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Mathematical Modeling of Transient and Steady-State Groundwater Flow in a Confined Aquifer

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ABSTRACT: Groundwater flow problems can be expressed mathematically with the employ of partial differential equations and which could be solved analytically or numerically. The results obtained showed that the direction of flow of groundwater is from a region of higher hydraulic head to a lower hydraulic head, and that pumping groundwater from the well faster than it is recharged leads to dry wells. The numerical methods used for the analysis of Transient and steady-state groundwater flow produced efficient, effective and accurate solution and can be used for all real-life problems.

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Groundwater refers to all the water occupying the voids, pores inside geological formations. It is the water found underneath the ground in the cracks and spaces in soil, sand and rock. Groundwater is almost everywhere below the earth's surface (Kumar and Singh, 2015). Groundwater modeling according to Baalousha (2011) is used to represent the system in a different form in order to study the system's response under certain conditions or to predict the system's behavior in the near future. Groundwater modeling is a powerful tool for managing, protecting and restoring water resources. Decision makers use models to predict the behavior of groundwater systems prior to project implementation or the implementation of a recovery plan. Mathematical models are ways to describe the physical system using mathematical equations. They are based on solving an equation or a system of equations that describe the physical phenomenon. Such equations are called governing equations of the specified phenomenon (Mango *et al.*, 2014). To develop models, it is helpful to first understand the general equation and how it is related to the underlying physical principles. The general equation has several different forms depending on whether the flow is saturated or unsaturated, two-dimensional or three-dimensional, isotropic or anisotropic, and transient or steady state (Waghmare, 2016). According Nkurunziza *et al.* (2014), one needs hydrological inputs, hydraulic parameters, and initial and boundary conditions for the calculations in groundwater modeling. Groundwater flow models simulate either steady or unsteady states (transient flow). In steady-state systems, inputs (recharge) and outputs (discharge) are in equilibrium so that there is no net change in the system with time. In unsteady state or transient simulations, the inputs (recharge) and outputs (discharge) are not in equilibrium so there is a net change in the systems with

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time that is the flow velocity and pressure are changing with time. For groundwater flow, the governing equation is the combination of Darcy's law and the principle of conservation of mass, thus, combining these two principles gives the general groundwater flow equation, which is a partial differential equation (Atangana and Botha, 2013), (Waghmare, 2016), (Wang and Zheng, 2015). Hence, the objective of this paper was to evaluate the mathematical modeling of transient and steady-state groundwater flow in a confined aquifer.

MATERIALS AND METHODS

The basic groundwater flow equation is:

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial h}{\partial z}\right)-Q=S_s\frac{\partial h}{\partial t} \quad (1)$$

h is the hydraulic head or piezometric head, K_x, K_y, K_z are the hydraulic conductivity along x, y, z axes, Q is the volumetric source or sink and S_s is the specific storage coefficient.

From equation 1, the governing equation for groundwater flow through an isotropic, homogeneous medium under steady-state condition in three-dimensions is

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}+\frac{\partial^2 h}{\partial z^2}=0 \quad (2)$$

Allowing the possibility of a sink (for example, a pumping well) or a source of water (for example, an injection well or recharge) which is expressed as volume of per area of aquifer per time, R, so equation (2) becomes

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}+\frac{\partial^2 h}{\partial z^2}=\frac{R}{T} \quad (3)$$

Equation (3) is called Poisson equation, which is the equation for steady-state flow equation with sinks/sources while equation (2) is a very famous equation called the Laplace equation.

Re-writing equation (1) without source/sink in an isotropic, homogeneous medium becomes

$$K\left[\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}+\frac{\partial^2 h}{\partial z^2}\right]=S_s\frac{\partial h}{\partial t} \quad (4)$$

Saturated thickness, *b*, is not dependent on head, *h*, and assuming the aquifer thickness is constant, both sides of equation (4) can be multiplied by the aquifer thickness to give equation (5)

$$Kb\left[\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}+\frac{\partial^2 h}{\partial z^2}\right]=S_s b\frac{\partial h}{\partial t} \quad (5)$$

Then by using the definition of Transmissivity, *T*, and Storativity, $T=Kb$, $S_s b=S$ and dividing both sides by T, equation (5) can be written in another form in equation (6).

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}+\frac{\partial^2 h}{\partial z^2}=\frac{S}{T}\frac{\partial h}{\partial t} \quad (6)$$

This is the Transient or unsteady-state equation in three-dimensions without sinks/sources.

We must allow for possibility of a sink (for example, a pumping well) or source of water (for example, an injection well or recharge) which is expressed as volume per area of aquifer per time, R, so equation (6) becomes

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}+\frac{\partial^2 h}{\partial z^2}=\frac{S}{T}\frac{\partial h}{\partial t}-\frac{R}{T} \quad (7)$$

Methods for Steady-State Flow

Method1: Successive Over-Relaxation Method (SOR)

Successive over-relaxation method is an improved Gauss-Seidel method for solving a linear system that leads faster convergence. In SOR, the Gauss-Siedel method is improved by introducing the previous results

$$H_{m,n}^{k+1}=H_{m,n}^{k}+\omega\frac{1}{4}(H_{m-1,n}^{k+1}+H_{m,n-1}^{k+1}+H_{m+1,n}^{k+1}+H_{m,n+1}^{k+1}-H_{m,n}^{k} \quad (8)$$

The convergence can be increased by introducing a parameter, ω (omega)

$$H_{m,n}^{k+1}=H_{m,n}^{k}+\omega\frac{1}{4}(H_{m-1,n}^{k+1}+H_{m,n-1}^{k+1}+H_{m+1,n}^{k+1}+H_{m,n+1}^{k+1}-\omega H_{m,n}^{k} \quad (9)$$

Re-writing equation (9), the new value $H_{m,n}^{k+1}$ is given by

$$H_{m,n}^{k+1}=(1-\omega)H_{m,n}^{k}+\omega\left(\frac{H_{m-1,n}^{k+1}+H_{m,n-1}^{k+1}+H_{m+1,n}^{k}+H_{m,n+1}^{k}}{4}\right) \quad (10)$$

With sinks/sources, equation (10) becomes

$$H_{m,n}^{k+1} = (1-\omega)H_{m,n}^{k} + \omega\left(\frac{H_{m-1,n}^{k+1} + H_{m,n-1}^{k+1}\ H_{m+1,n}^{k} + H_{m,n+1}^{k}}{4} - \frac{R(\Delta x)^2}{T}\right) \quad (11)$$

If 0 < ω < 1, then the updated head is under relaxed. If ω = 1, equation (11) reduces to Gauss- Siedel formula but for optimum result and in general 1 ≤ ω ≤2 (Wang and Anderson, 1992).

Method 2: Alternating Direction Implicit Method (ADI)
The ADI Method was first introduced by Peaceman and Rachford for solving the time-dependent heat equation in two space dimensions. It was quickly recognized that the unconditional stability of the method might render it effective as a steady-state (hence, elliptic) solver due to the possibility of employing large time steps for pseudo-time marching to a steady state (Imam, 2015).

Consider the two-dimensional steady state groundwater flow equation without sinks/sources below:

$$\frac{\delta^2 H}{dx^2} + \frac{\delta^2 H}{dy^2} = 0$$

Finite difference approximation, assuming $\Delta x = \Delta y$ is

$$H_{m+1,n} - 4H_{m,n} + H_{m-1,n} + H_{m,n+1} + H_{m,n-1} = 0 \quad (12)$$

For fix row formula, we have

$$H_{m-1,n}^{(p+1)} - 4H_{m,n}^{(p+1)} + H_{m+1,n}^{(p+1)} = -H_{m,n+1}^{(p)} - H_{m,n-1}^{(p)} \quad (13)$$

In the next iteration, we alternate the direction by using the formula for fix column

$$H_{m,n+1} - 4H_{m,n} + H_{m,n-1} = -H_{m+1,n} - H_{m+1,n} \quad (14)$$

Substituting *p+1th* approximation (the results from equation (13)) is substituted on the right as:

$$H_{m,n-1}^{(p+2)} - 4H_{m,n}^{(p+2)} + H_{m,n+1}^{(p+2)} = -H_{m-1,n}^{(p+1)} - H_{m+1,n}^{(p+1)} \quad (15)$$

With sinks/sources, we have for row and column

$$H_{m-1,n}^{(p+1)} - 4H_{m,n}^{(p+1)} + H_{m+1,n}^{(p+1)} = -H_{m,n+1}^{(p)} - H_{m,n-1}^{(p)} - \frac{R(\Delta x)^2}{T} \quad (16)$$

$$H_{m,n-1}^{(p+2)} - 4H_{m,n}^{(p+2)} + H_{m,n+1}^{(p+2)} = -H_{m-1,n}^{(p+1)} - H_{m+1,n}^{(p+1)} - \frac{R(\Delta x)^2}{T} \quad (17)$$

Method 3: Improved Alternating Direction Implicit Method (IADI)
To improve the convergence of ADI, we introduce a parameter k (Imam, 2015).
From equations (13) and (15), we have the improved formula for row and column

$$H_{m-1,n}^{(p+1)} - (2+k)H_{m,n}^{(p+1)} + H_{m+1,n}^{(p+1)} = -H_{m,n-1}^{(p)} - H_{m,n+1}^{(p)} + (2-k)H_{m,n}^{(p)} \quad (18)$$

$$H_{m,n-1}^{(p+2)} - (2+k)H_{m,n}^{(p+2)} + u_{m,n+1}^{(p+2)} = -H_{m-1,n}^{(p+1)} - H_{m+1,n}^{(p+1)} + (2-k)H_{m,n}^{(p+1)} \quad (19)$$

With sinks/sources, we have

From equations (18) and (19), we introduce the parameter k and add the source/sinks

$$H_{m-1,n}^{(p+1)} - (2+k)H_{m,n}^{(p+1)} + H_{m+1,n}^{(p+1)} = -H_{m,n+1}^{(p)} - H_{m,n-1}^{(p)} + (2-k)H_{m,n}^{(p)} - \frac{R(\Delta x)^2}{T} \quad (20)$$

$$H_{m,n-1}^{(p+2)} - (2+k)H_{m,n}^{(p+2)} + H_{m,n+1}^{(p+2)} = -H_{m-1,n}^{(p+1)} - H_{m+1,n}^{(p+1)} + (2-k)H_{m,n}^{(p+1)} - \frac{R(\Delta x)^2}{T} \quad (21)$$

Methods For Transient Flow

Method 1: Explicit Finite Difference Method (EFDM)

Explicit Finite Difference uses forward differencing for the time derivative and central differencing for the space derivatives. (Hoffmann and Chiang, 2000).

Consider the two-dimensional transient groundwater flow equation below:

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = \frac{S}{T}\frac{\partial H}{\partial t} \tag{22}$$

Using the forward difference approximation for the time derivative and central difference approximation for the space derivative, equation (22) becomes

$$\frac{H^{p}_{m+1,n} - 2H^{p}_{m,n} + H^{p}_{m-1,n}}{(\Delta x)^2} + \frac{H^{p}_{m,n+1} - 2H^{p}_{m,n} + H^{p}_{m,n-1}}{(\Delta y)^2} = \frac{S}{T}\frac{H^{p+1}_{m,n} - H^{p}_{m,n}}{\Delta t} \tag{23}$$

assume $\Delta x = \Delta y = b$ and let $\gamma = \frac{T\Delta t}{Sb^2}$

$$H^{p+1}_{m,n} = H^{p}_{m,n} + \gamma\left(H^{p}_{m+1,n} - 4H^{p}_{m,n} + H^{p}_{m-1,n} + H^{p}_{m,n+1} + H^{p}_{m,n-1}\right) \tag{24}$$

Collecting like terms, equation (24) becomes

$$H^{p+1}_{m,n} = H^{p}_{m,n}(1 - 4\gamma) + \gamma\left(H^{p}_{m+1,n} + H^{p}_{m-1,n} + H^{p}_{m,n+1} + H^{p}_{m,n-1}\right) \tag{25}$$

For the solution to be stable, γ must be kept sufficiently small so for two-dimensional case where $\Delta x = \Delta y$, the value of γ must be less than or equal to 0.25 while for one-dimensional case, γ must be less than or equal to 0.5 (Wang and Anderson, 1996).

Method 2: Crank Nicolson Method (CNM)

Crank Nicolson Method according to (Fadugba *et al.*, 2013) was developed by John Crank and Phyllis Nicolson in the mid-20th century. This method is always used in dealing with complex problems of science and technology. (Jamaluddin *et al*., 2020), (Arunachalam, 2023), (Hoffmann and Chiang, 2000)

Considering equation (22)

Using forward difference for the time derivative and central difference for the space derivative along at time level t and t+1

$$\frac{H^{t}_{m+1,n} - 2H^{t}_{m,n} + H^{t}_{m-1,n}}{(\Delta x)^2} + \frac{H^{t}_{m,n+1} - 2H^{t}_{m,n} + H^{t}_{m,n-1}}{(\Delta y)^2} = \frac{S}{T}\frac{H^{t+1}_{m,n} - H^{t}_{m,n}}{\Delta t} \tag{26}$$

$$\frac{H^{t+1}_{m+1,n} - 2H^{t+1}_{m,n} + H^{t+1}_{m-1,n}}{(\Delta x)^2} + \frac{H^{t+1}_{m,n+1} - 2H^{t+1}_{m,n} + H^{t+1}_{m,n-1}}{(\Delta y)^2} = \frac{S}{T}\frac{H^{t+1}_{m,n} - H^{t}_{m,n}}{\Delta t} \tag{27}$$

Taking the average of equations (26) and (27) and assume $\Delta x = \Delta y$,

$$\frac{S}{T}\frac{H^{t+1}_{m,n} - H^{t}_{m,n}}{\Delta t} = \frac{1}{2}\left(\frac{H^{t+1}_{m+1,n} + H^{t+1}_{m-1,n} + H^{t+1}_{m,n+1} + H^{t+1}_{m,n-1} - 4H^{t+1}_{m,n}}{(\Delta x)^2} + \right.$$
$$\left. + \frac{H^{t}_{m+1,n} - 4H^{t}_{m,n} + H^{t}_{m-1,n} + H^{t}_{m,n+1} + H^{t}_{m,n-1}}{(\Delta x)^2}\right) \tag{28}$$

Re-arranging and Let $\beta = \frac{T\Delta t}{S(\Delta x)^2}$, it becomes

$$H^{t+1}_{m,n} = H^{t}_{m,n} + \frac{1}{2}\beta\left(H^{t+1}_{m+1,n} + H^{t+1}_{m-1,n} + H^{t+1}_{m,n+1} + H^{t+1}_{m,n-1} - 4H^{t+1}_{m,n} + H^{t}_{m+1,n} - \right.$$
$$\left. - 4H^{t}_{m,n} + H^{t}_{m-1,n} + H^{t}_{m,n+1} + H^{t}_{m,n-1}\right) \tag{29}$$

With sinks/sources

$$\frac{S}{T}\frac{\partial H}{\partial t} = \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{R}{T}$$

Using equation (29), we have

$$H_{m,n}^{t+1} = H_{m,n}^{t} + \frac{1}{2}\,\beta\Big(H_{\;m+1,n}^{\,t+1} + H_{\;m-1,n}^{\,t+1} + H_{\;m,n+1}^{\,t+1} + H_{\;m,n-1}^{\,t+1} - 4H_{m,n}^{t+1} + H_{\;m+1,n}^{\,t} - $$
$$- 4H_{m,n}^{t} + H_{\;m-1,n}^{\,t} + H_{\;m,n+1}^{\,t} + H_{\;m,n-1}^{\,t} + \frac{2R(\Delta x)^2}{T}\Big) \tag{30}$$

Method3: Fractional Step Method (FSM)

Fractional step method (FSM) is similar to ADI method with time splitting resulting in two sets of tridiagonal problem per time. This method splits the multi-dimensional equation into a series of one-space dimensional equations and solves them sequentially. (Note that Crank-Nicolson scheme is used) (Hoffmann and Chiang, 2000)

Splitting equation (28) into 2, assuming $\Delta x = \Delta y$ and take $\gamma = \frac{T\Delta t}{S(\Delta x)^2}$, we have 2 equations

$$H_{m,n}^{t+1} = H_{m,n}^{t} + \frac{1}{4}\gamma\Big(H_{\;m+1,n}^{\,t+1} - 2H_{m,n}^{t+1} + H_{\;m-1,n}^{\,t+1} + H_{\;m+1,n}^{\,t} - 2H_{m,n}^{t} + H_{\;m-1,n}^{\,t}\Big) \tag{31}$$

$$H_{m,n}^{t+2} = H_{m,n}^{t+1} + \frac{1}{4}\gamma\Big(H_{\;m,n+1}^{\,t+2} - 2H_{m,n}^{t+2} + H_{\;m,n-1}^{\,t+2} + H_{\;m,n+1}^{\,t+1} - 2H_{m,n}^{t+1} + H_{\;m,n-1}^{\,t+1}\Big) \tag{32}$$

Adding sinks/sources to equations (31) and (32), we have the following equations

$$H_{m,n}^{t+1} = H_{m,n}^{t} + \frac{1}{4}\gamma\left(H_{\;m+1,n}^{\,t+1} - 2H_{m,n}^{t+1} + H_{\;m-1,n}^{\,t+1} + H_{\;m+1,n}^{\,t} - 2H_{m,n}^{t} + H_{\;m-1,n}^{\,t} - \frac{2R(\Delta x)^2}{T}\right) \tag{33}$$

$$H_{m,n}^{t+2} = H_{m,n}^{t+1} + \frac{1}{4}\gamma\left(H_{\;m,n+1}^{\,t+2} - 2H_{m,n}^{t+2} + H_{\;m,n-1}^{\,t+2} + H_{\;m,n+1}^{\,t+1} - 2H_{m,n}^{t+1} + H_{\;m,n-1}^{\,t+1} - \frac{2R(\Delta x)^2}{T}\right) \tag{34}$$

Fractional step method (FSM) is unconditionally stable and is of order $[(\Delta t)^2,\ (\Delta x)^2, (\Delta y)^2\ ]$ (Hoffmann and Chiang, 2000). This method may be applied to any multidimensional problem to provide the approximate factorization of the partial differential equations.

Numerical applications

Problem 1: A single well is pumped to steady-state in a confined isotropic, homogeneous aquifer. A square 500m x 500m grid is imposed as shown below and heads are measured in wells along the boundary of the grid. Compute the heads at the nine (9) interior nodes (Karvonen, 2002).

8.26	8.33	8.43	8.55	8.68
7.99	H1	H2	H3	8.55
7.65	H4	H5	H6	8.43
7.22	H7	H8	H9	8.33
6.68	7.22	7.65	7.99	8.26

Method 1: SOR

using equation (10) with $\omega = 1.2$ and stating the iterations at k = 0

$$H_{m,n}^{k+1} = (-0.2)H_{m,n}^{k} + \omega\left(\frac{H_{m-1,n}^{k+1} + H_{m,n-1}^{k+1}\ H_{m+1,n}^{k} + H_{m,n+1}^{k}}{4}\right) \tag{35}$$

The results converge after 7 iterations:

H1= 8.10 H2 = 8.24 H3 = 8.39 H4 = 7.83 H5 = 8.03

H6 = 8.24 H7 = 7.52 H8 = 7.82 H9 = 8.10

Method 2: ADI

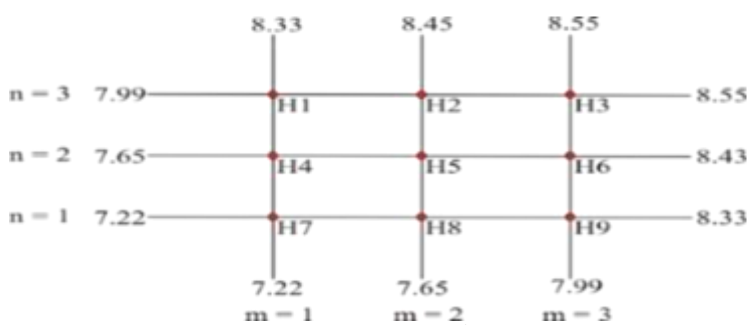

Fig 1: Schematic Diagram for ADI

Using equation (13) and figure 1 for the row iterations

$7.22 - 4H_7^{(p+1)} + H_8^{(p+1)} = -7.22 - H_4^{(p)}$ for m = 1

$H_7^{(p+1)} - 4H_8^{(p+1)} + H_9^{(p+1)} = -7.65 - H_5^{(p)}$ for m = 2

$H_8^{(p+1)} - 4H_9^{(p+1)} + 8.33 = -7.99 - H_6^{(p)}$ for m = 3

For Row n = 2

$7.65 - 4H_4^{(p+1)} + H_5^{(p+1)} = -H_7^{(p)} - H_1^{(p)}$ for m = 1

$H_4^{(p+1)} - 4H_5^{(p+1)} + H_6^{(p+1)} = -H_8^{(p)} - H_2^{(p)}$ for m = 2

$H_5^{(p+1)} - 4H_6^{(p+1)} + 8.43 = -H_9^{(p)} - H_3^{(p)}$ for m = 3

For Row n = 3

$7.99 - 4H_1^{(p+1)} + H_2^{(p+1)} = -H_4^{(p)} - 8.33$ for m =1

$H_1^{(p+1)} - 4H_2^{(p+1)} + H_3^{(p+1)} = -H_5^{(p)} - 8.43$ for m = 2

$H_2^{(p+1)} - 4H_3^{(p+1)} + 8.56 = -H_6^{(p)} - 8.55$ for m = 3

Using equation (15) and figure (1), we have the following equations for the column iterations

For Column m = 1

$7.22 - 4H_7^{(p+1)} + H_4^{(p+1)} = -H_8^{(p)} - 7.22$ for n =1

$H_7^{(p+1)} - 4H_4^{(p+1)} + H_1^{(p+1)} = -H_5^{(p)} - 7.65$ for n = 2

$H_4^{(p+1)} - 4H_1^{(p+1)} + 8.33 = -H_2^{(p)} - 7.99$ for n = 3

For Column m = 2

$7.65 - 4H_8^{(p+1)} + H_5^{(p+1)} = -H_7^{(p)} - H_9^{(p)}$ for n = 1

$H_8^{(p+1)} - 4H_5^{(p+1)} + H_2^{(p+1)} = -H_4^{(p)} - H_6^{(p)}$ for n = 2

$H_5^{(p+1)} - 4H_2^{(p+1)} + 8.43 = -H_1^{(p)} - H_3^{(p)}$ for n = 3

For Column m = 3

$7.99 - 4H_9^{(p+1)} + H_6^{(p+1)} = -H_8^{(p)} - 8.33$ for n = 1

$H_9^{(p+1)} - 4H_6^{(p+1)} + H_3^{(p+1)} = -H_5^{(p)} - 8.43$ for n = 2

$H_6^{(p+1)} - 4H_3^{(p+1)} + 8.55 = -H_2^{(p)} - 8.56$ for n = 3

The results converge after 11 iterations, the results are

H1 = 8.10	H2 = 8.24	H3 = 8.40	H4 = 7.83	H5 = 8.03
H6 = 8.24	H7 = 7.52	H8 = 7.83	H9 = 8.10	

Method 3: IADI

Using equations (18) and (19), we have the following equations for the iterations

Assume k = 1.5, starting at p = 0

For Row n = 1

$7.22 - 3.5H_7^{(p+1)} + H_8^{(p+1)} = -7.22 - H_4^{(p)} + 0.5H_7^{(p)}$ for m =1

$H_7^{(p+1)} - 3.5H_8^{(p+1)} + H_9^{(p+1)} = -7.65 - H_5^{(p)} + 0.5H_8^{(p)}$ for m =2

$H_8^{(p+1)} - 3.5H_9^{(p+1)} + 8.33 = -7.99 - H_6^{(p)} + 0.5H_9^{(p)}$ for m =3

For Row n = 2

$7.65 - 3.5H_4^{(p+1)} + H_5^{(p+1)} = -H_7^{(p)} - H_1^{(p)} + 0.5H_4^{(p)}$ for m =1

$H_4^{(p+1)} - 3.5H_5^{(p+1)} + H_6^{(p+1)} = -H_8^{(p)} - H_2^{(p)} + 0.5H_5^{(p)}$ for m =3

$H_5^{(p+1)} - 3.5H_6^{(p+1)} + 8.43 = -H_9^{(p)} - H_3^{(p)} + 0.5H_6^{(p)}$ for m =3

For Row n = 3

$7.99 - 3.5H_1^{(p+1)} + H_2^{(p+1)} = -H_4^{(p)} - 8.33 + 0.5H_1^{(p)}$ for m =1

$H_1^{(p+1)} - 3.5H_2^{(p+1)} + H_3^{(p+1)} = -H_5^{(p)} - 8.43 + 0.5H_2^{(p)}$ for m =2

$H_2^{(p+1)} - 3.5H_3^{(p+1)} + 8.56 = -H_6^{(p)} - 8.55 + 0.5H_3^{(p)}$ for m =3

For Column m = 1

$7.22 - 3.5H_7^{(p+1)} + H_4^{(p+1)} = -H_8^{(p)} - 7.22 + 0.5H_7^{(p)}$ for n =1

$H_7^{(p+1)} - 3.5H_4^{(p+1)} + H_1^{(p+1)} = -H_5^{(p)} - 7.65 + 0.5H_4^{(p)}$ for n =2

$H_4{}^{(p+1)} - 3.5H_1^{(p+1)} + 8.33 = -H_2^{(p)} - 7.99 + 0.5H_1^{(p)}$ for n =3

For Column m = 2

$7.65 - 3.5H_8^{(p+1)} + H_5^{(p+1)} = -H_7^{(p)} - H_9^{(p)} + 0.5H_8^{(p)}$ for n =1

$H_8{}^{(p+1)} - 3.5H_5^{(p+1)} + H_2^{(p+1)} = -H_4^{(p)} - H_6^{(p)} + 0.5H_5^{(p)}$ for n =2

$H_5{}^{(p+1)} - 3.5H_2^{(p+1)} + 8.43 = -H_1^{(p)} - H_3^{(p)} + 0.5H_2^{(p)}$ for n =3

For Column m = 3

$7.99 - 3.5H_9^{(p+1)} + H_6^{(p+1)} = -H_8^{(p)} - 8.33 + 0.5H_9^{(p)}$ for n =1

$-3.5H_6^{(p+1)} + H_3^{(p+1)} = -H_5^{(p)} - 8.43 + 0.5H_6^{(p)}$ for n =2

$H_6{}^{(p+1)} - 3.5H_3^{(p+1)} + 8.55 = -H_2^{(p)} - 8.56 + 0.5H_3^{(p)}$ for n =3

The results converged after 6 iterations are

H1 = 8.10	H2 = 8.24	H3 = 8.40	H4 = 7.83	H5 = 8.03
H6 = 8.24	H7 = 7.53	H8 = 7.82	H9 = 8.10	

Problem 2: A domain is bounded at the top and bottom by Neumann boundary with no flow and we allow water to infiltrate at the piezometric head at 1m and 2m per day on the left and right boundary respectively.
These can be expressed mathematically as:

$\nabla^2 = 0$ $\{(x, y)$: $0 < x < 1, 0 < y < 1$

BC: $H(0, y) = 1$ $0 \le y \le 1$

$H(1, y) = 2$ $0 \le y \le 1$; $\frac{\partial H(x,0)}{\partial y} = 0$ $0 \le x \le 1$

$\frac{\partial H(x,1)}{\partial y} = 0$ $0 \le x \le 1$ $N_y = 6, N_x = 6$

Fig 2: Finite Difference Grid Diagram

Using figure 2

For the no-flow Boundary(Neumann Boundary), we use the central difference formula for the first derivative. $\frac{dy}{dx} = \frac{H_{i+1,j} - H_{i-1,j}}{2\Delta x} = 0$ which implies that $H_{i+1,j} = H_{i-1,j}$ for example $H_{2,1} = H_{0,1}$.

Method 1: SOR

Considering problem 2 with Neumann boundaries on the top and bottom and taking omega ω = 1.4, SOR formula is stated below

$$H_{m,n}^{k+1} = (1-\omega)H_{m,n}^k + \frac{\omega\left(H_{m-1,n}^{(k+1)} + H_{m,n-1}^{(k+1)} + H_{m+1,n}^k + H_{m,n+1}^k\right)}{4} \quad (36)$$

taking omega $(\omega) = 1.4$, we have and using central difference formula for Neumann boundary that is for the derivative

$$H_{m,n}^{k+1} = (-0.4)H_{m,n}^k + \frac{1.4\left(H_{m-1,n}^{(k+1)} + H_{m+1,n}^k + H_{m,n-1}^{(k+1)} + H_{m,n+1}^k\right)}{4} \quad (37)$$

$$H_{m,n}^{k+1} = (-0.4)H_{m,n}^k + \frac{1.4\left(H_{m-1,n}^{(k+1)} + H_{m+1,n}^k + 2H_{m,n-1}^{(k+1)}\right)}{4} \quad (38)$$

$$H_{m,n}^{k+1} = (-0.4)H_{m,n}^{k} + \frac{1.4\left(H_{m-1,n}^{(k+1)} + H_{m+1,n}^{k} + 2H_{m,n+1}^{k}\right)}{4} \quad (39)$$

For the Bottom Boundary (H20 to H24), use equation (39) , For the Top Boundary (H1 to H4), use equation (38) , Other Nodes (H5 to H19), use equation (37)

The results after 11 iterations are

H1= 1.20	H2 = 1.40	H3 = 1.60	H4 = 1.80	H5 = 1.20	H6 = 1.40
H7=1.60	H8 = 1.80	H9 = 1.20	H10 = 1.40	H11 = 1.60	H12 = 1.80
H13 = 1.20	H14 = 1.40	H15 = 1.60	H16 = 1.80	H17 = 1.20	H18 = 1.40
H19 = 1.60	H20 = 1.80	H21 = 1.20	H22 = 1.40	H23 = 1.60	H24 = 1.80

Method 2: ADI

using figure 3 and equation (13) for the row iteration and equation (15) for the column iterations and starting at p = 0 and apply the boundary using the central difference for first derivative that is $H_{i+1,j} = H_{i-1,j}$ and $H_{i,j+1} = H_{i,j-1}$

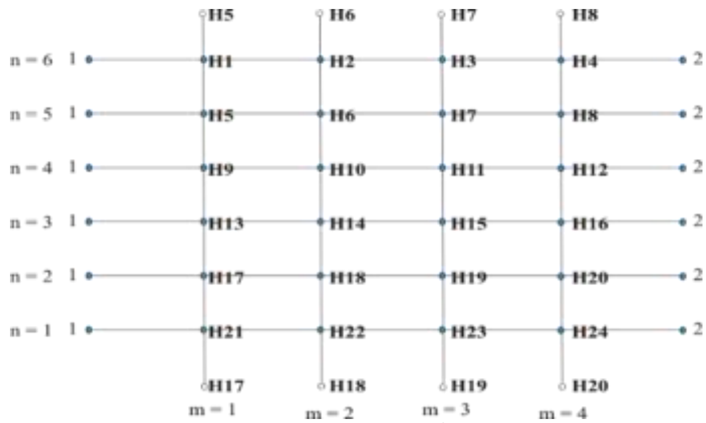

Fig 3: ADI schematics Diagram for Problem 2

$$H_{m-1,n}^{(1)} - 4H_{m,n}^{(1)} + H_{m+1,n}^{(1)} = -H_{m,n+1}^{(0)} - H_{m,n-1}^{(0)} \quad (40)$$

$$H_{m,n-1}^{(2)} - 4H_{m,n}^{(2)} + H_{m,n+1}^{(2)} = -H_{m-1,n}^{(1)} - H_{m+1,n}^{(1)} \quad (41)$$

We continue the iterations until the results converge, the results converged after iteration 18, the results are:

H1= 1.20	H2 = 1.40	H3 = 1.60	H4 = 1.80	H5 = 1.20	H6 = 1.40
H7=1.60	H8 = 1.80	H9 = 1.20	H10 = 1.40	H11 = 1.60	H12 = 1.80
H13 = 1.20	H14 = 1.40	H15 = 1.60	H16 = 1.80	H17 = 1.20	H18 = 1.40
H19 = 1.60	H20 = 1.80	H21 = 1.20	H22 = 1.40	H23 = 1.60	H24 = 1.80

Method 3: IADI

using figure 3 and equations (18) and (19) for the iterations and assume k = 1.2

for Row iteration

$$H_{m-1,n}^{(p+1)} - (3.2)H_{m,n}^{(p+1)} + H_{m+1,n}^{(p+1)} = -H_{m,n-1}^{(p)} - H_{m,n+1}^{(p)} + (0.8)H_{m,n}^{(p)} \quad (42)$$

Column iterations,

$$H_{m,n-1}^{(p+2)} - (3.2)H_{m,n}^{(p+2)} + u_{m,n+1}^{(p+2)} = -H_{m-1,n}^{(p+1)} - H_{m+1,n}^{(p+1)} + (0.8)H_{m,n}^{(p+1)} \quad (43)$$

the results converge after iteration 8, the results are:

H1= 1.20	H2 = 1.40	H3 = 1.60	H4 = 1.80	H5 = 1.20	H6 = 1.40
H7=1.60	H8 = 1.80	H9 = 1.20	H10 = 1.40	H11 = 1.60	H12 = 1.80
H13 = 1.20	H14 = 1.40	H15 = 1.60	H16 = 1.80	H17 = 1.20	H18 = 1.40
H19 = 1.60	H20 = 1.80	H21 = 1.20	H22 = 1.40	H23 = 1.60	H24 = 1.80

Problem 3 (Well Drawdown in A Confined Aquifer): A well fully penetrates a horizontal isotropic aquifer of thickness 30m. The confined aquifer is assumed to be of circular shape and the discharging well is located at the centre of the aquifer, the radius of the homogeneous, isotropic aquifer is 1100 m. The transmissivity, T is 400 m^2/d and the pumping rate Q from the well is 2000 m^3/d. Before pumping, the static water level in the well is 30 m, the hydraulic head along the circular boundary of the aquifer is 30 m and it is assumed that drawdown extends out to a radial distance of 1100 m from the well. That is the static water level remains unaffected for distances that exceed 1100 m from the well. Take $\Delta x = 100$. (Karvonen, 2002).

Mathematically,

$\Delta x = \Delta y = 100$, $T = 400\ m^2/d$,

$R = 0.2 m/d$, $Q = 2000\ m^3/d$

$$R = -\frac{Q}{(\Delta x)^2} = \frac{2000}{10000} = -0.2$$

$$\frac{R(\Delta x)^2}{T} = 0.2 * 100 * \frac{100}{400} = 5$$

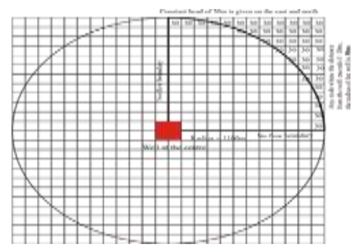

Figure 4: Schematic Diagram

As shown in Figure 4, only one quadrant will be considered and use rectangular coordinates for the solution. Because of symmetry, it is required that the line x and y axes be no-flow boundary. We can also Figure 4 and table 1 that all the nodes that the distance from the well is greater than 1100m takes the value 30m. There is No flow boundary on the left and bottom of the quadrant and constant head on the right and top boundary = 30m.

Table 1: The finite difference grid

30	30	30	30	30	30	30	30	30	30	30	30
H1	H2	H3	H4	H5	30	30	30	30	30	30	30
H6	H7	H8	H9	H10	H11	H12	30	30	30	30	30
H13	H14	H15	H16	H17	H18	H19	H20	30	30	30	30
H21	H22	H23	H24	H25	H26	H27	H28	H29	30	30	30
H30	H31	H32	H33	H34	H35	H36	H37	H38	H39	30	30
H40	H41	H42	H43	H44	H45	H46	H47	H48	H49	30	30
H50	H51	H52	H53	H54	H55	H56	H57	H58	H59	H60	30
H61	H62	H63	H64	H65	H66	H67	H68	H69	H70	H71	30
H72	H73	H74	H75	H76	H77	H78	H79	H80	H81	H82	30
H83	H84	H85	H86	H87	H88	H89	H90	H91	H92	H93	30
H94	H95	H96	H97	H98	H99	H100	H101	H102	H103	H104	30

Method 1: SOR

Using equation (10) for the nodes without the pump and equation (11) for the pumping node with $\omega = 1.5$, starting the iteration at k = 0 and applying the boundary at the left and the bottom using

$\frac{dy}{dx} = \frac{H_{i+1,j} - H_{i-1,j}}{2\Delta x} = 0$ which implies that $H_{i+1,j} = H_{i-1,j}$ for example $H_{2,1} = H_{0,1}$

$$H_{m,n}^{k+1} = (-0.5)H_{m,n}^{k} + 1.5\left(\frac{H_{m-1,n}^{k+1} + H_{m,n-1}^{k+1}\ H_{m+1,n}^{k} + H_{m,n+1}^{k}}{4}\right) \tag{44}$$

for the pumping node, H94

$$H_{m,n}^{k+1} = (-0.5)H_{m,n}^{k} + 1.5\left(\frac{H_{m-1,n}^{k+1} + H_{m,n-1}^{k+1}\ H_{m+1,n}^{k} + H_{m,n+1}^{k}}{4} - 5\right) \tag{45}$$

The results converge after iteration 93

Table 2: Results of SOR after 93 iterations

30	**30**	**30**	**30**	**30**	**30**	**30**	**30**	**30**	**30**	**30**	**30**
29.91	29.92	29.92	29.93	29.96	**30**	**30**	**30**	**30**	**30**	**30**	**30**
29.82	29.83	29.84	29.86	29.89	29.92	29.95	**30**	**30**	**30**	**30**	**30**
29.73	29.73	29.75	29.78	29.81	29.86	29.9	29.95	**30**	**30**	**30**	**30**
29.62	29.63	29.65	29.69	29.73	29.78	29.83	29.89	29.95	**30**	**30**	**30**
29.49	29.5	29.54	29.58	29.64	29.71	29.77	29.83	29.9	29.96	**30**	**30**
29.35	29.36	29.41	29.47	29.55	29.62	29.7	29.78	29.85	29.92	**30**	**30**
29.17	29.19	29.26	29.35	29.45	29.55	29.64	29.73	29.81	29.89	29.96	**30**
28.93	28.98	29.09	29.22	29.35	29.47	29.58	29.69	29.78	29.86	29.93	**30**
28.6	28.71	28.9	29.09	29.26	29.41	29.54	29.65	29.75	29.84	29.92	**30**
28.03	28.37	28.71	28.98	29.19	29.36	29.5	29.63	29.73	29.83	29.92	**30**
26.78	28.03	28.6	28.93	29.17	29.35	29.49	29.62	29.73	29.83	29.91	**30**

Method 2: ADI

Using figure 4 and equations (13) and (15), starting the iteration at p = 0 and applying the boundary conditions on the left and at the bottom since they are no flow boundaries.

$\frac{dy}{dx} = \frac{H_{i+1,j} - H_{i-1,j}}{2\Delta x} = 0$ which implies that $H_{i+1,j} = H_{i-1,j}$ and $H_{i,j+!} = H_{i,j-1}$

$H_{2,1} = H_{0,1}$ and $H_{1,2} = H_{1,0}$

For Row Iteration at p = o,

$$H^{(1)}_{m-1,n} - 4H^{(1)}_{m,n} + H^{(1)}_{m+1,n} = -H^{(0)}_{m,n-1} - H^{(0)}_{m,n+1} \quad (46)$$

For Column Iteration at p = o,

$$H^{(1)}_{m,n-1} - 4H^{(1)}_{m,n} + H^{(1)}_{m,n+1} = -H^{(0)}_{m-1,n} - H^{(0)}_{m+1,n} \quad (47)$$

And for the pumping node, using equations (16) and (17)

For Row Iteration at p = o,

$$H^{(1)}_{m-1,n} - 4H^{(1)}_{m,n} + H^{(1)}_{m+1,n} = -H^{(0)}_{m,n-1} - H^{(0)}_{m,n+1} - 5 \quad (48)$$

For Column Iteration at p = o,

$$H^{(1)}_{m,n-1} - 4H^{(1)}_{m,n} + H^{(1)}_{m,n+1} = -H^{(0)}_{m-1,n} - H^{(0)}_{m+1,n} - 5 \quad (49)$$

Alternating the row and the column, the results converge after iteration 209

Table 3: Results of ADI after 209 iterations

30	**30**	**30**	**30**	**30**	**30**	**30**	**30**	**30**	**30**	**30**	**30**
29.91	29.92	29.92	29.93	29.96	**30**	**30**	**30**	**30**	**30**	**30**	**30**
29.82	29.83	29.84	29.86	29.89	29.92	29.96	**30**	**30**	**30**	**30**	**30**
29.73	29.73	29.75	29.78	29.81	29.85	29.9	29.95	**30**	**30**	**30**	**30**
29.62	29.63	29.65	29.68	29.73	29.78	29.83	29.89	29.95	30	**30**	**30**
29.49	29.5	29.54	29.58	29.64	29.7	29.77	29.83	29.9	29.96	**30**	**30**
29.35	29.36	29.41	29.47	29.55	29.62	29.7	29.78	29.85	29.92	**30**	**30**
29.16	29.19	29.26	29.35	29.45	29.55	29.64	29.73	29.81	29.89	29.96	**30**
28.93	28.98	29.09	29.22	29.35	29.47	29.58	29.68	29.78	29.86	29.93	**30**
28.6	28.71	28.9	29.09	29.26	29.41	29.54	29.65	29.75	29.84	29.92	**30**
28.03	28.37	28.71	28.98	29.19	29.36	29.5	29.63	29.73	29.83	29.92	**30**
26.78	28.03	28.6	28.93	29.16	29.35	29.49	29.62	29.73	29.82	29.91	**30**

Method 3: IADI

Using Table 1 and equations (18) and (19), generate the equations for the Improved Alternating Direction implicit method (ADI) At p = 0 and k = 1.1, applying the boundary condition as in the second method the row equation becomes

$$H^{(1)}_{m-1,n} - (3.1)H^{(1)}_{m,n} + H^{(1)}_{m+1,n} = -H^{(0)}_{m,n-1} - H^{(0)}_{m,n+1} + (0.9)H^{(0)}_{m,n} \quad (50)$$

the column equation becomes

$$H^{(2)}_{m,n-1} - (3.1)H^{(2)}_{m,n} + u^{(2)}_{m,n+1} = -H^{(1)}_{m-1,n} - H^{(1)}_{m+1,n} + (0.9)H^{(1)}_{m,n} \quad (51)$$

At the point of the well, for p = 0, k=1.1, using equations (20) for row iteration and equation (21) for column iteration

$$H^{(1)}_{m-1,n} - (3.1)H^{(1)}_{m,n} + H^{(1)}_{m+1,n} = -H^{(0)}_{m,n+1} - H^{(0)}_{m,n-1} + (0.9)H^{(0)}_{m,n} - 5 \quad (52)$$

While the column iteration uses

$$H^{(2)}_{m,n-1} - (3.1)H^{(2)}_{m,n} + H^{(2)}_{m,n+1} = -H^{(1)}_{m-1,n} - H^{(1)}_{m+1,n} + (0.9)H^{(1)}_{m,n} - 5 \quad (53)$$

Alternating the row and the column, the result after iteration 48 is presented in Table 4

Table 4: Results of ADI after

30	**30**	**30**	**30**	**30**	**30**	**30**	**30**	**30**	**30**	**30**	**30**
29.91	29.92	29.92	29.93	29.96	**30**	**30**	**30**	**30**	**30**	**30**	**30**
29.83	29.83	29.84	29.86	29.89	29.92	29.96	**30**	**30**	**30**	**30**	**30**
29.73	29.73	29.75	29.78	29.81	29.85	29.9	29.95	**30**	**30**	**30**	**30**
29.62	29.63	29.65	29.68	29.73	29.78	29.83	29.89	29.95	**30**	**30**	**30**
29.49	29.51	29.54	29.58	29.64	29.7	29.77	29.83	29.9	29.96	**30**	**30**
29.35	29.36	29.41	29.47	29.55	29.63	29.7	29.78	29.85	29.92	**30**	**30**
29.17	29.19	29.26	29.35	29.45	29.55	29.64	29.73	29.81	29.89	29.96	**30**
28.93	28.98	29.09	29.22	29.35	29.47	29.58	29.69	29.78	29.86	29.93	**30**
28.6	28.71	28.9	29.09	29.26	29.41	29.54	29.65	29.75	29.84	29.92	**30**
28.03	28.37	28.71	28.98	29.19	29.36	29.51	29.63	29.73	29.83	29.92	**30**
26.78	28.03	28.6	28.93	29.17	29.35	29.49	29.62	29.73	29.83	29.91	**30**

Problem 4: Transient Groundwater Flow
A rectangular aquifer is divided into a 7x7 grid with uniform grid-distances of 100m in both x and y direction. The Transmissivity is homogeneous at a value of $0.1m^2/s$. The Storage coefficient is 0.001. The North and South Boundaries are impervious with zero flux. The west and East boundaries are constant head boundaries with head values at 50m, which is also the initial head at all nodes at time t = 0. The discretized recharge due to precipitation is zero (0). There is a pumping well, starting operation at time t = 0 at a constant rate of $1m^3/s$ and it is located in the node (4,4), take $\Delta t = 10s$, Compute heads for the interior nodes at time t = 50s and 100s (Kinzelbach, 1986).

Given: $T = 0.1m^2/s$; $S = 0.001$; $R = 1m^3/s$; H (east and west) = 50m; H_y (north and south) = 0; $\Delta t = 10s$, and $\Delta x = \Delta y = 100m$

Table 5: Mathematical application of Transient flow

	100	200	300	400	500	600	700
100	50	H1	H2	H3	H4	H5	50
200	50	H6	H7	H8	H9	H10	50
300	50	H11	H12	H13	H14	H15	50
400	50	H16	H17	well	H19	H20	50
500	50	H21	H22	H23	H24	H25	50
600	50	H26	H27	H28	H29	H30	50
700	50	H31	H32	H33	H34	H35	50

$\frac{R}{T} = \frac{1}{0.1} = 10,\ \gamma = \frac{T\Delta t}{S(\Delta x)^2} = \frac{0.1*10}{0.001*10000} = 0.1$ and for pumping, $\frac{R(\Delta x)^2}{T} = \frac{0.0001*100^2}{0.1} = 10$
Method1: EFDM
Using table 5 and equation (54) for the nodes without a well at time t = 0, $\gamma = 0.1$

$$H^1_{m,n} = H^0_{m,n} + 0.1\left(H^0_{m+1,n} + H^0_{m-1,n} + H^0_{m,n+1} + H^0_{m,n-1} - 4H^0_{m,n}\right) \quad (54)$$

And equation (55) for the nodes with the pumping well (Note that pumping takes place at node 18 that is H18)

$$H^1_{m,n} = H^0_{m,n} + 0.1\left(H^0_{m+1,n} + H^0_{m-1,n} + H^0_{m,n+1} + H^0_{m,n-1} - 4H^0_{m,n} - 10\right) \quad (55)$$

All initial values = 50 and applying the boundary conditions
Using the result of time t = 0, compute the results at the next time step of time t = 10, and the results at time t = 100 seconds are presented in the tables 6.

Table 6: Result of EFDM at t = 100secs

50.00	50.00	49.98	49.96	49.98	50.00	50.00
50.00	49.97	49.91	49.80	49.91	49.97	50.00
50.00	49.91	49.64	49.14	49.64	49.91	50.00
50.00	49.81	49.14	49.85	49.14	49.81	50.00
50.00	49.91	49.64	49.14	49.64	49.91	50.00
50.00	49.97	49.91	49.80	49.91	49.97	50.00
50.00	50.00	49.98	49.96	49.98	50.00	50.00

Method2: Crank Nicolson Method

Using equation (56) for the nodes without the pumping well at time t = 0, $\varphi = 0.1$

$$H_{i,j}^{1} = H_{i,j}^{0} + \frac{1}{2}0.1\big(H_{i+1,j}^{1} + H_{i-1,j}^{1} + H_{i,j+1}^{1} + H_{i,j-1}^{1} - 4H_{i,j}^{1} + \; + H_{i+1,j}^{0} + + H_{i-1,j}^{0} + H_{i,j+1}^{0} + H_{i,j-1}^{0} - 4H_{i,j}^{0}\big) \tag{56}$$

And equation (57) for the nodes with the pumping well (Note that pumping takes place at node 18 that is H18)

$$H_{i,j}^{1} = H_{i,j}^{0} + \frac{1}{2}0.1\big(H_{i+1,j}^{1} + H_{i-1,j}^{1} + H_{i,j+1}^{1} + H_{i,j-1}^{1} - 4H_{i,j}^{1} + + H_{i+1,j}^{0} + + H_{i-1,j}^{0} + H_{i,j+1}^{0} + H_{i,j-1}^{0} - 4H_{i,j}^{0} - 2(10)\big) \tag{57}$$

Use the result of time t = 0 to compute the results at the next time step of time t = 10 seconds, and the results of distribution of heads at time t = 100s are presented in the tables 7.

Table 7: Result of CNM at t = 100secs

50.00	49.99	49.97	49.95	49.97	49.99	50.00
50.00	49.97	49.90	49.80	49.90	49.97	50.00
50.00	49.91	49.65	49.17	49.65	49.91	50.00
50.00	49.81	49.17	46.91	49.17	49.81	50.00
50.00	49.91	49.65	49.17	49.65	49.91	50.00
50.00	49.97	49.90	49.80	49.90	49.97	50.00
50.00	49.99	49.97	49.95	49.97	49.99	50.00

Method3: Fractional Step Method

Applying the boundary condition and using equation (58) is for row iteration without pumping well while equation (59) is for row iteration with pumping well. at time t = 0, $\alpha = 0.1$

$$H_{i,j}^{1} = H_{i,j}^{0} + 0.25(0.1)\big(H_{i+1,j}^{1} - 2H_{i,j}^{1} + H_{i-1,j}^{1} + H_{i+1,j}^{0} - 2H_{i,j}^{0} + H_{i-1,j}^{0}\big) \tag{58}$$

$$H_{i,j}^{1} = H_{i,j}^{0} + 0.25(0.1)\left(H_{i+1,j}^{1} - 2H_{i,j}^{1} + H_{i-1,j}^{1} + H_{i,j+1}^{0} - 2H_{i,j}^{0} + H_{i+1,j}^{0} - 2(10)\right) \tag{59}$$

Using equation (60) is for column iteration without pumping well while equation (61) is for column iteration with pumping well. at time t = 0, $\alpha = 0.1$

$$H_{i,j}^{1} = H_{i,j}^{0} + 0.25(0.1)\big(H_{i,j+1}^{1} - 2H_{i,j}^{1} + H_{i,j-1}^{1} + H_{i,j+1}^{0} - 2H_{i,j}^{0} + H_{i,j-1}^{0}\big) \tag{60}$$

$$H_{i,j}^{1} = H_{i,j}^{0} + 0.25(0.1)\left(H_{i,j+1}^{1} - 2H_{i,j}^{1} + H_{i,j-1}^{1} + H_{i,j+1}^{0} - 2H_{i,j}^{0} + H_{i,j-1}^{0} - 2(10)\right) \tag{61}$$

Impose boundary conditions and apply equations (58) to (61) alternatively and the results at time t =100 seconds are shown in the table 8

Table 8: Result of FSM at t = 100secs

50	50	50	50	50	50	50
50	50	50	49.97	50	50	50
50	50	49.95	49.62	49.95	50	50
50	49.98	49.7	46.73	49.7	49.98	50
50	50	49.95	49.62	49.95	50	50
50	50	50	49.97	50	50	50
50	50	50	50	50	50	50

RESULTS AND DISCUSSION

The results of the problems were plotted and presented in figures 5, 6, 7 and 8 respectively
Problem 1:

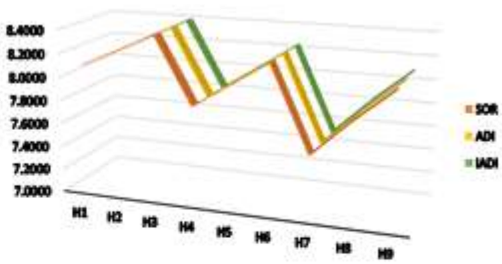

Table 9: No of iterations of the 3 methods of prob 1

Method	No. of iteration
SOR	8
ADI	11
IADI	6

Fig 5. Comparison of the three methods in problem 1

For the steady-state flow, Figure 5 showed that the results of the three methods used yield the same results but in table 9, IADI converged faster than the other two methods, this showed IADI is the best method of solution.
Problem 2:

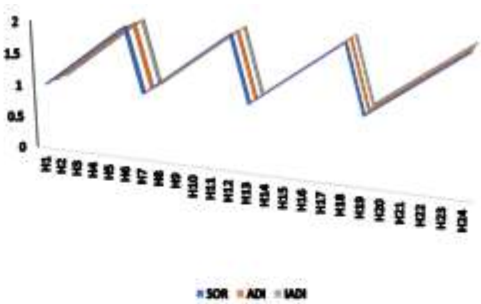

Table 10: No of iterations of the 3 methods (prob 2)

Method	No. of Iteration
SOR	11
ADI	18
IADI	8

Fig 6. Comparison of the three methods in problem 2

Figure 6 showed that the results of the three methods used yield the same results but in table 10, IADI converged faster than the other two methods, this showed IADI is the best method of solution.

Problem 3:

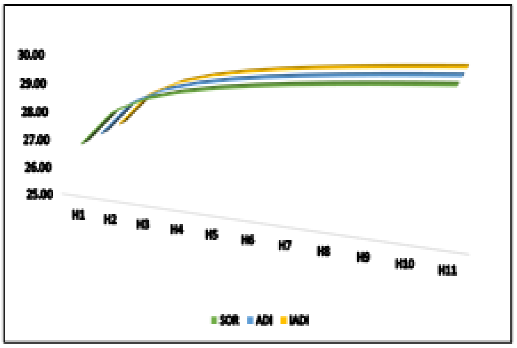

Fig 7. Comparison of the three methods in problem 3

Table 11: No of iterations of the 3 methods of prob 3

Method	No. Of Iteration
SOR	93
ADI	213
IADI	48

Figure 7 showed that the results of the three methods used yield the same results but in table 11, IADI converged faster than the other two methods, this showed IADI is the best method of solution.
Problem 4:

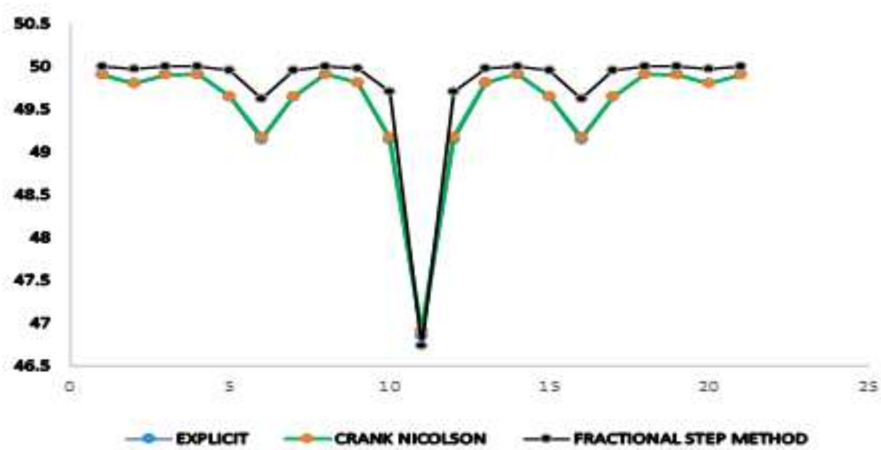

Fig 8: Comparison of the three methods of problem 4

For the transient flow, in figure 8, the three methods are reliable for the solution of groundwater flow but the explicit finite difference method gives the same result as the existing solution.

Conclusion: In this study we have been able to combine the mathematical modeling of both the transient and steady-state groundwater flow and applied three different methods of solutions for each state. The problems considered showed that the three methods used are efficient and reliable in solving transient and steady-state groundwater flow and can be used for all partial differential equations of real-life problems and that the direction of flow of groundwater is from higher hydraulic head to lower hydraulic head.

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