Research Article

(n,t)-Copresented Modules and (n,t)-Cocoherent Rings

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ABSTRACT

In this paper, for some hereditary torsion theory (T, F) with associated torsion radical t, the concepts of t-finitely cogenerated (t-fcg) modules and t-finitely copresented (t-fcp) modules are introduced as duals of t-finitely generated modules and t-finitely presented modules, respectively, of M. F. Jones (1982). These concepts also generalize the notions of cofinitely generated and cofinitely related modules.

Using the idea of t-finitely cogenerated module, the notion of (n, t)-copresented modules is introduced for some non-negative integer n. This notion of (n, t)-copresented modules is dual to (n, t)-presented modules studied by Dor and Mbuntum (2015) and generalizes the notion of n-copresented modules by Bennis et al (2012). In this process, we characterize t-finitely copresented modules (t-fcp), (n, t)-copresented modules, (n, t)-cocoherent rings and (n, 0,t)-projective modules.

Key Words: t-finitely cogenerated modules, t-finitely copresented modules, (n,t)-copresented modules, (n,t)-coherent rings, (n,0,t)-projective modules, (n,t)-cocoherent rings MSC2000:16D10, 16D80, 16E30, 16E60, 16S90, 18G05

RESUME

Dans cet article, pour certaine théorie de torsion héréditaire (T, F) associée au radical t, les notions de modules t-finiment coengendré et modules t-finiment coprésentés sont introduites comme des duaux de modules t-finiment engendré et modules t-finiment présentés de Jones (1982) respectivement. Ce notions généralisent aussi les concepts de modules cofiniment engendré et cofiniment relies.

Se basant sur l'idée de module t-finiment coengendré, la notion de module (n, t) – coprésenté est introduite pour des entiers positifs n. Cette notion de module (n, t) – coprésenté est duale de celle de module (n, t) – présenté considérée par Dor et Mbuntum (2015) et généralise la notion de module n –coprésenté de Bennis et al (2012).

Dans cette optique, nous caractérisons les modules t-finiment coprésentés, les modules (n, t) coprésentés, les anneaux (n, t) – cocohérents et les modules (n,0,t)-projectifs.

Mots Clés: modules t-finiment coengendré, modules t-finiment copéesentés, modules (n, t) –coprésentés, modules (n,0,t)-projectifs, anneaux (n, t) – cocohérents

1 Introduction

Let R be a ring. An R-module M is said to be finitely presented (f.p.) if there exists an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ with K finitely generated (f.g.) and F f.g. and free. A ring for which every f.g. right ideal is f.p. is called a right coherent ring. Coherent rings were first studied by Chase [3]. Chase's characterizations of coherent rings has led to several similar characterizations of coherence relative to a hereditary torsion theory. In particular, Jones [11] studied "Coherence Relative to an Hereditary Torsion Theory".

Recall that a subclass $\mathbb{T}$ of right R-modules is called a hereditary torsion class if it is closed under submodules, homomorphic images, extensions and direct sums. $\mathbb{T}$ uniquely determines a torsion-free class
$\mathbb{F} = \{F|\ Hom_R(T,\ F) = 0,\ \text{for all } T \in \mathbb{T}\}$.
$\mathbb{F}$ is closed under submodules, extensions, injective hulls and direct products. The pair $(\mathbb{T},\ \mathbb{F})$ is called a hereditary torsion theory for right R-modules. There is a left exact torsion radical t associated with each hereditary torsion theory $(\mathbb{T},\ \mathbb{F})$. For each R-module M, t(M) denotes the largest submodule of M in $\mathbb{T}$. M is torsion (M$\in \mathbb{T}$) if and only if T(M) = M, while M is torsion-free (M$\in \mathbb{F}$) if and only if t(M) = 0.

Jones [11] defined an R-module M to be t-finitely generated (t-fg) if there exists a f.g. submodule N of M such that $M/N \in \mathbb{T}$. M is said to be t-finitely presented (t-fp) if there exists an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ with F f.g. and free and K t-fg. A ring R is said to be right t-coherent if every f.g. right ideal of R is t-fp, where t is the torsion radical corresponding to the hereditary torsion theory $(\mathbb{T},\ \mathbb{F})$.

The concept of finitely presented modules has been generalized to the concept of n-presented modules which has led to the study of n-coherent rings by several authors (see for example Costa [5], Chen and Ding [4], Zhanmin Zhu [17], [18], D. Zhou [16]). Dor and Mbuntum [6] combined the notions of t-coherent rings and n-coherent rings to introduce and study (n,t)-presented modules and (n,t)-coherent rings.

Recently Bennis et al [2] introduced and studied n-copresented modules and n-cocoherent rings, concepts which arise from the notion of finitely cogenerated modules which are dual to finitely generated modules.

P. Vámos in 1968 defined and studied finitely embedded modules as a dual to finitely generated modules. J.P. Jans in his paper of 1969 on co-Noetherian rings called them cofinitely generated modules. V.A. Hiremath in 1982 obtained a categorical justification for finitely embedded modules as duals of finitely generated modules and derived some more properties of cofinitely generated modules. Hiremath in the same paper also introduced the notion of cofinitely related modules as duals to finitely related modules.

For a hereditary torsion $(\mathbb{T}, \mathbb{F})$ with associated torsion radical t, submodules N of a module M for which $M/N \in \mathbb{F}$ are said to be t-pure by Golan in [8] (1975) . Teply in [13] (1986), calls them t-closed submodules.

We use a hereditary torsion theory to define new notions in relative homological algebra and using ideas from both torsion theory and homological algebra, we consider some properties of these notions.
All rings are associative with identity and modules are unitary right R-modules unless otherwise stated. For any module M, E(M) denotes the injective envelope of M.
We begin with some definitions

1. An R-module M is finitely cogenerated (fcg) if for every family $\{A_\lambda\}_\Lambda$ of submodules of M with $\bigcap_\Lambda A_\lambda = 0$, there is a finite subset $E \subset \Lambda$ such that $\bigcap_E A_\lambda = 0$.

   Proposition 1.1. *(Proposition [2])*
   The following statements are equivalent for an R-module M:

   (a) M is finitely cogenerated (fcg).

(b) *For every set* $f_\alpha : M \longrightarrow U_\alpha$ $(\alpha \in A)$ *with* $\bigcap_A Ker f_\alpha = 0$*, there is a finite subset* $F \subset A$ *with* $\bigcap_F Ker f_\alpha = 0$

(c) *For every index set* $\{U_\alpha\}_{\alpha \in A}$ *and monomorphism* $0 \longrightarrow M \longrightarrow \prod_A U_\alpha$*, there is a finite subset* $F \subset A$ *and a monomorphism* $0 \longrightarrow M \longrightarrow \prod_F U_\alpha$

(d) *There is a finite set* $\{S_i,\ i = 1, 2, \ldots, n\}$ *of simple R-modules, such that* $E(M) = E(S_1) \oplus E(S_2) \oplus \cdots \oplus E(S_n)$*, where for any module N,* $E(N)$ *is the injective hull of N.*

(e) *There is a finite set* $\{S_i,\ i = 1, 2, \ldots, n\}$ *of simple R-modules, such that M is isomorphic to a submodule of* $E(S_1) \oplus E(S_2) \oplus \cdots \oplus E(S_n)$*.*

For the proof of this proposition and more properties of fcg modules see [1] and [15].

2. A module M is called finitely copresented (fcp) if there exists an exact sequence $0 \longrightarrow M \longrightarrow K \longrightarrow L \longrightarrow 0$ where K is finitely cogenerated cofree and L is finitely cogenerated. Bennis et al in [2] showed that this is equivalent to showing that there exists an exact sequence $0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1$ with each I_i fcg and injective, $i = 0, 1$. For more properties of fcp modules see [15].
3. Let ($\mathbb{T}$, $\mathbb{F}$) be a hereditary torsion theory with corresponding radical t. Jones in [11] calls an R-module M t-finitely generated if there exists a finitely generated submodule N of M such that $M/N \in \mathbb{T}$ and M is said to be t-finitely presented if there exists an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ with F finitely generated and free and K t-finitely generated.
4. An R-module M is said to be n-copresented for some non-negative integer n if there exists an exact sequence $0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \ldots \longrightarrow I_n$ where each I_i is injective and finitely cogenerated. For more properties of n-copresented modules see [2] and [18]
5. Let ($\mathbb{T}$, $\mathbb{F}$) be a hereditary torsion theory with corresponding radical t. An R-module M is said to be (n,t)-presented if there exists an exact sequence $F_n \longrightarrow F_{n-1} \longrightarrow \ldots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ with each F_i free and t-finitely generated. See [6] for properties of (n,t)-presented modules.

2 t-finitely Cogenerated Modules and t-finitely copresented Modules

Definition 2.1. Let $(\mathbb{T}, \mathbb{F})$ be a hereditary torsion theory with corresponding radical t. An R-module M is t-finitely cogenerated (t-fcg) if there exists a finitely cogenerated submodule N of M such that $M/N \in \mathbb{F}$.

Remark 2.1.

1. Every finitely cogenerated module M is t-finitely cogenerated since $M/M = 0 \in \mathbb{F}$.
2. If an R-module M has a finitely cogenerated t-pure submodule N then $M/N \in \mathbb{F}$ and hence M is t-fcg.
3. If $\mathbb{F} = \{0\}$, then M is t-fcg if and only if M is fcg.
4. Every torsion-free module is t-fcg.

Example 2.1. *Let R be a commutative integral domain (e.g.* $R = \mathbb{Z}$*). A left R-module M is said to be torsion-free if* $0 \neq r \in R$ *and* $0 \neq m \in M$ *implies that* $0 \neq rm$*. The left(right) R-modules satisfying this condition form a torsion-free class* $\mathbb{F}$*. The torsion theory* ($\mathbb{T}$, $\mathbb{F}$) *cogenerated by* $\mathbb{F}$ *is referred to in [[8]: Example 1, p. 305] as the "ancestor" of torsion theories. If* $R = \mathbb{Z}$*, the* $\mathbb{Z}$*-module* $\mathbb{Z}_{p^\infty}$ *(or* $\mathbb{Z}(p^\infty)$*) is fcg, [[7], p. 16].* $\mathbb{Z}_{p^\infty}$ *(or* $\mathbb{Z}(p^\infty)$*) is thus t-fcg by Remark 2.1 (1), with respect to this torsion theory.*

Lemma 2.1. *1. Every submodule of a t-fcg module is t-fcg.*

2. A direct summand of a t-fcg module is t-fcg.

Proof:

1. Suppose M is t-fcg and M_1 is a submodule of M. Let N be a finitely cogenerated (fcg) submodule of M such that $M/N \in \mathbb{F}$. Then $M_1 \cap N$ is fcg as a submodule of the fcg module N. Also $M_1 \cap N \subseteq M_1$ and $M_1/(M_1 \cap N) \cong (M_1 + N)/N$. Moreover $(M_1 \cap N)/N \in \mathbb{F}$ since $(M_1 \cap N)/N \subseteq M/N$ and $\mathbb{F}$ is closed under submodules. So $M_1/(M_1 \cap N) \in \mathbb{F}$ and M_1 is t-fcg.
2. A direct summand of M is a submodule of M and hence result follows by (1). □

Lemma 2.2. *1. Let A and B be t-fcg modules. Then $A \oplus B$ is t-fcg.*

2. Let $B \xrightarrow{g} C \longrightarrow 0$ be an exact sequence of R-modules with C t-fcg. Then B is t-fcg.

Proof:

1. Let A' and B' be fcg submodules of A and B, respectively, such that $A/A',\ B/B' \in \mathbb{F}$. Then $A/A' \oplus B/B' \in \mathbb{F}$ since $\mathbb{F}$ is closed under direct sums. Moreover, $A' \oplus B'$ is fcg and there is an induced monomorphism $0 \longrightarrow (A \oplus B)/(A' \oplus B') \longrightarrow A/A' \oplus B/B'$. $\mathbb{F}$ is closed under submodules and therefore $(A \oplus B)/(A' \oplus B') \in \mathbb{F}$. Thus $A \oplus B$ is t-fcg.
2. Suppose C is t-fcg and let C' be a fcg submodule of C such that $C/C' \in \mathbb{F}$. Choose a fcg submodule B' of B such that $g(B') = C'$. Let $\bar{g} : B/B' \longrightarrow C/C'$ be the map induced by g. Then $\bar{g}$ is a well-defined isomorphism and hence $B/B' \in \mathbb{F}$. Thus B is t-fcg. □

Definition 2.2. An R-module M is said to be t-finitely copresented (t-fcp) if there exists an exact sequence $0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1$ where I_0 and I_1 are injective and t-finitely cogenerated and t is the torsion radical associated with some hereditary torsion theory ($\mathbb{T}$, $\mathbb{F}$).

Remark 2.2. 1. Every finitely copresented module is t-finitely copresented.

2. If $\mathbb{F} = \{0\}$, then M is t-fcp if and only if M is fcp.

Lemma 2.3. *[[1]: Lemma 18.9]*
Let M be an R-module and suppose $i : M \longrightarrow E$ an injective envelope of M. If Q is injective and $q : M \longrightarrow Q$ is a monomorphism, then Q has a decomposition $Q = E' \oplus E''$ such that

1. *$E' \cong E$*
2. *$Im\ q$ is a submodule of E'*
3. *$q : M \longrightarrow E'$ is an injective envelope of M*

Lemma 2.4. *If R is a hereditary ring and N is a submodule of an R-module M, then $E(M/N) = E(M)/N$. Thus if R is hereditary, the injective hull of a t-fcg R-module is t-fcg.*

Proof

If R is hereditary, then every homomorphic image of an injective module is injective. Thus $E(M)/N$ is injective and $E(M/N) = E(M)/N$. The last statement follows easily.

Proposition 2.5. *If M is t-fcp then for any exact sequence $0 \longrightarrow M \longrightarrow L \longrightarrow N \longrightarrow 0$ with L t-finitely cogenerated, N is also t-finitely cogenerated.*

Proof

M t-fcp implies there exists an exact sequence $0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1$ with I_0 and I_1 injective and t-finitely cogenerated. Let $K = Im(I_0 \longrightarrow I_1)$. Then K is t-fcg as a submodule of the t-fcg module I_1. Suppose there exists an exact sequence $0 \longrightarrow M \longrightarrow L \longrightarrow N \longrightarrow 0$ with L t-fcg. Then we can construct the following commutative pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & I_0 \arrow[r] \arrow[d] & K \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & K \arrow[r] & 0 \\
 & N \arrow[r, equal] \arrow[d] & N \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Since K is t-fcg, P is also t-fcg by Lemma 2.2(2). Also, I_0 injective implies that the sequence $0 \longrightarrow I_0 \longrightarrow P \longrightarrow N \longrightarrow 0$ splits and hence N is t-fcg as a direct summand of P by Lemma 2.1 (2). $\square$

Proposition 2.6. *Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of R-modules. Then*

1. *If A and C are t-fcp then B is also t-fcp.*
2. *If A is t-fcp and B is t-fcg then C is t-fcg.*
3. *If B is t-fcp and if the injective hull of a t-fcg R-module is t-fcg, then A is t-fcp.*

Proof

1. Suppose A and C are t-fcp. Then we have the exact sequences $0 \longrightarrow A \longrightarrow A_0 \longrightarrow A_1$ and $0 \longrightarrow C \longrightarrow C_0 \longrightarrow C_1$ with $A_0,\ A_1,\ C_0,\ C_1$ injective and t-fcg. Let $B_0 = A_0 \oplus C_0$ and $B_1 = A_1 \oplus C_1$. Then B_0 and B_1 are t-fcg. By simultaneous resolution, we obtain the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_0 \arrow[r] \arrow[d] & A_0 \oplus C_0 \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_1 \arrow[r] & A_1 \oplus C_1 \arrow[r] & C_1 \arrow[r] & 0
\end{tikzcd}$$

   and hence B is t-fcp.

2. Follows from Proposition 2.5.
3. Suppose B is t-fcp. Then there exists an exact sequence $0 \longrightarrow B \longrightarrow B_0 \longrightarrow B_1$, where

B_0 and B_1 are injective and t-fcg. We obtain the following pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & A \arrow[r] \arrow[d] & A_0 \arrow[r] \arrow[d] & A_1 \arrow[d] \\
0 \arrow[r] & B \arrow[r] \arrow[d] & B_0 \arrow[r] \arrow[d, "\beta"] & B_1 \arrow[d, "\beta_1"] \\
0 \arrow[r] & C \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d] & C_1 \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd}$$

where $A_0 = Ker\beta$ and $A_1 = Ker\beta_1$. Since B_0 and B_1 are t-fcg, A_0 and A_1 are t-fcg. We have the exact sequence $0 \longrightarrow A \longrightarrow E(A_0) \longrightarrow E(A_1)$. By hypothesis $E(A_0)$ and $E(A_1)$ are injective and t-fcg. Hence A is t-fcp.

3 (n,t)-copresented Modules

Definition 3.1. Let ($\mathbb{T}$, $\mathbb{F}$) be an hereditary torsion theory with corresponding radical t. An R-module M is said to be (n,t)-copresented if there exists an exact sequence
$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \ldots \longrightarrow I_n$ where each I_k is injective and t-fcg, $k = 0, 1, \ldots, n$.

Remark 3.1.

1. Every n-copresented module is (n,t)-copresented since every fcg module is t-fcg.
2. By definition, an R-module is (1,t)-copresented if and only if it is t-fcp.
3. It is clear that if M is (n,t)-copresented then M is (m,t)-copresented for every positive integer $m \leq n$.
4. If $\mathbb{F} = \{0\}$, then an R-module is (n,t)-copresented if and only if it is n-copresented.

Example 3.1. *Referring to the torsion theory in Example 2.1, consider the exact sequence* $0 \longrightarrow \mathbb{Z}_{p^k}(\mathbb{Z}(p^k)) \longrightarrow \mathbb{Z}_{p^\infty}(p^\infty) \longrightarrow \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$.
$\mathbb{Z}_{p^\infty}$ *and* $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ *are injective modules and they are fcg by [[7]: Theorem 25.1]. Hence they are t-fcg and by Remark 2.1 (1). Thus* $\mathbb{Z}_{p^k}$ *is t-fcp and (1,t)-copresented.*

Proposition 3.1. *If an R-module is (0,t)-copresented then it is t-fcg. The converse holds if the injective hull of a t-fcg module is t-fcg; in particular if R is hereditary.*

Proof

If M is (0,t)-copresented, then there exists an exact sequence $0 \longrightarrow M \longrightarrow I_0$, where I_0 is injective and t-fcg. Since every submodule of a t-fcg module is t-fcg, M is t-fcg.

Conversely, suppose the injective hull of a t-fcg module is t-fcg. If M is t-fcg, then M has a fcg submodule N such that $M/N \in \mathbb{F}$. Consider the exact sequence $0 \longrightarrow M \longrightarrow E(M)$. By hypothesis, E(M) is t-fcg and so M is (0,t)-copresented.

We use a method similar to that used by Zhu in [18] to characterize (n,t)-copresented modules.

Lemma 3.2. *Let A and B be R-modules and n a non-negative integer. Then* $A \oplus B$ *is (n,t)-copresented if and only if A and B are (n,t)-copresented.*

Proof

Suppose A and B are (n,t)-copresented. Then there exist exact sequences $0 \longrightarrow A \longrightarrow A_0 \longrightarrow \ldots \longrightarrow A_n$ and $0 \longrightarrow B \longrightarrow B_0 \longrightarrow \ldots \longrightarrow B_n$ with each A_i and B_i injective and t-fcg, $i = 0, 1 \ldots, n$. Using the Horse Shoe Lemma for injectives we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & A \oplus B & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_0 & \longrightarrow & A_0 \oplus B_0 & \longrightarrow & B_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_n & \longrightarrow & A_n \oplus B_n & \longrightarrow & B_n & \longrightarrow & 0
\end{array}$$

and so $A \oplus B$ is (n,t)-copresented since $A_i \oplus B_i$ is injective and t-fcg.
Conversely suppose $A \oplus B$ is (n,t)-copresented. Then we have an exact sequence
$0 \longrightarrow A \oplus B \xrightarrow{\varepsilon} E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} E_n$ with each E_i injective and t-fcg.
By Lemma 2.3, we have the exact sequence
$0 \longrightarrow A \longrightarrow E(\varepsilon(A)) \longrightarrow E(Im(d_0 i_0)) \longrightarrow E(Im(d_1 i_1)) \ldots \longrightarrow E(Im(d_{n-1} i_{n-1}))$
where $E(\varepsilon(A))$ is a direct summand of E_0, $E(Imd_k i_k)$ is a direct summand of E_{k+1}, i_0 natural injection from $E(\varepsilon(A))$ to E_0, i_k natural injection from $E(Imd_k i_k)$ to E_{k+1}, $k = 0, 1, \ldots, n-1$. By Lemma 2.1, A is (n,t)-copresented. Similarly B is (n,t)-copresented.

The following theorem is a characterization of (n,t)-copresented modules and generalizes Proposition 1.2 of [18]. It is also a dual of Theorem 1 and Theorem 2 of [6].

Theorem 3.3. *If the injective hull of a t-fcg module is t-fcg (e.g. if R is hereditary), then the following are equivalent for an R-module M:*

1. *M is (n,t)-copresented*
2. *There exists an exact sequence* $0 \longrightarrow M \longrightarrow I_0 \longrightarrow \ldots \longrightarrow I_{n-1} \longrightarrow L \longrightarrow 0$ *with each* I_k *injective and t-fcg and L t-fcg.*
3. *M is (n-1, t)-copresented and if there exists an exact sequence* $0 \longrightarrow M \longrightarrow E_0 \longrightarrow \ldots \longrightarrow E_{n-1} \longrightarrow L \longrightarrow 0$ *with each* E_k *injective and t-fcg then L is t-fcg.*
4. *There exists an exact sequence* $0 \longrightarrow M \longrightarrow E \longrightarrow L \longrightarrow 0$ *with E injective and t-fcg and L (n-1, t)-copresented*
5. *M is t-fcg and if the sequence* $0 \longrightarrow M \longrightarrow E \longrightarrow L \longrightarrow 0$ *is exact with E t-fcg then L is (n-1, t)-copresented.*

Proof

(1) $\Longrightarrow$ (2): Suppose M is (n,t)-copresented. Then there exists an exact sequence
$0 \longrightarrow M \longrightarrow E_0 \longrightarrow \cdots \longrightarrow E_{n-1} \xrightarrow{f} E_n$ with each E_i injective and t-fcg, $i = 0, 1, \ldots, n$.

Let $L = Imf$. Then L is t-fcg as a submodule of E_n and hence we have the exact sequence $0 \longrightarrow M \longrightarrow E_0 \longrightarrow \cdots \longrightarrow E_{n-1} \longrightarrow L \longrightarrow 0$ with each E_i injective and t-fcg and L t-fcg.
(2) $\Longrightarrow$ (3): Suppose there exists an exact sequence
$0 \longrightarrow M \longrightarrow E_0 \longrightarrow \cdots \longrightarrow E_{n-1} \longrightarrow L \longrightarrow 0$ with each E_i injective and t-fcg and L t-fcg. Then M is (n-1, t)-copresented. If
$0 \longrightarrow M \longrightarrow I_0 \longrightarrow \cdots \longrightarrow I_{n-1} \longrightarrow K \longrightarrow 0$ with each I_i injective and t-fcg, then by Schanuel's Lemma for injectives K is t-fcg.
(3) $\Longrightarrow$ (1): Assume (3). Then there exists an exact sequence
$0 \longrightarrow M \longrightarrow E_0 \longrightarrow \cdots \longrightarrow E_{n-2} \xrightarrow{g} E_{n-1}$. Let $K = kerg$ and $L = E_{n-1}/K$. Then we have the exact sequence
$0 \longrightarrow M \longrightarrow E_0 \longrightarrow \cdots \longrightarrow E_{n-1} \longrightarrow L \longrightarrow 0$ and by hypothesis L is t-fcg. Let $E_n = E(L)$, the injective envelope of L. L t-fcg implies $E(L)$ is also t-fcg by hypothesis. Hence we have the exact diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & E_0 \arrow[r] & \cdots & E_{n-1} \arrow[rr] \arrow[dr] & & E_n \\
 & & & & & L \arrow[ur] \arrow[dr] & \\
 & & & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

and so M is (n,t)-copresented.

(1) $\Longrightarrow$ (4): Suppose M is (n,t)-copresented. Then we have an exact sequence
$0 \longrightarrow M \longrightarrow E_0 \xrightarrow{f} E_1 \longrightarrow \cdots \longrightarrow E_n$ with E_i injective and t-fcg. If $L = Imf$ then the sequence
$0 \longrightarrow L \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots \longrightarrow E_n$ is exact and thus L is (n-1, t)-copresented. Also M is t-fcg since it is a submodules of E_0. Therefore we have the exact sequence
$0 \longrightarrow M \longrightarrow E_0 \longrightarrow L \longrightarrow 0$ with E_0 injective and t-fcg and L (n-1, t)-copresented.
(4) $\Longrightarrow$ (5): Assume (4). Then we have an exact sequence
$0 \longrightarrow M \longrightarrow E \longrightarrow L \longrightarrow 0$ with E injective and t-fcg and L (n-1, t)-copresented. M is t-fcg as a submodule of E. If $0 \longrightarrow M \longrightarrow E' \longrightarrow K \longrightarrow 0$ with E' injective and t-fcg then by Schanuel's lemma for injectives and Lemma 3.2, K is (n-1, t)-copresented.
(5) $\Longrightarrow$ (1): Assume (5). M t-fcg implies E(M) is t-fcg. Thus we have the exact sequence $0 \longrightarrow M \longrightarrow E(M) \longrightarrow E(M)/M \longrightarrow 0$ and by hypothesis $E(M)/M$ is (n-1, t)-copresented. Therefore we have the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & E(M) \arrow[rr] \arrow[dr] & & E_1 \arrow[r] & \cdots \arrow[r] & E_n \\
 & & & E(M)/M \arrow[ur] \arrow[dr] & & & \\
 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

and so M is (n,t)-copresented. □

The next theorem considers the behavior of (n,t)-copresented modules on short exact sequences.

Theorem 3.4. *Let R be a ring, $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence R-modules and n a non-negative integer. Then*

1. *If A and C are (n,t)-copresented, then B is (n,t)-copresented.*
2. *If C is (n-1, t)-copresented and B is (n,t)-copresented, then A is (n,t)-copresented.*
3. *If A is (n+1,t)-copresented and B (n,t)-copresented, then C is (n,t)-copresented*

Proof

1. The proof is similar to the proof of the first part of Lemma 3.2
2. Suppose C is (n-1, t)-copresented and B is (n,t)-copresented. Then we have the exact sequence $0 \longrightarrow B \longrightarrow B_0 \longrightarrow B_1 \longrightarrow \ldots \longrightarrow B_n$ with each B_i injective and t-fcg, $i = o, 1, \ldots, n$. From this sequence we obtain the following two exact sequences:
   $0 \longrightarrow B \longrightarrow B_0 \longrightarrow K \longrightarrow 0$ and $0 \longrightarrow K \longrightarrow B_1 \longrightarrow \ldots \longrightarrow B_n$ where
   $K = Im(B_0 \longrightarrow B_1) = Ker(B_1 \longrightarrow B_2)$.
   We then construct the pushout diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B_0 \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
 & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

   C and K are (n-1, t)-copresented implies by (1) that D is (n-1, t)-copresented. Thus we have the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B_0 \arrow[rr] \arrow[dr] & & D_0 \arrow[r] & D_1 \arrow[r] & \cdots & \arrow[r] & D_{n-1} \\
 & & & D \arrow[ur] \arrow[dr] & & & & & \\
 & & 0 \arrow[ur] & & 0 & & & &
\end{tikzcd}$$

   and hence A is (n,t)-copresented since we have the exact sequence
   $0 \longrightarrow A \longrightarrow B_0 \longrightarrow \quad \longrightarrow D_0 \longrightarrow \cdots \qquad \longrightarrow D_{n-1}$
   with B_0 and D_i, $\; i = 0, 1, \ldots, n-1$ injective and t-fcg.
3. Suppose A is (n+1, t)-copresented and B is (n,t)-copresented. A is (n+1, t)-copresented implies there exists an exact sequence $0 \longrightarrow A \longrightarrow A_0 \longrightarrow A_1 \longrightarrow \ldots \longrightarrow A_{n+1}$ with each A_i injective and t-fcg, $i = 0, 1, \ldots, n+1$. From this sequence we obtain the exact sequences $0 \longrightarrow A \longrightarrow A_0 \longrightarrow K \longrightarrow 0$ and $0 \longrightarrow K \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \ldots \longrightarrow A_{n+1}$ where $K = Im(A_0 \longrightarrow A_1) = Ker(A_1 \longrightarrow A_2)$ i.e. K is (n,t)-copresented. We

construct the pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_0 \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

K and B (n,t)-copresented implies by (1) that D is also (n,t)-copresented. A_0 injective implies that the exact sequence $0 \longrightarrow A_0 \longrightarrow D \longrightarrow C \longrightarrow 0$ splits. Hence $D = A_0 \oplus C$ and by Lemma 3.2 C is (n,t)-copresented.

Remark 3.2. By Remark 3.1(4), if $\mathbb{F} = \{0\}$, then an R-module is (n,t)-copresented if and only if it is n-copresented. Thus some of the results obtained in [2] and [18] are special cases of our results. In particular [[18], Proposition 1.2] is a special case of our Theorem 3.3, [[2]: Theorem 2.4 (1), (2), (3)] is a special case of our Theorem 3.4 and [[14]: Proposition 3*(1)] is a special case of our Lemma 2.1, when $\mathbb{F} = \{0\}$.

4 (n,t)-Cocoherent Rings

Definition 4.1. Let ($\mathbb{T}$, $\mathbb{F}$) be a hereditary torsion theory with radical t. For a positive integer n, a ring R is called right (n,t)-cocoherent if every (n,t)-copresented right R-module is (n+1, t)-copresented.

Theorem 4.1. *The following statements are equivalent for a ring R provided that the injective hull of every t-fcg module is t-fcg.*

1. *R is right (n,t)-cocoherent.*
2. *If the sequence*
   i) $0 \longrightarrow M \xrightarrow{d_0} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} \cdots \quad E_{n-1} \xrightarrow{d_n} E_n$
   is exact where each E_i is a t-fcg and injective right R-module, then there exists an exact sequence of right R-modules
   ii) $0 \longrightarrow M \xrightarrow{d_0} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} \cdots \quad E_{n-1} \xrightarrow{d_n} E_n \xrightarrow{d_{n+1}} E_{n+1}$
   where each E_i is t-fcg and injective.
3. *Every (n-1, t)-copresented factor module of a t-fcg injective right R-module is (n,t)-copresented.*

Proof

$1 \Longrightarrow 2$:

Suppose R is (n,t)-cocoherent and $0 \longrightarrow M \xrightarrow{d_0} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} \cdots \quad E_{n-1} \xrightarrow{d_n} E_n$ is exact with each E_i t-fcg and injective. Then we have the exact sequence

$0 \longrightarrow M \xrightarrow{d_0} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} \cdots \quad E_{n-1} \xrightarrow{d_n} E_n \longrightarrow E_n/Imd_n \longrightarrow 0$. By Theorem 3.3, E_n/Imd_n is t-fcg and by hypothesis $E_{n+1} = E(E_n/Imd_n)$ is injective and t-fcg. Hence we have the exact sequence $0 \longrightarrow M \xrightarrow{d_0} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} \cdots \quad E_{n-1} \xrightarrow{d_n} E_n \longrightarrow E_{n+1}$

with each E_i t-fcg and injective.
$2 \Longrightarrow 1$ is clear.
$1 \Longleftrightarrow 3$ follows from Theorem 3.3. □

Proposition 4.2. *If R is an (n,t)-cocoherent ring, then every (n,t)-copresented R-module M is infinitely t-copresented, in the sense that M is (m,t)-copresented for every positive integer m.*

Proof
Suppose M is (n,t)-copresented. Then there exists an exact sequence
$0 \longrightarrow M \xrightarrow{d_0} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} \cdots \quad E_{n-1} \xrightarrow{d_n} E_n$ with each E_i t-fcg and injective. This gives rise to the exact sequence $0 \longrightarrow M \longrightarrow E_0 \longrightarrow M_1 \longrightarrow 0$, where $M_1 = Imd_1 = Kerd_2$. Since R is (n,t)-cocoherent, M is (n+1, t)-copresented and hence M_1 is (n,t)-copresented. M_1 (n,t)-copresented implies M_1 is (n+1, t)-copresented since R is (n,t)-cocoherent and therefore M is (n+2, t)-copresented. Continuing this way, we find that M is (m,t)-copresented for every $m \geq n$ and so M is infinitely t-copresented. □

Proposition 4.3. *Let n be a positive integer. If R is (n, t)-cocoherent, then R is (m, t)-cocoherent for every positive integer $m \geq n$.*

Proof
Let M be an R-module and m and n positive integers with $m \geq n$. Suppose M is (m, t)-copresented. Then M is (n, t)-copresented since $m \geq n$. R (n, t)-cocoherent implies that M is infinitely t-copresented by Proposition 4.2. In particular, M is (m+1, t)-copresented and thus R is (m, t)-cocoherent. □

Definition 4.2. Let n and d be non-negative integers. An R-module M is said to be (n, d, t)-projective if $Ext_R^{d+1}(M,\ A) = 0$ for every (n, t)-copresented R-module A.

Proposition 4.4. *Let $\{M_i\}_{i\in I}$ be a family of R-modules. Then $\bigoplus_{i\in I} M_i$ is (n, d, t)-projective if and only if each M_i is (n, d, t)-projective.*

Proof
$Ext_R^{d+1}(M_i,\ A) = 0$ if and only if $0 = \prod_{i\in I} Ext_R^{d+1}(M_i,\ A) = Ext_R^{d+1}(\bigoplus_{i\in I} M_i,\ A)$.□

Proposition 4.5. *Let P be a projective R-module and K a submodule of P. If P/K is (n, d, t)-projective, then K is (n+1, d, t)-projective.*

Proof
Let A be an (n+1, t)-copresented R-module. Then there exists an exact sequence $0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$, where E is a t-fcg injective module and B is (n, t)-copresented module. This yields the following two exact sequences:
$$0 = \quad Ext_R^{d+1}(P,\ A) \longrightarrow Ext_R^{d+1}(K,\ A) \longrightarrow Ext_R^{d+2}(P/K,\ A) \longrightarrow Ext_R^{d+2}(P,\ A) = 0$$
and $0 = Ext_R^{d+1}(P/K,\ E) \longrightarrow Ext_R^{d+1}(P/K,\ B) \longrightarrow Ext_R^{d+2}(P/K,\ A) \longrightarrow Ext_R^{d+2}(P/K,\ E) = 0$
since P is projective and E is injective and (n, t)-copresented. Hence $Ext_R^{d+1}(K,\ A) \cong Ext_R^{d+1}(P/K,\ B) = 0$ since B is (n, t)-copresented and P/K is (n, d, t)-projective. Thus K is (n+1, d, t)-projective. □

Definition 4.3. 1. A short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \qquad (4.1)$$

is t-copure if

$$0 \longrightarrow Hom_R(C,\ M) \longrightarrow Hom_R(B,\ M) \longrightarrow Hom_R(A,\ M) \longrightarrow 0 \qquad (4.2)$$

is exact for every t-copresented R-module M.

2. If the sequence 4.2 is exact for every (n, t)-copresented R-module M, the sequence 4.1 is said to be (n, t)-copure
3. A submodule A of B is said to be t-copure in B if the sequence $0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$ is t-copure. A factor module N of B is said to be t-copure if there is a t-copure short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow N \longrightarrow 0$.

Proposition 4.6. *Let n and d be non-negative integers with $n \geq d+1$. Then every t-copure factor module of an (n, d, t)-projective module is (n, d, t)-projective.*

Proof

Let N be a t-copure factor module of an (n, d, t)-projective module M. Then there exists a t-copure exact sequence $0 \longrightarrow K \xrightarrow{f} M \longrightarrow N \longrightarrow 0$. Let A be an (n, t)-copresented R-module. Then there exists an exact sequence

$$0 \longrightarrow A \longrightarrow I_0 \xrightarrow{g_0} I_1 \xrightarrow{g_1} \cdots \longrightarrow I_{n-1} \xrightarrow{g_{n-1}} I_n$$

where each I_i is injective and t-fcg. Since $n \geq d_{n+1}$, we can let $L = Img_{d-1}$. Then L is t-finitely copresented. Hence

$$Ext^1_R(M,\ L) \cong Ext^{d+1}_R(M,\ A) = 0$$

and we obtain the exact sequence

$$0 \longrightarrow Hom_R(N,\ L) \longrightarrow Hom_R(M,\ L) \xrightarrow{f^*} Hom_R(K,\ L) \longrightarrow Ext^1_R(N,\ L) \longrightarrow Ext^1_R(M,\ L) = 0$$

. Since N is t-copure, $Ext^1_R(N,\ L) = 0$. Thus

$$Ext^{d+1}_R(M,\ A) = Ext^1_R(N,\ L) = 0$$

and so N is (n, d, t)-projective.

The following theorem gives some characterizations of (n, 0, t)-projective modules.

Theorem 4.7. *Let n be a positive integer and M an R-module. Then the following statements are equivalent:*

1. *M is (n, 0, t)-projective.*
2. *For every exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

with A (n, t)-copresented, the sequence

$$0 \longrightarrow Hom(M,\ A) \longrightarrow Hom(M,\ B) \longrightarrow Hom(M,\ C) \longrightarrow 0$$

is exact.

3. *If N is (n-1, t)-copresented factor module of a t-fcg injective R-module I, then every R-homomorphism f from M to N can be lifted to a homomorphism from M to I.*
4. *Every exact sequence*

$$0 \longrightarrow M'' \longrightarrow M' \longrightarrow M \longrightarrow 0$$

is (n, t)-copure.

5. *There exists an (n, t)-copure exact sequence*

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

of R-modules with P projective.

6. *There exists an exact sequence*

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

of R-modules with P (n, 0, t)-projective.

Proof

$1 \Longrightarrow 2$: Since M is (n, o, t)-projective, the sequence

$$0 \longrightarrow Hom(M,\ A) \longrightarrow Hom(M,\ B) \longrightarrow Hom(M,\ C) \longrightarrow Ext^1_R(M,\ A) = 0$$

is exact and (2) follows.

$2 \Longrightarrow 3$: Let N be an (n-1, t)-copresented factor module of a t-fcg injective module I. Then there exists an exact sequence $I \xrightarrow{\eta} N \longrightarrow 0$. Let $K = Ker\eta$. The K is (n, t)-copresented. From the exact sequence $0 \longrightarrow K \longrightarrow I \longrightarrow N \longrightarrow 0$ we obtain, by (2), the exact sequence

$$0 \longrightarrow Hom(M,\ K) \longrightarrow Hom(M,\ I) \longrightarrow Hom(M,\ N) \longrightarrow 0$$

and (3) follows.

$3 \Longrightarrow 1$: Let A be any (n,t)-copresented module. Then there exists an exact sequence $0 \longrightarrow A \longrightarrow I \longrightarrow N \longrightarrow 0$, where I is t-fcg injective and N is (n-1, t)-copresented. This yeilds an exact sequence

$$Hom(M,\ I) \longrightarrow Hom(M,\ N) \longrightarrow Ext^1_R(M,\ A) \longrightarrow Ext^1_R(M,\ I) = 0.$$

Hence $Ext^1_R(M,\ A) = 0$ by (3).

$1 \Longrightarrow 4$: Assume (1). Then from the sequence $0 \longrightarrow M'' \longrightarrow M' \longrightarrow M \longrightarrow 0$, we have the exact sequence

$$0 \longrightarrow Hom(M,\ A) \longrightarrow Hom(M',\ A) \longrightarrow Hom(M'',\ A) \longrightarrow Ext^1_R(M,\ A) = 0.$$

for every (n, t)-copresented R-module A and (4) follows.

$4 \Longrightarrow 5 \Longrightarrow 6$ is clear.

$6 \Longrightarrow 1$: By (6), there is an (n, t)-copure exact sequence

$$0 \longrightarrow K \xrightarrow{f} P \longrightarrow M \longrightarrow 0 \tag{4.3}$$

of R-modules with P (n, 0, t)-projective. Thus for every (n, t)-copresented R-module A, we have the exact sequence

$$0 \longrightarrow Hom(M,\ A) \longrightarrow Hom(P,\ A) \xrightarrow{f^*} Hom(K,\ A) \longrightarrow Ext^1_R(M,\ A) \longrightarrow Ext^1_R(P,\ A) = 0\ .$$

Since f^* is epic as the sequence 4.3 is (n, t)-copure, we must have $Ext^1_R(M,\ A) = 0$ and (1) follows. □

Conclusion We have used a hereditary torsion theory to define new notions in relative homological algebra and using ideas from both torsion theory and homological algebra, we have proved some properties of these notions, which certainly lend themselves to further research.

The authors wish to thank the referee for the valuable comments and suggestions that has improved the quality of the work.

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Received: 02/11/2015
Accepted: 19/12/2015