Klamka, 1976), then a point \( y_1 \) on the boundary of \( R(t_1, t_0) \) implies that, there is a support plane \( \Pi \) of \( R(t_1, t_0) \) through \( y_1 \), that is \( c^T (y - y_1) \leq 0 \) for each \( y \in R(t_1, t_0) \) where \( c \neq 0 \) is an outward normal to \( \Pi \). If \( u_1 \) is the control corresponding to \( y_1 \), we have
\[
c^T \int_{t_0}^{t_1} X(t_1, s + h) C(s + h) u(s) ds \leq c^T \int_{t_0}^{t_1} X(t_1, s + h) C(s + h) u_1(s) ds
\]
for each \( u \in U \). Since \( U \) is a unit sphere, this last inequality holds for each \( u \in U \), if and only if
\[
c^T \int_{t_0}^{t_1} X(t_1, s + h) C(s + h) u(s) ds \leq \int_{t_0}^{t_1} c^T X(t_1, s + h) C(s + h) u_1(s) ds
\]
and \( u_1(t) = \text{sgn } c^T \left( X(t_1, s + h) C(s + h) \right) \) as \( y_1 \) is on the boundary. Since we always have \( 0 \in R(t_1, t_0) \). If \( 0 \) were not in the interior of \( R(t_1, t_0) \) then \( 0 \) is on the boundary, hence, from the foregoing, this implies \( 0 = \int_{t_0}^{t_1} c^T X(t_1, s + h) C(s + h) ds \) so that
\[
c^T \left[ X(t_1, s + h) C(s + h) \right] = 0 \text{ almost everywhere } t \in [t_0, t_1].
\]
This by our definition implies that the system is not proper since \( c^T \neq 0 \). This completes the proof.

**Theorem 2 -** System (3) is relatively controllable if and only if \( 0 \in \text{int } R(t_1, t_0) \) for each \( t_1 > t_0 \).

**Proof -** By lemma 1, system (3) is relatively controllable on \([t_0, t_1], t_1 > t_0\) if and only if, it is proper on \([t_0, t_1]\). Therefore, by theorem 1, system (3) is relatively controllable if and only if \( 0 \in \text{int } R(t_1, t_0) \).

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Theorem 3 - If system (3) is relatively controllable on \([t_0, t_1]\) for each \(t_1 > t_0\), then the domain null controllability of system (3) contains zero in its interior.

Proof - Assume that system (3) is relatively controllable on \([t_0, t_1]\), \(t_1 > t_0\) then by theorem 2, \(0 \in \text{int} \ R(t_1, t_0)\), for each \(t_1 > t_0\). Since \(x = 0\) is a solution of system (3) with \(u = 0\), we have \(0 \in D\). Hence, \(0 \notin \text{int} \ D\), then there exists a sequence \(x_m \subseteq E^n\) such that \(x_m \to 0\) as \(m \to \infty\) and no \(x_m\) is in \(D\), that is \(x_m \neq 0\). From (5), we have

\[
0 \neq x(t_1) = X(t_1, t_0) x_m(t_0) + \int_{t_0}^{t_1} X(t_1, s) C(s) u(s + h) ds
\]

for any \(t_1 > t_0\) and any \(u \in U\). Hence, for \(u = 0\), \(z_m = x(t_1) = X(t_1, t_0) x_m(t_0)\), is not in \(R(t_1, t_0)\) for any \(t_1 > t_0\). Therefore the sequence \(z_m \subseteq E^n\) is such that \(z_m \notin R(t_1, t_0)\), \(z_m \neq 0\), but \(z_m \to 0\) as \(m \to \infty\). Therefore, \(0 \notin \text{int} \ R(t_1, t_0)\) a contradiction. Hence, \(0 \in \text{int} \ D\).

Theorem 4 - Assume

(i) System (3) is relatively controllable on \([t_0, t_1]\) for each \(t_1 > t_0\)

(ii) The zero solution of system (6) is uniformly asymptotically stable, so that the solution of (6) satisfies

\[
\|x(t)\| \leq k \|x_0\| e^{-\alpha(t-t_0)}, \quad \alpha > 0, \quad k > 0
\]

are constants. Then system (3) is null controllable with constraints.

Proof - By (i) and theorem 3, the domain \(D\) of null controllability of system (3) contains zero in its interior. Therefore there exists a ball \(B_1\) such that \(0 \in B_1 \subseteq D\). By (ii), every solution of system (3) (with \(u = 0\)) satisfies \(x(t) \to 0\) as \(t \to \infty\). Hence at some \(t_1 < \infty\) (with \(u = 0\)) \(x(t_1) \in B_1 \subseteq D\), for \(t_1 > t_0\). Therefore, using \(t_1\) and \(x_1 = x(t_1)\) at \(u = 0\) as initial data there exists a \(u \in U\) and some \(t_2 > t_1\) such that the solution \(x(t)\) of system (3) satisfies \(x(t_2) = 0\). Thus proving the theorem.

Lemma 2

System (3) is relatively controllable on \([t_0, t_1]\) if and only if \(\text{rank} \ W(t_0, t_1) = n\).

The proof is quite standard and simple, for similar proof see Klamka (1976).

Theorem 5 - System (3) is null controllable with constraints if

(i) \(\text{Rank} \ W(t_0, t_1) = n\), for \(t_1 > t_0\)

(ii) The zero solution of (6) is uniformly asymptotically stable

Proof - Immediately from theorem 4 and lemma 1

Theorem 6 - The system (6) is null controllable with constraints if

(i) \(\text{Rank} \ W(t_0, t_1) = n\), for \(t_1 > t_0\)

(ii) If all the characteristics roots have negative real part

Proof: Immediately.

Conclusion: In this paper we have developed and proved sufficient conditions for the controllability and null controllability of linear systems with delay in the state and control. That is, if the uncontrolled system is asymptotically stable and the controlled system is relatively controllable then, the system is null controllable with constraints.
REFERENCES


