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Solution of One-Dimensional Contaminant Flow Problem Incorporating the Zero Order Source Parameter by Method of Eigen-Functions Expansion

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ABSTRACT: A semi – analytical study of a time dependent one – dimensional advection – dispersion equation (ADE) with Neumann homogenous boundary conditions for studying contaminants flow in a homogenous porous media is presented. The governing equation which is a partial differential equation incorporates the advection, hydrodynamic dispersion, first order decay and a zero order source effects in the model formulation. The velocity of the flow was considered exponential in nature. The solution was obtained using Eigen function expansion technique after a suitable transformation. The results which investigate the effect change in the parameters on the concentration were discussed and represented graphically. The study revealed that as the zero order source coefficient increases, the contaminant concentration decreases with time.

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Increase in the needs for water supply all over the world due to fast growing population and rapid industrialisation has resulted in a growing dependency on underground water sources. Hence, groundwater is generally accepted and has become a major source of water supply. Large number of complicated and often interactive physical, chemical and microbiological processes govern the solute migration and flow in the subsurface environment. Groundwater can become contaminated from natural sources or various types of human activities like residential, commercial, industrial and agricultural activities especially in areas of high population density where land use is intensive. Through these activities, chemical or contaminant may be released to the environment intentionally or accidentally polluting groundwater and causing contamination, thereby affecting groundwater quality. Soils that are porous and permeable tend to transmit water and certain types of contaminants easily to the aquifer and any other water supply near a source of contamination may also be contaminated because of the interrelation between groundwater and surface waters. These consequences of groundwater contamination include; poor drinking water quality, loss of water supply, degraded surface water systems, high clean-up costs, high costs for alternative water supplies and potential health problems. In order to account for these problems, many attempts such as developing analytical transport models in porous media which considered the simultaneous effect of hydrodynamic dispersion, molecular diffusion, sorption and first order decay has been made in the past (Shackelford and Rowe, 1998; Genucheten and Alves, 1982; Singh *et al.* ,2009; Guerrero and Skaggs, 2010). Solute transport in porous media is commonly described by advection – dispersion equation (ADE) (Jaiswal *et al.*, (2009). Many solutions comprising various initial and boundary conditions through one, two and three dimensional transport problems in porous formation can be obtained analytically (Warrick *et al.*, 1972; Guerrero *et al.*, 2013; Singh *et al.*, 2009; Jaiswal *et al.*, 2009; Yadav *et al.*, 2010; Kumar *et al.*, 2010; Sanskrityayn *et al.*, 2017, Yang *et al.*, 2016) and semi - analytically or numerically using various methods.

These methods are increasingly being applied to solve these problems today. Previously, the importance of contaminants transport model and restoration process has been widely investigated by numerical techniques (Dehgan, 2004; Peter, *et al.*, 2010; Das *et al.*, 2017 and Gharib *et al.*, 2017). In this paper, the solution of a time dependent one – dimensional contaminant flow problem which incorporates the zero-order source parameter under Neumann boundary conditions obtained by Eigen-Functions expansion method is presented.

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MATERIALS AND METHODS

Mathematical Model and Derivations: Considering the transport of a contaminant through a homogenous finite aquifer of length l under transient – state flow, it is assumed that initially (at time $t = 0$), the flow is not clean. Let c_i be the initial contaminant concentration in the aquifer describing the distribution of the concentration at all points of the flow domain. The time- dependent source concentration is assumed at the origin $x = 0$. At the end of boundary $x = l$, it was assumed, there was no solute flux. Let $c(x,t)$ be the contaminant concentration in the aquifer at time t, $u(t)$, the velocity of the medium transporting the contaminants, and $D(t)$ the solute dispersion parameter. Following the work of Kumar *et al.* (2010), Jimoh *et al*. (2017) and Olayiwola *et al.* (2013), the zero-order source parameter $\mu(t)$ was incorporated and obtained the model equation under consideration:

$$\frac{\partial c}{\partial t}+\frac{\partial s}{\partial t}=D(t)\frac{\partial^2 c}{\partial x^2}-u(t)\frac{\partial c}{\partial x}-\gamma(t)c+\mu(t) \quad (1)$$

with initial and boundary conditions:

$$\left.\begin{aligned} &c(0,t)=c_0\left(1+e^{-qt}\right)\\ &\frac{\partial c(l,t)}{\partial x}=0\\ &c(x,0)=c_i \end{aligned}\right\} \quad (2)$$

*Method of Solution***:** Assuming the parameters are functions of time t, i.e.,

$D=D_0 f(t),\ \ u=u_0 f(t),\ \ \gamma=\gamma_0 f(t),$ and $\mu_0=\mu_0 f(t),$ (3)

Equation (1) becomes:

$$(1+k_d)\frac{\partial c}{\partial t}=D_0 f(t)\frac{\partial^2 c}{\partial x^2}-u_0 f(t)\frac{\partial c}{\partial x}-\gamma_o f(t)c+\mu_o f(t) \quad (4)$$

Equation (4) simplifies to:

$$\frac{\partial c}{\partial t}=D_0 f(t)\frac{\partial^2 c}{\partial x^2}-u_0 f(t)\frac{\partial c}{\partial x}-\gamma_o f(t)c+\mu_o f(t) \quad (5)$$

where $k_d=\frac{\partial s}{\partial c}$ is the contaminant distribution coefficient and $R=1+k_d$ is the retardation factor. Assuming $k_d=0$ then $R=1$ and the following equation results:

$$\frac{1}{f(t)}\frac{\partial c}{\partial t}=D_0\frac{\partial^2 c}{\partial x^2}-u_0\frac{\partial c}{\partial x}-\gamma_o c+\mu_o \quad (6)$$

A new time variable is introduced as follows:

$$\tau=\int_0^t f(s)ds \quad (7)$$

Where $f(t)=e^{-qt}$

$$\left.\begin{aligned} &\frac{d\tau}{dt}=f(t)\\ &\frac{dt}{d\tau}=\frac{1}{f(t)} \end{aligned}\right\} \quad (8)$$

By substituting equation (8) in equation (6), the contaminant flow model (1) with the corresponding initial and boundary condition is obtained below:

$$\left.\begin{aligned} &\frac{\partial c}{\partial \tau}=D_0\frac{\partial^2 c}{\partial x^2}-u_0\frac{\partial c}{\partial x}-\gamma_o c+\mu_o\\ &c(x,0)=c_0\left(2-q\tau\right),\ \tau>0\\ &\frac{\partial c(l,t)}{\partial x}=0,\ \tau\ge 0\\ &c(x,0)=c_i \end{aligned}\right\} \quad (9)$$

Non-dimensionalisation: Equation (9) is non-dimensionalized using the following dimensionless variables:

$$x'=\frac{x}{l},c'=\frac{c}{c_0},\tau'=\frac{D_0\tau}{l^2},u_0'=\frac{u_0 l}{D_0},q=\frac{ql^2}{D_0},\gamma_0'=\frac{\gamma_0 l^2}{D_0},\mu_0'=\frac{\mu_0\tau}{c_0} \quad (10)$$

After non-dimensionalisation, the following initial and boundary value problem was obtained. The primes were dropped for convenience.

$$\left.\begin{aligned} &\frac{\partial c}{\partial \tau}=\frac{\partial^2 c}{\partial x^2}-u_0\frac{\partial c}{\partial x}-\gamma_0 c+\mu_0\\ &c(0,\tau)=\frac{c_0}{c_0}\left(2-q\tau\right),\ \ \tau>0\\ &\frac{\partial c(l,\tau)}{\partial x}=0,\quad \tau\ge 0\\ &c(x,0)=\frac{c_i}{c_0},\ \ 0\le x\le 1 \end{aligned}\right\} \quad (11)$$

Though the flow of contaminants is a seasonal event in a tropical region, the velocity of the flow shall be considered exponential in nature; that is,

$$\mathrm{f}(t) = e^{-qt} \tag{12}$$

where q is the flow resistance coefficient.

For the velocity $\mathrm{f}(t)$, the dimensionless time variable may be written as;

$$\tau = \frac{D_0}{l^2} \int_0^t \mathrm{v}(s)\ ds \tag{13}$$

that is,

$$\tau = \frac{D_0}{ql^2}\left(1 - e^{-qt}\right),\ t \geq 0 \tag{14}$$

as used in Olayiwola *et.al,* (2013)

Solution of the Model Equation: The parameter expanding method is applied to break the modeled problem into simpler equations as follows.

Let

$$c(x,\tau) = \gamma_0^0 c_0(x,\tau) + \gamma_0^1 c_1(x,\tau) + \gamma_0^2 c_2(x,\tau) + h.o.t \tag{15}$$

and $u_0 = a_0\gamma_0 + a_1\gamma_0^2 + h.o.t$ (16)

where h.o.t reads "higher order terms" in γ_0

Substituting equation (15) and (16) in equation (11), we have the following:

$$\left.\begin{aligned} &\frac{\partial}{\partial \tau}({\gamma_0}^0 c_0 + \gamma_0 c_1 + \gamma_0^2 c_2 + \ldots) = \frac{\partial^2}{\partial x^2}({\gamma_0}^0 c_0 + \gamma_0 c_1 + \gamma_0^2 c_2 + \ldots) \\ &-(a_0\gamma_0 + a_1\gamma_0^2 + \ldots)\left(\frac{\partial}{\partial x}({\gamma_0}^0 c_0 + \gamma_0 c_1 + \gamma_0^2 c_2 + \ldots)\right) - \gamma_0({\gamma_0}^0 c_0 + \gamma_0 c_1 + \ldots) + \mu_0 \end{aligned}\right\} \tag{17}$$

Equating coefficients of corresponding terms on both sides of the above equation (17) we obtain:

Order zero $\left(\gamma_0^{(0)}\right)$:

$$\left.\begin{aligned} &\frac{\partial c_0}{\partial \tau} = \frac{\partial^2 c_0}{\partial x^2} + \mu_0 \\ &c_0(x,0) = \frac{c_i}{c_0}, \\ &c_0(0,\tau) = 2 - q\tau, \\ &\left.\frac{\partial c_0}{\partial x}\right|_{x=1} = 0 \end{aligned}\right\} \tag{18}$$

Order one $\left(\gamma_0^{(1)}\right)$:

$$\left.\begin{aligned} &\frac{\partial c_1}{\partial \tau} = \frac{\partial^2 c_1}{\partial x^2} - a_0\frac{\partial c_0}{\partial x} - c_0 \\ &c_1(x,0) = 0 \\ &c_1(0,\tau) = 0 \\ &\left.\frac{\partial c_1}{\partial x}\right|_{x=1} = 0 \end{aligned}\right\} \tag{19}$$

Equations (18) and (19) are transformed into homogenous boundary conditions with the help of the following transformation:

$$w_0(x,\tau) = x\beta(\tau) + \alpha(\tau) \tag{20}$$

Where $\beta(\tau) = 0 \quad \alpha(\tau) = 2 - \mathrm{q}\,\tau$

So that $c_0(x,\tau) = w_0(x,\tau) + v_0(x,\tau)$ (21)

Where $w_0(x,\tau)$ is a function that satisfies the boundary conditions.

$$c_0(0,\tau) = v_0(0,\tau) + x(0) + (2 - q\tau) = 2 - q\tau \tag{22}$$

After transformation, the following initial value problems were obtained.

$$\left.\begin{aligned} &\frac{\partial v_0}{\partial \tau} = \frac{\partial^2 v_0}{\partial x^2} + (\mu_0 + q) \\ &v_0(0,\tau) = 0 \\ &\frac{\partial}{\partial x} v_0(1,\tau) = 0 \\ &v_0(x,0) = \frac{c_i}{c_0} - 2 \end{aligned}\right\} \tag{23}$$

and

$$\left.\begin{aligned} &\frac{\partial v_1}{\partial \tau} = \frac{\partial^2 v_1}{\partial x^2} - a_0\frac{\partial v_0}{\partial x} - v_0 \\ &v_1(x,0) = 0 \\ &v_1(0,\tau) = 0 \\ &\left.\frac{\partial v_1}{\partial x}\right|_{x=1} = 0 \end{aligned}\right\} \tag{24}$$

Solving equation (23) and (24) using Eigen-function expansion technique, the following solutions were obtained.

$$v_0 = \left(2 - q\tau\right) + \sum_{n=1}^{\infty}\left(\frac{16(\mu_0 + q)}{(2n-1)^3\pi^3}\left[1 - e^{-\left(\frac{(2n-1)\pi}{2}\right)^2\tau}\right] + \frac{4}{(2n-1)\pi}\left(\frac{c_i}{c_0} - 2\right)e^{-\left(\frac{(2n-1)\pi}{2}\right)^2\tau}\right)\times \sin\left(\frac{2n-1}{2}\right)\pi x \tag{25}$$

$$v_1 = -\left(a_0 + 1\right)\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left[\begin{array}{l}\frac{16(\mu_0+q)}{(2n-1)^3\pi^3}\left(\frac{4}{(2n-1)^2\pi^2} - \frac{4}{(2n-1)^2\pi^2}e^{-\left(\frac{2n-1}{2}\right)^2\pi^2\tau} - \tau e^{-\left(\frac{(2n-1)\pi}{2}\right)^2\tau}\right)\\ + \frac{4}{(2n-1)\pi}\left(\frac{c_1}{c_0} - 2\right)\tau e^{-\left(\frac{(2n-1)\pi}{2}\right)^2\tau}\end{array}\right]\sin\left(\frac{2n-1}{2}\right)\pi x \tag{26}$$

The general solution of the contaminant flow problem (1) is

$$\therefore c\left(x,\tau\right) = c_0\left(x,\tau\right) + \gamma_0 c_1\left(x,\tau\right),$$

i.e.,

$$\begin{aligned} c(x,\tau) &= \left(2 - q\tau\right) + \sum_{n=1}^{\infty}\left(\frac{16(\mu_0+q)}{(2n-1)^3\pi^3}\left[1 - e^{-\left(\frac{(2n-1)\pi}{2}\right)^2\tau}\right] + \frac{4}{(2n-1)\pi}\left(\frac{c_i}{c_0} - 2\right)e^{-\left(\frac{(2n-1)\pi}{2}\right)^2\tau}\right)\times\sin\left(\frac{2n-1}{2}\right)\pi x \\ &\quad -\left(a_0+1\right)\gamma_0\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left[\begin{array}{l}\frac{16(\mu_0+q)}{(2n-1)^3\pi^3}\left(\frac{4}{(2n-1)^2\pi^2} - \frac{4}{(2n-1)^2\pi^2}e^{-\left(\frac{2n-1}{2}\right)^2\pi^2\tau} - \tau e^{-\left(\frac{(2n-1)\pi}{2}\right)^2\tau}\right)\\ + \frac{4}{(2n-1)\pi}\left(\frac{c_1}{c_0} - 2\right)\tau e^{-\left(\frac{(2n-1)\pi}{2}\right)^2\tau}\end{array}\right]\sin\left(\frac{2n-1}{2}\right)\pi x \end{aligned} \tag{27}$$

RESULTS AND DISCUSSIONS

The solution given in (27) are computed for the values of $c_i = 200, c_0 = 1, \ell = 100km, u_0 = 1, \;\; q = 0.9$ and represented graphically.

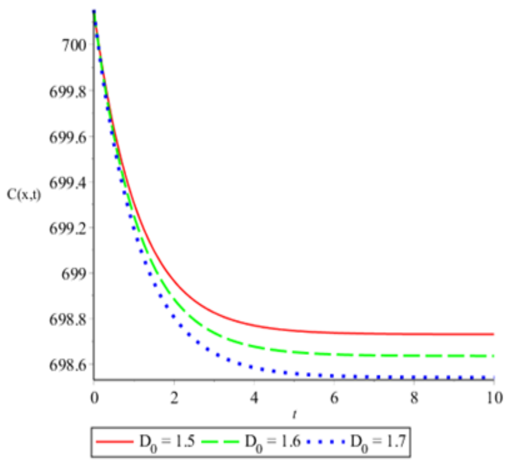

Fig 1: Variation of Contaminant concentration with Time for Different Values of Dispersion Coefficient

Figure 1 illustrates the time variation of the concentration profile with respect to the dispersion coefficient term. It is evident from the graph that the contaminant concentration decreases as the initial dispersion coefficient term increases. Also the contaminant concentration decreases with time.

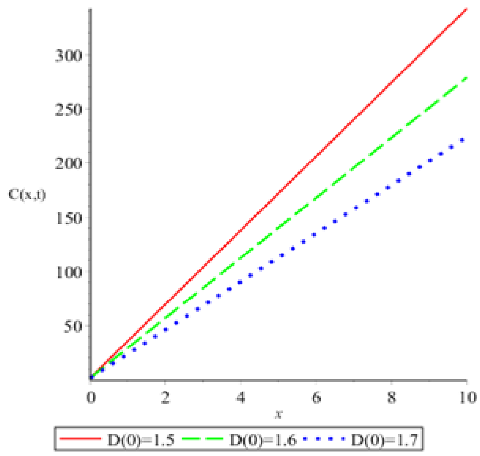

Fig 2: Variation of Contaminant concentration with Space for Different Values of Dispersion Coefficient

Figure 2 illustrates the spatial variation of the concentration profile with respect to the dispersion coefficient term. It is evident from the graph that the contaminant concentration decreases as the initial dispersion coefficient term increases. Also the contaminant concentration increases along the spatial direction. Figure 3 illustrates the time variation of the concentration profile with respect to the Decay parameter. It is evident from the graph that the contaminant concentration decreases as the decay parameter increases. Also the contaminant concentration decreases with time. Figure 4 illustrates the spatial variation of the concentration profile with respect to the decay parameter. It is evident from the graph that the contaminant concentration decreases as the decay parameter increases. Also the contaminant concentration increases along the one-dimensional spatial direction.

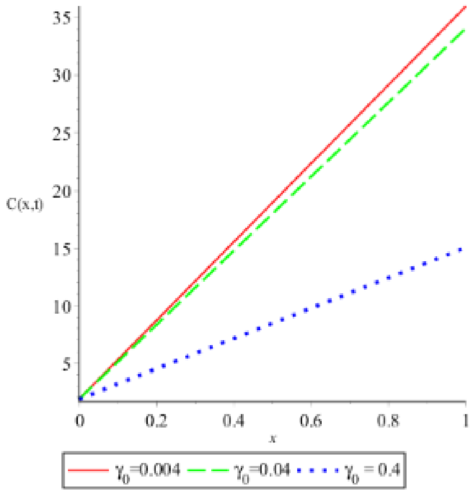

Fig 3: Variation of Contaminant concentration with Time for Different Values of Decay Parameter

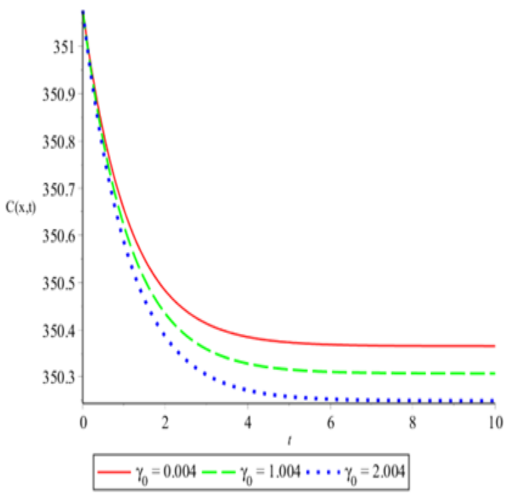

Fig 4: Variation of Contaminant concentration with Distance *x* for Different Values of Decay Parameter

Figure 5 illustrates the time variation of the concentration profile with respect to the zero order source coefficient. It is evident from the graph that the contaminant concentration decreases as the zero order source increases, but decreases faster as the zero order source coefficient decreases.

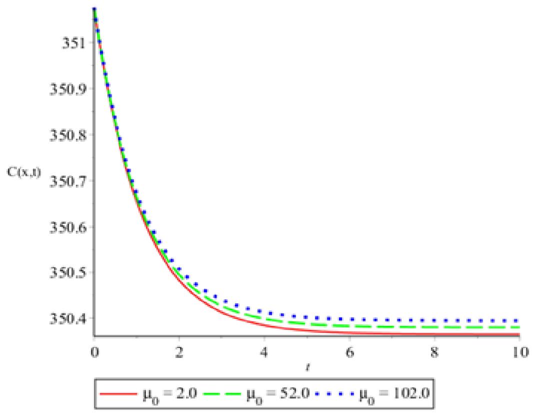

Fig 5: Variation of Contaminant concentration with time for Different Values of Zero Order Source Coefficient

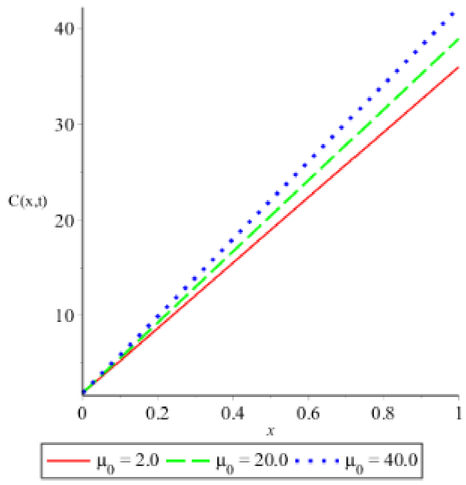

Fig 6: Variation of Contaminant concentration with Space for Different Values of Zero Order Source Coefficient

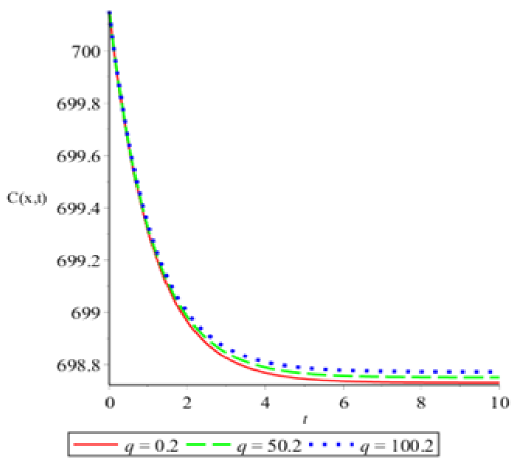

Fig 7: Variation of Contaminant concentration with Time for Different Values of Flow Resistance Parameter q

Figure 6 illustrates the spatial variation of the concentration profile with respect to the zero order source coefficient. It is evident from the graph that the contaminant concentration increases as the zero order source coefficient increases. Also the contaminant concentration increases along the one-dimensional spatial direction. Figure 7 illustrates the time variation of the concentration profile with respect to the growing rate of flow resistance parameter. It is evident from the graph that the contaminant concentration decreases as the flow resistance parameter decreases. Also the contaminant concentration decreases with time.

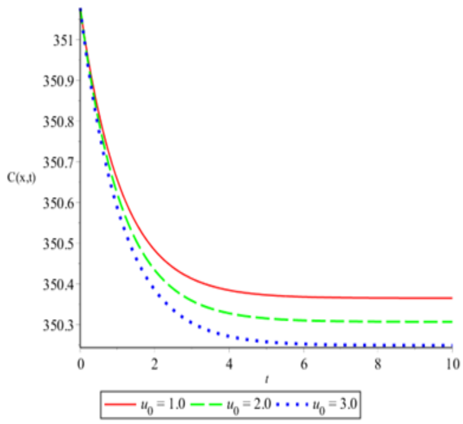

Fig 8: Variation of Contaminant concentration with Time for Different Values of Initial Velocity

Figure 8 illustrates the spatial variation of the concentration profile with respect to the initial velocity. It is evident from the graph that the contaminant concentration decreases as the initial velocity increases. Also the contaminant concentration decreases with time.

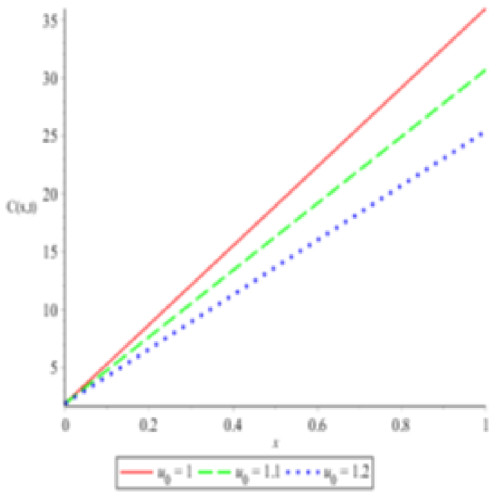

Fig 9: Variation of Contaminant concentration with Space for Different Values of Initial Velocity

Figure 9 illustrates the spatial variation of the concentration profile with respect to the initial velocity. It is evident from the graph that the contaminant concentration decreases as the initial velocity increases. Also the contaminant concentration increases along the one-dimensional spatial direction.

*Conclusion***:** Solute transport model with time dependent source concentration was formulated and solved semi-analytically to predict contaminant concentration along transient groundwater flow in a homogeneous finite aquifer. The governing equation which is a partial differential equation includes terms like advection, hydrodynamic dispersion, and first order decay processes incorporating a zero order source effects. The velocity of the flow was considered exponential in nature. The solution obtained using parameter expanding method and Eigen - function expansion technique which investigates the effect of change in the parameters on the concentration were represented graphically and discussed. Illustrations considering Neumann boundary condition revealed the effects of change in the parameters on the contaminant concentration level. It also reveals that the contaminant concentration decreases with respect to time and increases with respect to distance.

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