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Stationary Iterative Methods and Their Application to Some Problems Arising in Physics

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ABSTRACT: The objective of this paper is to evaluate the Stationary Iterative Methods and Their Application to Some Problems Arising in Physics. The solution of such problems depends on the diagonal dominance criterion for the convergence of such stationary iterative methods. The solution of such linear algebraic equations is discussed in this work. The iteration matrices of the Jacobi and Gauss-Seidel methods were derived via the splitting of the coefficient matrices. Their convergence was examined with the diagonal dominance criterion and the schemes were found to converge. The weak diagonal dominance criterion was thereafter derived. The iteration matrices were then used to obtain their respective spectral radii. The rates of convergence of the schemes were discussed in relation to spectral radii of the iteration matrices and structure of the coefficient matrices. The coefficient matrices of the examples used were of three categories: the strictly diagonally dominant, the weakly diagonally dominant and the non-diagonally dominant. The spectral radii of the iteration matrices for the strictly diagonally dominant systems were less than unity and the iterations for Jacobi, Gauss-Seidel and Successive overrelaxation methods converged. For the weakly diagonally dominant systems, only the Jacobi iteration matrix had a spectral radius higher than unity and the scheme did not converge

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Weak diagonal dominance in linear systems of algebraic equations exists when the absolute value of a diagonal entry in a square matrix is greater than or equal to the sum of the absolute values of the other elements in its row or column. This often arises in physics problems involving discretization by means of numerical methods such as finite difference or finite element methods yielding a system of linear equations where the coefficient matrix might be weakly diagonally dominant. When modeled vibration of structures like bridges are discretized, the resulting linear system has a coefficient matrix sometimes weak in its diagonal dominance, especially if the mesh is relatively coarse ((Barret, *et al*. 2002) Similarly, in fluid simulations, the equations governing fluid flow are often discretized, leading to a system of equations where the coefficient matrix might be weakly diagonally dominant, particularly when dealing with viscous or turbulent flows. (Elam, *et al*, 1995) Again, modeling heat transfer in a solid or fluid can also involve discretized equations that result in a system of linear equations with a weakly diagonally dominant matrix, especially when dealing with problems involving conduction or

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convection (Karaa, S and Zhang, J.,2002). Hitherto, direct methods have been the solution method to this class of problems. Due to the non-convergence nature of Jacobi iterative method for weakly diagonally dominant systems whose strongly diagonal dominance criterion is also used for all stationary iterative methods, the method has not been seen applied to these physics problems. Secondly, this class of physics problems usually result in a large scale sparse linear system which are better solved by iterative (Sa'ad, 1999).

Hence, the objective of this paper is to evaluate the Stationary Iterative Methods and Their Application to Some Problems Arising in Physics.

MATERIALS AND METHODS

Systems of algebraic linear equations often result from discretization of partial differential equations (PDEs) (Sa'ad, 1999) from practical problems. This class of equations are generally simplified in matrix form (Kelly, 1995). The feasibility of any method of solving a system of linear algebraic equations usually depends on the number of the equations and nature of the coefficient matrix. When the dimension of the coefficient matrix is large or sparse, direct methods of solution becomes inefficient (Strong, 2004), therefore the use of iterative methods as solution techniques becomes inevitable. Thus, the need for iterative methods (Meyer, 2000, Kharab, 2002).

Linear algebraic systems of equations are usually of the form

$$Ax = b \qquad (1)$$

where A is a square matrix, x and b are unknown and righ- hand vectors respectively (Olver, PJ, *et al* 2018).

An iterative method is a method in which the exact or the approximate solution is obtained by means of repeated improvement of a given initial approximate solution $x^{(0)}$. Precisely, an iterative method is a way of solving an equation or a system of equations in which an initial approximation $x^{(0)}$ is used to obtain a better approximation $x^{(1)}$, which in turn is used to find a still better approximations $x^{(2)}$, $x^{(3)}$, $\cdots$. This procedure is continued until the exact solution or near exact solution is arrived at (Barret, *et al*. 2002). There are essentially two broad categories of stationary iterative methods for solving equation 1; namely the Jacobi and the Gauss-Seidel (Meyer, 2000). The third method, Successive Over relaxation (SOR) method is simply a generalization of the latter when $\omega = 1$. As such the SOR is inherently similar to the Gauss-Seidel iteration. The Jacobi iteration operates on simultaneous corrections because for each $x^{(k+1)}$, all the $x_j^{(k)}$ are computed for all j before substituting for the computation before they are substituted for $j = 2$ while the Gauss-Seidel method uses the next round of computation, that is, the Gauss-Seidel method adopts the successive correction method. That is, each $x_j^{(k+1)}$ component is substituted for as soon as it computed (Yousef S. and Henk A. V., 2000). These schemes are defined as in equations 2, 3 and 4:

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(-\sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)} + b_i\right) \qquad (2)$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(-\sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)} + b_i\right) \qquad (3)$$

$$x_i^{(k+1)} = (1-\omega)x_j^{(k+1)} + \frac{\omega}{a_{ii}}\left(-\sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)} + b_i\right) \qquad (4)$$

By definition the efficiency of equation 3 is expected to be higher than equation 2 in terms of the type and speed of convergence. The speed of convergence would be related to some properties of the method under consideration (Strong, 2004).

Iterative schemes for finding solution to (1) generally entails splitting the coefficient matrix A into matrices M and N in such a way that $A = M + N$ where M is non-singular (Strong, 2004) thus giving $(M + N) = b$ and from equation 1.

$$Mx = -Nx + b$$

Pre-multiplying by M^{-1} gives

$$M^{-1}Mx = -M^{-1}Nx + M^{-1}b \qquad (5)$$

Hence

$$x = -M^{-1}Nx + M^{-1}b \qquad (6)$$

Which simplifies to

$$x = Px + q \qquad (7)$$

where $P = -M^{-1}N$ and $q = M^{-1}b$ and in iterative form is

$$x^{(k+1)} = Px^{(k)} + q \qquad (8)$$

Let $e^{(k)} = x^{(k)} - x$ and $e^{(k+1)} = x^{(k+1)} - x$ be the errors in the kth and $(k+1)th$ iterations respectively then

$$e^{(k+1)} = \left(Px^{(k)} + q\right) - (Px + q)$$

$$e^{(k+1)} = P\left(x^{(k)} - x\right) = Pe^{(k)}$$

Hence

$$\left\|e^{(k+1)}\right\| = \left\|Pe^{(k)}\right\|, \qquad k = 0,1,2,\cdots \quad (9)$$

Now,

$$\left\|e^{(k+1)}\right\| \leq \|P\| \cdot \left\|e^{(k)}\right\| \qquad (10)$$

This implies that

$$\left\|e^{(k+1)}\right\| \leq \left\|e^{(k)}\right\| \qquad (11)$$

therefore the iterative method is convergent.

If however, $\|P\|$ is small then $\left\|e^{(k+1)}\right\| \leq \left\|e^{(k)}\right\|$ indicated in equation 11. If eventually $\left\|e^{(k+1)}\right\| = \left\|e^{(k)}\right\|$ then iteration is terminated as there will be no further differences between successive iterates.

Hence

$$\left\|e^{(k+1)}\right\| \leq \left\|e^{(k)}\right\|$$

The Jacobi method (2) is of the form

$$x^{(k+1)} = -D^{-1}(L+D)x^{(k)} + D^{-1}b \qquad (12)$$

where $-D^{-1}(L+D)$ is the iteration matrix and $-D^{-1}(L+D) = \frac{a_{ij}}{a_{ii}}$

The Gauss-Seidel method on the other hand is defined as

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(-\sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)} + b_i\right)$$

and can be expressed as

$$x^{(k+1)} = -D^{-1}\left(Lx^{(k+1)} + Ux^{(k)} + b\right)$$

Since $DD^{-1} = I$, then we have

$$Dx^{(k+1)} = -Lx^{(k+1)} - Ux^{(k)} + b$$

Therefore

$$(L+D)x^{(k+1)} = -Ux^{(k)} + b$$

Since$(L+D)$ is non-singular the Gauss-Seidel iteration formula is

$$x^{(k+1)} = -(L+D)^{-1}Ux^{(k)} + (L+D)^{-1}b \qquad (13)$$

By definition, a square matrix $A = \left(a_{ij}\right), i,j = 1,2,3,\cdots,n$ is called strictly diagonally dominant if for every row (or column), the sum of the absolute values of the off-diagonal elements in a row (or column) is less than the absolute value of the diagonal element.. i.e.,

$$\sum_{\substack{i=1 \\ i\neq j}}^{n} \left|a_{ij}\right| < |a_{ii}|, \qquad i = 1,2,3,\cdots,n \qquad (14)$$

This is the strict diagonal dominance criterion.
On the other hand, the weak diagonal dominance criterion

$$\sum_{\substack{i=1 \\ i\neq j}}^{n} \left|a_{ij}\right| \leq |a_{ii}|, \qquad i = 1,2,3,\cdots,n \qquad (15)$$

can be viewed as an extension of equation 14 which implies that the sum of the absolute values in each row (or column) of the coefficient matrix does not exceed the absolute value of the diagonal entry.

Some of the properties of iteration matrices are prescribed in terms of their eigenvalues and for simplicity we consider a linear system of order 2.

The Jacobi iteration matrix equation 15 has all its diagonal entries as zero. Now, consider a (2×2) linear system with the iteration matrix

$$P_J = \begin{pmatrix} 0 & -\frac{a_{12}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 \end{pmatrix}$$

then the determinantal equation $\left|P_J - \lambda \mathrm{I}\right| = 0$ yields

$$\left|P_J\right| = \begin{vmatrix} -\lambda & -\frac{a_{12}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & -\lambda \end{vmatrix} = 0$$

$\lambda_J^2 - \frac{a_{12}}{a_{11}}\frac{a_{21}}{a_{22}} = 0$, $\lambda_J^2 = \frac{a_{12}}{a_{11}}\frac{a_{21}}{a_{22}}$and $\lambda_J = \sqrt{\frac{a_{12}}{a_{11}}\frac{a_{21}}{a_{22}}}$

Since
$\sum_{j=1}^{n}\frac{|a_{ij}|}{|a_{ii}|} < 1$ then $\frac{a_{12}}{a_{11}}\frac{a_{21}}{a_{22}} < 1$, that is $\lambda_J < 1$. If A is diagonally dominant then equation 2 is bound to converge and the rate of convergence is determined by λ_J (Bagnara, 1995). Similarly, the Gauss-Seidel iteration matrix for a (2×2) linear system is

$$P_{GS} = \begin{pmatrix} 0 & -\dfrac{a_{12}}{a_{11}} \\ 0 & -\dfrac{a_{12}}{a_{11}}\dfrac{a_{21}}{a_{22}} \end{pmatrix}$$

The determinantal equation $|P_{GS} - \lambda I| = 0$ yields

$$|P_{GS}| = \begin{vmatrix} -\lambda & -\dfrac{a_{12}}{a_{11}} \\ 0 & \dfrac{a_{12}}{a_{11}}\dfrac{a_{21}}{a_{22}} - \lambda \end{vmatrix} = 0$$

$\lambda_{GS}\left(\frac{a_{12}}{a_{11}}\frac{a_{21}}{a_{22}} - \lambda_{GS}\right) = 0$, $\lambda_{GS} = \frac{a_{12}}{a_{11}}\frac{a_{21}}{a_{22}}$ and $\lambda_{GS} < 1$
This implies that if each iteration in the Jacobi scheme causes the error to be halved, for example, then the Gauss-Seidel scheme causes the error to be quartered thus suggesting a faster convergence.

Since the eigenvalues of the iteration matrices of both (2) and (3) are less than 1 then the spectral radii (maximum eigenvalue) is less than 1 and the iterative schemes are guaranteed to converge (Barret *et al*, 1993, Yousef S. and Henk A. V., 2000, Bagnara, 1995).

RESULTS AND DISCUSSION

Three of the examples given (Example 1-3) were selected from diagonally dominant linear systems, two from weak diagonally dominant (Examples 4 and 5) and one non-diagonally dominant linear system (Example 6) in order to determine whether convergence is achievable with equations 14 and 15. The MATLAB software was used in the computation. The coefficient matrix A, the right-hand side vector b, the tolerance $\varepsilon = 10^{-3}$ as well as the maximum iterations n were supplied as required by the MATLAB m-files. For the SOR method, $\omega = \frac{2}{1=\sqrt{1-\rho^2}}$ was used as the relaxation (Kharab, 2002) parameter where ρ is the spectral radius of both the Jacobi and the Gauss-Seidel iteration matrix. Table 1 above is the table of examples considered in this work. The coefficient matrices of the examples are of three categories; the strictly diagonally dominant, the weakly diagonally dominant and the non- diagonally dominant. Table 2 is the solutions to the numerical computations of problems in Table 1. The diagonal dominance criterion for the convergence of stationary iterative methods was applied to each of the examples (1-6). While Example 1-3satisfied the strict diagonal dominance criterion, Examples 4 and 5 satisfied the criterion weakly.

Table 1: Table of examples

S/N	Coefficient Matrix A	Vector b	Properties of A
1	$\begin{pmatrix} 7 & 1 & 2 \\ 1 & 8 & 1 \\ 1 & 3 & 6 \end{pmatrix}$	$\begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}$	Strictly diagonally dominant
2	$\begin{pmatrix} 9 & 1 & -1 & 2 & 4 \\ 1 & 11 & 2 & 1 & -2 \\ 2 & 3 & 10 & 0 & 3 \\ 1 & 1 & -1 & 8 & 4 \\ -4 & 3 & 2 & 1 & 12 \end{pmatrix}$	$\begin{pmatrix} 42 \\ 6 \\ 9 \\ 44 \\ 44 \end{pmatrix}$	Strictly diagonally dominant
3	$\begin{pmatrix} 8 & 2 & -1 & 1 & 1 & 2 \\ 2 & 8 & 1 & 1 & 0 & 1 \\ 1 & -2 & 10 & 2 & 1 & -1 \\ 1 & 2 & 1 & 16 & 4 & 1 \\ 1 & 0 & 1 & 3 & 12 & 3 \\ 3 & 2 & -2 & 1 & 2 & 11 \end{pmatrix}$	$\begin{pmatrix} 31 \\ 17 \\ 19 \\ 22 \\ 8 \\ 49 \end{pmatrix}$	Strictly diagonally dominant
4	$\begin{pmatrix} 7 & 3 & 4 \\ 2 & 5 & 3 \\ 1 & 5 & 5 \end{pmatrix}$	$\begin{pmatrix} 14 \\ 10 \\ 11 \end{pmatrix}$	Non-strictly diagonally dominant
5	$\begin{pmatrix} 4 & 1 & 2 & -1 \\ 1 & 6 & 2 & 3 \\ 2 & 3 & 7 & 2 \\ 2 & 1 & 6 & 3 \end{pmatrix}$	$\begin{pmatrix} 7 \\ 12 \\ 14 \\ 12 \end{pmatrix}$	Non-strictly diagonally dominant
6	$\begin{pmatrix} 3 & 1 & 1 & 2 \\ -2 & 5 & 2 & 3 \\ -1 & -4 & 3 & 1 \\ 1 & 2 & 5 & 3 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 20 \\ 6 \\ 20 \end{pmatrix}$	Non-diagonally dominant

Table 2: Table of solutions to examples

S/N	Jacobi		Gauss-Seidel		SOR		ω		Exact	
	Solution Vector	*Number of Iteration*	*Spectral Radius*	*Solution Vector*	*Number of Iteration*	*Spectral Radius*	*Solution Vector*	*Number of Iteration*		
1	$\begin{pmatrix}0.9998\\0.9999\\0.9998\end{pmatrix}$	10	0.4212	$\begin{pmatrix}1.0000\\1.0001\\1.0000\end{pmatrix}$	4	0.1642	$\begin{pmatrix}1.0000\\1.0000\\1.0000\end{pmatrix}$	4	1.0488	$\begin{pmatrix}1\\1\\1\end{pmatrix}$
2	$\begin{pmatrix}2.0011\\1.0003\\-0.9993\\3.0009\\3.9998\end{pmatrix}$	9	0.3944	$\begin{pmatrix}2.0002\\1.0000\\-0.9999\\3.0001\\4.0001\end{pmatrix}$	7	0.1901	$\begin{pmatrix}1.9994\\1.0000\\-1.0001\\2.9997\\3.9998\end{pmatrix}$	8	1.0422	$\begin{pmatrix}2\\1\\-1\\3\\4\end{pmatrix}$
3	$\begin{pmatrix}2.9992\\0.4994\\1.9999\\0.9994\\-1.0007\\3.9992\end{pmatrix}$	16	0.6648	$\begin{pmatrix}3.0005\\0.4996\\2.0000\\1.0003\\-1.0003\\4.0000\end{pmatrix}$	6	0.4498	$\begin{pmatrix}3.0001\\0.4997\\1.9998\\1.0002\\-1.0001\\3.9999\end{pmatrix}$	6	1.1194	$\begin{pmatrix}3\\0.5\\2\\1\\-1\\4\end{pmatrix}$
4	No convergence	Not Available	25	$\begin{pmatrix}1.0004\\1.0012\\0.9988\end{pmatrix}$	11	0.5663	No convergence	Not Available	1.7435 - 0.6688i	$\begin{pmatrix}1\\1\\1\end{pmatrix}$
5	No convergence	Not Available	25	$\begin{pmatrix}0.9981\\1.0014\\1.0008\\0.9981\end{pmatrix}$	12	0.1891	No convergence	Not Available	1.4058 - 0.9140i	$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$
6	No convergence	Not Available	9	No convergence	Not Available	2.6222	No convergence	Not Available	1.0835 - 0.9965i	$\begin{pmatrix}-1\\1\\2\\3\end{pmatrix}$

For these two examples the Jacobi and SOR schemes did not converge only the Gauss-Seidel method converged with high number of iterations. Example 6 is non-diagonally dominant and as such none of the three schemes converged.

The spectral radii of the iteration matrices considered were found to be related; that is the square of spectral radius of Jacobi iteration matrix equals that of the Gauss-Seidel iteration matrix (Strong, 2004) which explains the difference in their speed of convergence, i.e., the Gauss-Seidel and SOR methods converged faster than the Jacobi method. Theoretically, both the Gauss-Seidel and SOR methods were expected to converge twice as fast as the Jacobi scheme (Meyer, 2000) but from the results, it is observable that while in Example 2 the difference is not much, in Examples 1 and 3 the speed of convergence of Gauss-Seidel and SOR was about thrice that of the Jacobi scheme.

Conclusion: The coefficient matrices of examples 1 to 3 were strictly diagonally dominant with the spectral radii of their iteration matrices being less than 1 and all the three methods converged. The weak diagonally dominant linear systems of examples 4 and 5 converged for only the Gauss-Seidel method as the spectral radii of the iteration matrices were less than unity. The Jacobi method did not converge spectral radii of the iteration matrices were greater than unity while the SOR method did not converge because the relaxation parameter $0 < \omega < 2$ was not satisfied. The spectral radius of the iteration matrix of example 6 was bigger than unity and none of the schemes converged. The spectral radii of the iteration matrices also influence convergence. Also, weakly diagonally dominance criterion which hitherto was not considered made the Gauss-Seidel scheme to stand out a method that can find application in solving problems in physics.

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REFERENCES

Bagnara. RO (1995): Unified Proof for the Convergence of Jacobi and Gauss-Seidel Methods. *SIAM*. 37(1): 93-97

Barrett, R; Berry, M; Chan, TF; Demmel, J; Donato, JM; Dongarra, J; Eijkhout, V; Pozo, R; Romine, C; Vorst, HV (1993) Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods. http://www.siam.org/books. pp. 7-11

Elman, HC, Sylvester, DJ, Wathen, AJ (1995) Iterative Methods for Problems in Computational Fluid Dynamics. Oxford University Computing Laboratory Numerical Analysis Group Report no. 96/19, pp. 1-57

Karaa, S and Zhang, J. (2003) Analysis of Stationary Iterative Methods for the Discrete Convection–Diffusion Equation with a 9-point Compact Scheme. J. of Comp. and Appl. Math. No. 154, pp. 447-476

Kelly, (1999) CT Iterative Methods for Linear and Nonlinear Equations. SIAM. Philadelphia., pp. 1-7

Kharab, A; Guenther, RB (2019) An Introduction to Numerical Methods: A MATLAB Approach. CRC London. pp.132-141

Meyer, CD (2000.) Matrix Analysis and Applied Linear Algebra. SIAM. Philadelphia pp. 620-622

Nandakumar, K. (1999) Numerical Solutions of Engineering Problems Lecture Notes Department of Chemical and Materials Engineering University of Alberta

Olver, PJ; Shakiban, C. (2018) Applied Linear Algebra. Springer, New York, pp 1-15

Sa'ad, Y. (1999) Iterative Methods for Sparse Linear System. New Science Foundation. London., pp. 45-48

Strong DM (2004) Iterative Methods for Solving Linear Systems. *J. Math Ass. Amer*, pp. 1-7

Yousef S; Henk AV (2000) Iterative Solution of Linear Systems in the *20th Cent. J. Comp. and App. Math.* pp.1-33