Finite element approximation of the stokes problem in domains with corners and influence of the polygonal approximations of domains.

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ABSTRACT

The paper is concerned with the finite element analysis of the stokes problem in domain with corners. The error estimate is established for the nonconforming P1/P0 approximation by adding special basis. associated to the corners. The influence of the polygonal approximation of a smooth domain on the error estimates of the finite element approximation is studied. We established that for the P2/P1 conforming approximation of the stokes problem through polygonal approximation,we have $IIu-u_h II_{1,} W = O(h^{8/5-}e)$ for any positiv real e which is better than the $O(h^{3/2})$ obtained so far for scalar elliptic ,equations.This result can be extended to any second order elliptic problem.

Keywords: Finite element approximation - Stokes problem - Polygonal approximations of domains - Domains with corners.

RESUME

Le problème de Stokes dans un domaine avec singularité est approximée par la méthode des éléments finis. Les estimations des erreurs sont données à l' aide de bases spéciales associées aux singularités pour l' approximation non conforme P1/P0. L influence de l' approximation polygonal de domaines réguliers sur les estimations d 'erreurs est évaluée. Nous obtenons pour l approximation conforme P2/P1 que ; $IIu-u_h II_{1,} W = O(h^{8/5-}e)$ Pour tout réel e>0 qui améliore l estimation $O(h^{3/2})$ admise jusqu' à présent.. Ces estimations peuvent s' étendre à tous les problèmes elliptiques de second ordre.

Mots cles : Problème de Stokes-Eléments finis-Estimations d'erreurs-domaines avec singularités-Approximation polygonale.

Introduction

The finite element method is one of the most used numerical methods in solving partial differential equations generally obtained in the modelisation of natural phenomena (Biochemistry, fluid flow, economic sciences, etc...).

The finite element analysis of elliptic problem (linear and non linear) has drawn the attention of many autors, including : Ciarlet (1978), Glowinski (1984), Pironneau (1989), Feistauer (1987, 1993, 1999).

The effects of polygonal approximation of nonpolygonal domains and of numerical integration has been studied by Strang and Berger (1973), Thomee (1973) Feistauer (1990). Stang established in particular that for finite element approximation of second order and for convex domains we have the following error estimates $IIu - u_h II_{1,} W = O(h^{3/2})$.

We establised for a general smooth domain the estimate $IIu - u_h II_{1,} W = O(h^{8/5-}e)$ for any positive real e which is better than the $O(h^{3/2})$ obtained so far.

Finite element analysis of the stokes problem in domains with corners is studied.

The classical result known for the scalar second order elliptic problem is extended to the finite element approximation of the stokes problem.

We proved in particular by adding some special basis associated to the corners that the error estimate for the linear non conforming finite element approximation of the stokes problem is of order O (h) although the solution is only in $H^{2,}a(W)$ $1 < a < 2$.

Numerical results and experiments in confirming these theorical results are given in TCH [13].

II.1. Preliminaries

Let $X = H_0^1(W)$, $Q = L^2(W)$ X_h and Q_h two discret spaces with

$$X_h \;Ì\; L^2(W_h),\quad Q_h \;Ì\; L^2(W_h)$$

Define

(2.1.1) $b(u,v) = òW \operatorname{div}(u).\, v dx$ for $u\, Î\, X$

We can express b (u,v) as
$b(u,v) = (Bu,v)_{Q',Q}$ with $\| Bu \|_{Q'} \leq c \| u \|_X$
B is a bounded operator from X to Q' and b (u,v) a continuous bilinear form on X x Q
set

$$(2.1.2)\quad M = \sup_{u\in X, v\in Q} \frac{b(u,v)}{\|u\|_X . \|v\|_Q}$$

For any space X_h satisfaying assumption (H2)
We set
$XX_h = X_h + X$ and $b_h(u,v) = \sum_{T\in \Im_h} \int_T \operatorname{div}(u)\, v dx$ for all $u \in XX_h$, $v \in Q$

$b_h(u,v)$ is a continuous bilinear form on XX_h x Q with $\| b_h \| \leq M$
M given by (2.1.2.)

$\Im_h$ any triangulation of $\bar{\Omega}$
As in the continuous case we have $b_h(u,v) = (B_h u,v)_{Q',Q}$ and B_h is a bounded operator from XX_h to Q'
Assuming that the condition (H3) is satisfied i.e

$$(2.1.3)\quad \sup_{\substack{u\in X_n \\ v\in Q_h}} \frac{b(u,v)}{\|u\|_h . \|u\|_{0,2}} \geq \beta \qquad \beta>0$$

β is independent of h and u
V_h being the kernel of B_h in X_h

(2.1.3) implies that B_h is an isomorphism from $V_h^{\perp}$ into Q' and

(2.1.4) $\| B_h^{-1} \|_{Q'} \leq 1/\beta$

Following Girault [5] for the conforming case we can state :

Lemma 2.1.1
There exists a constant c > 0 such that
$\inf \| u - v_h \|_h \leq c. \inf \| u - \phi_h \|_h$ for all $u \in V$
c independent of u and h.
Proof
Let $u \in V$
For any $v_h \in X_h'$ there exists a $\phi_h \in V_h^{\perp}$ such that
$B_h \phi_h = B_h(u-v_h)$ and by (2.1.4) We have $\| \phi_h \| \leq 1/\beta \| u-v_h \|_{X_h}$
$\leq c \| u-v_h \|$ (*) by
lemma 1.5.1
in TCH[13]
Setting $W_h = 0_h + v_h$, we have
$(B_h(0_h+v_h), q_h) = (B_h(u-v_h), q_h) + (B_h v_h, q_h)$

$= \langle B_h u, q_h \rangle = 0$ since $u \in V$ and

$0_h + v_h \in V_h$
We have
$\| u - (0_h + v_h) \| \leq \| u - v_h \|_h + \| Z_h \|_h$
$\leq c \| u-v_h \|$ by (*)

The result follows □

Lemma 2.1.2.
Under the assumptions (H1) and (H2) we have :
For all p $\in H^1(\Omega)$, u $\in H^2(\Omega) \cap H^1_0(\Omega)$,

$$(2.1.2) I - \sum_{T\in\Im_h} \int_T p.div(v_h)dx + \int_{\partial T} \frac{\partial u}{\partial n}.v_h ds + \int_{\partial T} p.(v_h.\overset{\mapsto}{n})ds I$$

$$\leq c\, h \; (II\, p\, II_{H^1(\Omega)} + II\, u\, II_{H2(\Omega)})\, II\, v_h\, II$$

for all $v_h \in Xo_h$

c independant of u, p and h..

Proof

By hypothesis (H1) we have $\int_k [v]ds = 0$ for any interior side K of triangle and any $v \in X_{oh}$
Where $[v] = v+ - v^-$
and also $\int_k vds = 0$ for any boundary side .
we have

$$I \int_K \frac{\partial u}{\partial n}.v_h dsI = I \int_K \left(\frac{\partial u}{\partial n} - \left(\overline{\frac{\partial u}{\partial n}}\right)\right).(v_h - \bar{v}_h)dsI \leq II \frac{\partial u}{\partial n} - \overline{\left(\frac{\partial u}{\partial n}\right)} II_{L^2(K)} \; .IIv_h - \overline{v_h} II_{L^2(K)}$$

by Cauchy Schwarz inequality

where

$$\bar{\theta} = \frac{1}{IKI}\int_K \theta\, dx$$

From the fact that

$II\,\theta\, II^2{}_{L^2(K)} \approx II\,\theta\, II_{L^2(T)}$ where T is any triangle having K as side
and the estimates.

$$IIu - \bar{u}II_{L^p(\Omega)} \leq \left(\frac{\omega_n}{I\Omega I}\right)^{1-1/n} IIDuII_{L^p(\Omega)} \; for \quad any \quad u \in H^1(\Omega)$$

Ω convex (see GT [1])
We obtain

$$IIu - \bar{u}\, II_{L^2(K)} \leq ch^{1/2} II\nabla uII_{L^2(T)}$$

using the same argument we obtain

$$II \frac{\partial u}{\partial n} - \overline{\left(\frac{\partial u}{\partial n}\right)} II_{L^2(K)} \leq ch^{1/2} II\nabla^2 uII_{L^2(\Omega)}$$

and the estimates

(2.1.3) $I\int_K v_h.\frac{\partial u}{\partial n}dsI \leq chIIuII_{H^2}.IIv_hII_h$

With lemma 1.5.1

Using assumption (H2) and the definition of V_h we have

$\int_T p.div(v_h)dx = \int_T (p - J_h p).div(v_h)dx + \int_T J_h p div(v_h)\, dx$ for all

$v_h \in V_h$ but $\int_T q_h.div(v_h)dx = 0$ for all $q_h \in Q_h$

hence $I\int_T pdiv(v_h)dxI \leq cIIp - J_h pII_{L^2(T)}.IIv_hII_{H^1(T)}$

$$\leq chIIpII_{H^1(T)}.IIv_hII_{H^1(T)}$$

Summing on all triangles we obtain together with lemma 1.5.1

(2.1.4) $I\sum_{T\in\Im_h}\int_T pdiv(v_h)dx \leq chIIpII_{H^1(\Omega)}.IIpII_{H^1(\Omega)}.IIv_hII_h.$

Using the same argument as above we have

(2.1.5) $I\sum_T \int_{\partial T} p.(v_h.\vec{n})dsI \leq chIIpII_{H^1(\Omega)}.IIv_hII_h$

From (2.1.3);(2.1.4) and (2.1.5) we obtain the result □

Remark 2.13

The terms in lemma (2.1.2) are identically zero for conforming approximation.

An immediat consequence of the preceeding lemmata is:

Theorem 2.13

Under the assumption (H1),(H2) and (H3) we have

i) $IIu - u_hII_h \leq c \inf_{v_h \in X_h} IIu - v_hII_h$

ii) $IIp - p_hII_{L^2(\Omega_h)} \leq c\left[\inf IIu - v_hII + \inf IIp - q_hII_{L^2(\Omega_h)}\right]$

Where $(u;p) \in (H_0^1 \cap H^2)^2 x H^1(\Omega)$

$(u_h;p_h) \in V_h x Q_h^0$ *being*

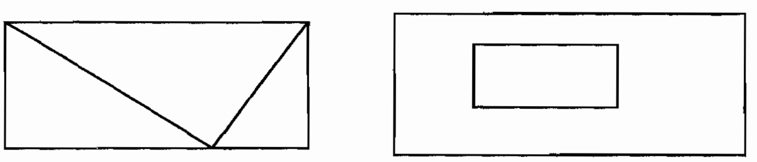
respectively the solutions of the problems (P) and (P_h).

c independant of, u.

II.2 - Finite elements analysis of the stokes problem in domain wit corners

It is to note that the flows in singular domains are more natural and the analysis of great importance. We can cite between others the flow between the wheels of a car, the flow in a lock as shown in figure. We can note for this last problem that the theoretical setting is not complete.

Fig 2.1



A

We generalise in this section the result obtained in the case of simple strong elliptic problem to the case of stokes problem for non-conforming approximation.

The regularity of the solution of the stokes problem in domain with corners has been studied in Kellog [9] Grisvard [11] Dauge [10] and Jensen [12].

One way of recovering the loss of accuracy in F.E.M in this case is to augment the trial space by some singular solutions of the equations (supposedly known) at the cost of more complexity in the discret system (sparsity and condition number of the stiffness matrix could change).

This method does not seem to work in the case of the stokes problem. This is due mainly to the incompressibility condition div (u) = 0.

We recall that the structure of the singular solution as given in [11] is :

(2.1.1) $u = u_0 + u_s$ where $u_o \in H^2(\Omega)$ and $u_s = b\ r^\alpha \quad 0<\alpha<1$

With $r=\|x-a\|$ a is a corner.

The preceding technique consist of considering the space $W_h = V_h \oplus T_{rh}$ *where* V_h *is* a regular trial space and

$T_{rh}=\{\lambda r^\alpha;\lambda \in \mathbb{R}^2\}$ $0<\alpha<1$ a one dimensional vector space

In (2.1.1) u_0 and λr^α are not necessary divergence free in which case

$\|u_h-u_0\|_h$ is not optimal

Our investigation in the direction of mesh refinements is in consideration of those difficulties.

2.1 <u>Some appropriate functional spaces</u> :

We introduce the following Banach spaces which characterization can be found in [11].

We suppose that Ω is a polygonal domain with a finite number of corners :a1;a2;....an

We set $H^{2;\alpha}(\Omega)=\{u\in H^1(\Omega)/r_1{}^\alpha\sum D^\beta u\in L^2(\Omega)\}$

$r_1=\|a_1-x\|$;$x\in\Omega$

Since all the spaces are topologically isomorphic, we recall the main properties of

$H_0^{2;\alpha}(\Omega) = H^{2;\alpha}$

It should be noted that

(2.1.2) $\|u\|^2_{H^{2;\alpha}} = \|u\|^2_{H^1(\Omega)} + \sum_{|\beta|=2} \|r^\alpha D^\beta u\|^2_{L^2(\Omega)}$

is a norm on $H_0^{2;\alpha}$

The following lemma is partially proved in [11]

<u>Lemma 2.1.1.</u>

The space $H_0^{2;\alpha}$ is compactly imbedded in $H^1(\Omega)$ and continuously in

$W_p^2(\Omega)$ for $1<p<<2/(1+\alpha)$.

<u>Proof</u>

Let $u\in H_0^{2;\alpha}(\Omega)$ and $|\beta|=2;\beta\in \mathbb{N}^2$

$$\|D^\beta u\|_{L^p(\Omega)} = \sup(D^\beta u; v)\,;\, p' = \frac{p}{p-1}; p>1$$

$$v \in L^{p'}(\Omega); \|v\|_{L^{p'}(\Omega)} = 1$$

$$but \quad \left|\int_\Omega D^\beta u.v dx\right| \le \|r^\alpha D^\beta u\|_{L^2} . \|v\|_{L^{p'}} . \left\|\frac{1}{r^\alpha}\right\|_{L^q}$$

$$with \quad (2.1.2) \quad \frac{1}{p}+\frac{1}{q}=\frac{1}{2}$$

by Hölder inequality.

But $\|1/r^\alpha\|_{L^q}<+\infty$ if and only if $\alpha q<n$ i.e $q<2/\alpha$ for n=2

But $q=2p/(2-p)$ valid for $1<p<2$;which yields $1<p<2/(1+\alpha)$

For $|\beta|\le 1$ we obtain by Hölder inequality

$$\|D^\beta u\|_{L^p(\Omega)} \le \|D^\beta u\|_{L^2(\Omega)} .[mes(\Omega)]^{1/q} < +\infty$$

$$q = \frac{2p}{2-p}$$

and the result follows.

The compact imbedding in $H^1(\Omega)$follows by Sobolev imbedding theorem.
Remark 2.1
By Sobolev imbedding theorem we have
$H^{2;\alpha}(\Omega) \mapsto W_p^2(\Omega) \to C^0(\Omega)$ all the imbedding being compact.

2.2 Interpolation operator in $H^{\alpha;2}(\Omega)$
Let us recall the following theorem proved in Gri [11] p. 392.
Theorem 2.2.1
There exists constant c >0 such that
set of linear function on T
0 being a vectice of T for every u independent of u and T.
Definition of an interpolation operator

Let $\hat{T}$ be a reference triangle that we suppose for simplicity with a vectex at O.
We define the operator :

$\hat{\Pi}$ from $H^{2,\alpha}(\hat{T})$ into $P_1(T)$ by :

$\hat{\Pi}\, u \in P_1(T)$ and $\hat{\Pi}\, u = u$ at midsides of $\hat{T}$ (this has a sense since

$H^{2,\alpha}(\hat{T}) \to C°(\overline{\hat{T}})$

(2.2.0) is obviously a continuous operator from $H^{2,\alpha}(T)$ into $H^1(T)$

noting that $(I-\hat{\Pi})P = 0$ for all $p \in P_1(T)$. We deduce from the preceding result that

there exists a $\hat{c} > 0$ such that :

(2.2.1) $\left\| u - \hat{\Pi} \right\| H^1(\hat{T}) \le \hat{c} \| u - p \| H^{2,\alpha}(\hat{T})$ for all $p \in P_1(\hat{T})$. It follows that

(2.2.2) $\left\| u - \hat{\Pi} \right\| H^1(\hat{T}) \le \hat{c} \inf_{p \in P_1(T)} \| u - p \| H^{2,\alpha}(\hat{T})$

Let $\Im_h$ be a regular triangulation of Ω

All the triangles of $\Im_h$ are affine equivalent to a single reference triangle $\hat{T}$ that we suppose without loss of generality with a vectice O.
This means that for all T of $\Im_h$ there exists a non-singular matrix B_T and a vector $b \in \Re^2$ such that :

$F : \hat{T} \to T$

$\hat{x} \to B_T\, \hat{x} + b$

F transforming vectices of $\hat{T}$ into those of T.

Define $\Pi_T u = \hat{\Pi}(u \circ F^{-1})$
Using relation (2.2.0), chain rule and change of variable formula we can obtain the following lemma.
See also Gri [11] p. 391.

Lemma 2.2.1

There exists a constant c 0 independent of h such that :

$$\| v(u - \Pi_T u) \|^2 \le c \| B_T^{-1} \|^{2+2\alpha} . \| B_T \|^4 . \int_T |x - a|^{2\alpha} |D^\beta u|^2 \, dx$$

For all u in $H^{2,\alpha}(T)$

We state now a result on interpolation of elements of $H^{2,\alpha}(T)$
under some refinement assumptions.
We suppose in all the sequel that Ω is a polygonal domain with a single corner a $\in \Gamma$ with opening $\pi < \omega < 2\pi$.
Set $\alpha = 1 - \pi/\omega$

Assumption A_1
(a) The triangulation is regular say $\Im_h$
(b) There exist constants $c_1 > 0$, $c_2 > 0$ such that
If $d(a,T) \le r_0$ then $c_1 h^{1/(1-\alpha)} \le h_T \le c_2 h^{1/(1-\alpha)}$
c_1, c_2 independant of h.

Assumption A_2
(a) The triangulation $\Im_h$ of Ω is regular
(b) For each triangle with a vectice at the corner we have
$c_1 h^{1/(1-\alpha)} \le h_T \le c_2 h^{1/(1-\alpha)}$
c_1, c_2 two positive constants independent of h.

Comment 2.2.3

We state assumption A_1 to recover the order of accuracy for linear elements for domain with corners.

In case where the corners appear by approximation of nonconvex smooth domains by polygons and for which $\alpha = 1 - \pi/\omega$ tends to zero for fine grids, the refinement is merely done for triangles with vectices at a corner. This is done in view of reducing the number of triangles which could become very large under assumption A_1. That is the aim of assumption A_2.

Under assumption A1 we obtain an order of approximation h for nonconforming linear elements and $h^{1-\varepsilon}$ with $\varepsilon = \dfrac{\alpha}{1-\alpha}$ for assumption A_2. Unfortunately not interest. $1-\varepsilon \le 1-\alpha$

We also recall that for $u \in H^2(T)$, a counterpart of lemma 2.2.1 is given by :

(2.2.3) $\|u - \Pi_T u\|_h \le c\|B_T^{-1}\|^2 . \|B_T\|^4 . \|D^2 u\|^2 L^2(\Omega)$

We state now the main result on interpolation of $H^{2,\alpha}$ functions.

Theorem 2.2.4

Let there be a bounded polygonal with one corner and $\Im_h$ a regular triangulation of Ω.

1) If $\Im_h$ satisfies assumption A_1, then there exists a constant c>0, independent of h such that :
$\|u - \Pi_h u\|_h \le ch\|u\|H^{2,\alpha}(\Omega)$ for all $u \in H^{2,\alpha}(\Omega)$
2) If assumption A_2 is satisfied then there exists a constant c>0, independent of h such that :
$\|u - \Pi_h u\|_h \le ch^{1-\alpha/(1-\alpha)}\|u\|H^{2,\alpha}(\Omega)$ for all $u \in H^{2,\alpha}(\Omega)$
3) If $\Im_h$ is merely regular we have
4) $\|u - \Pi_h u\|_h \le ch^{1-\alpha}\|u\|H^{2,\alpha}(\Omega)$ for all $u \in H^{2,\alpha}(\Omega)$

Proof

Let T be a triangle of such that : d (a,T)> r_0 in case of assumption A_1 and with no vectice at the corner a. We obtain by (2.2.3)

$$\int_T |\nabla(u - \Pi_T u)|^2 dx \le c\|B_T^{-1}\|^2 \|B_T\|^4 . \|u\|^2 H^{2,\alpha}(T)$$

$$\le \frac{c}{r_0^{2\alpha}} \|B_T^{-1}\|^2 . \|B_T\|^4 . \|u\|^2 H^{2,\alpha}(T)$$

in case of A_1

$$\le ch^{-2\alpha/(1-\alpha)} \|B_T^{-1}\|^2 . \|B_T\|^4 . \|u\|^2 H^{2,\alpha}(T)$$

Using the estimates $\left\|B_T^{-1}\right\| \leq ch^{-1}$

and we obtain

$$\int_T \left|\nabla(u - \Pi_T u)\right|^2 \leq ch^2 \|u\|^2 H^{2,\alpha} \text{ in case } A_2 \text{ and}$$

$$\leq ch^{2(1-\alpha/1-\alpha))} \text{ in case } A_2$$

For satisfying d (a,T) $\leq r_0$ in case A_1 and with a vectice a we obtain by lemma and assumption A_1 or A_2.

$$\int_T \left|\nabla(u - \Pi_T u)\right|^2 dx \leq c\left\|B_T^{-1}\right\|^{2+2\alpha} .\|B_T\|^4 .\|u\|^2 H^{2,\alpha}(T)$$

$$\leq ch^2 \|u\|^2 H^{2,\alpha}(T)$$

The result follows by summing up on all the triangles.

3) The result in case of regular triangulation follows exactly as above replacing $h^{1/(1-\alpha)}$ by h.

2.3 An interpolation operator in $H^{1,\alpha}(\Omega)$

We define $H^{1,\alpha}(\Omega) = \{u \in L^2(\Omega), r^\alpha \nabla u \in L^2(\Omega)\}$ and

2.3.1 $J_h u = \dfrac{1}{|T|} \int_T u dx$ for all $u \in H^{1,\alpha}(T)$

It is easily verified that J_h is a continuous operator from $H^{1,\alpha}(T)$ into $L^2(T)$

We have the following result

Theorem 2.3.1

If $\Im_h$ is a regular triangulation of then

1) There exists a constant c > 0, independent of h such that : $\|u - J_h u\| L^2(\Omega) \leq ch^{1,\alpha} \|u\| H^{1,\alpha}(\Omega)$ for all $u \in H^{1,\alpha}(\Omega)$

2) If $\Im_h$ satisfies assumption A1 then

$\|u - J_h u\| L^2(\Omega) \leq ch\|u\| H^{1,\alpha}(\Omega)$ for all $u \in H^{1,\alpha}(\Omega)$ c independent of h and u.

Proof

We can prove as in theorem 2.2.4. that

(2.3.2) $H^{1,\alpha}(\Omega) \to W_p^1(\Omega)$ continuously for $1<p<2/(1+\alpha)$. Appling the classical result

$$\|u - J_h u\| L^P(T) \leq \left(\frac{\omega_n}{|T|}\right)^{1-1/2} .(diamT)^2 .\|\nabla u\| L^P$$

$$\leq \left(\frac{\omega_n}{|T|}\right)^{1/2} .(diamT)^2 .\|u\| H^{1,\alpha}(T) \text{ by (2.3.3)}$$

$$\leq ch\|u\| H^{1,\alpha}(\Omega)$$

for regular triangulation or satisfying A_1 using the imbedding

$$W_p^1(\Omega) \to L^q(\Omega)$$

$$p \leq q \leq 2/(2-p)$$

and interpolation proprieties of type $\|u\| L^q \leq \|u\|_p^\lambda .\|u\|_r^{1-\lambda}$ $p \leq q \leq r$ and $\lambda = (1/p - 1/q)/(1/q - 1/r)$

we obtain $\|u - J_h u\| L^2(\Omega) \leq ch\|u\| H^{1,\alpha}(\Omega)$

2.4 Influence of polygonal approximation on the accuracy in F.E.M.

We note that most theoretical analysis on stokes and Navier-Stokes equations have been done for polygonal domain. Our aim is to give the exact error analysis when approximating smooth domains by polygonals.

Any polygonal approximation of non convex smooth domain yields domain with non convex angle.

The discret solution on Ω_h is in reality an approximation of the continuous problem posed on Ω_h which solution did not possess the required regularity.
Due to this fact we study particularly the lowest order ; the P^1/P^0 nonconforming approximation and the P^2/P^1 conforming approximation.

It is proved that $\inf \left\| \tilde{u} - v_h \right\|_h \le c \left[\phi_{h \in H_0^1(\Omega_h)} \inf \left\| \tilde{u} - \phi_h \right\| H^1(\Omega_h) + \left\| div(\tilde{u}) \right\| L^2(\Omega_h) \right]$

When Ω is convex $\Omega_h \subset \Omega$ and we take $\tilde{u} = u$ and $\tilde{u}$ denotes a regular extension of u.
We have in particular

(2.4.0) $\int_{\Omega_h} \left| div(\tilde{u}) \right|^2 dx = \int_{\Omega_h \setminus \Omega} \left| div(\tilde{u}) \right|^2 dx \le ch^2 \left\| \tilde{u} \right\|^2 H^2(\Omega_\delta)$

at the best for $\tilde{u} \in H^2(\Omega_\delta)$ such that $div\, \tilde{u} = 0$ in Ω
We foresee in (2.4.0) a necessity in case of higher approximation to construct an extension in such a way that

$div\, \tilde{u} = 0$ in Ω_δ

Ω_δ being some small neighbourhood of
The existence of such extension is given in theorem 2.4.1
We use mainly theorem 2.4 of BS[3] which we recall here.

Theorem 2.4.1

Let Ω be a bounded Lipschitz domain of $\Re^n$. Let $1 < r < +\infty$,
$m \in$ IN, then there exists a linear operator $R = R_\Omega^{m,r}$ from $H_0^{m,r}(\Omega)$ into $\left(H_0^{m+1,r}(\Omega)\right)^n$ with the following proprieties :

(i) divRf=f for all f$\in H_0^{m,r}(\Omega)$ with $\int_\Omega f dx = 0$

(ii) $\left\| \nabla^{m+1} Rf \right\|_r \le c \left\| \nabla^m f \right\|_r$ for all f$\in H_0^{m,r}(\Omega)$ where c=c(Ω, r)

Remark 2.4.2

It is easy to verify that if u$\in \left(H_0^{m+1}(\Omega)\right)^n$ then $\int_\Omega \mathrm{div(u)dx} = 0$ by divergence theorem.
In all the sequel when Ω is a nonconvex domain, we choose a neighbourhood Ω_δ of Ω such that any polygonal approximation Ω_h of Ω satisfies :

(2.4.2) $\Omega_h \subset \Omega_\delta$ for $0 < h \le h_0$ for some $\delta > 0$ and h_0 fixed with o< h_0 < 1

Theorem 2.4.3

Let Ω Ω be a bounded smooth domain of $\Re^n$ u $\in \left(H^2(\Omega)\right)$, p$\in H^1(\Omega)$, f $\in L^2(\Omega)$ with divu=0 in Ω Ω

then there exist extensions $\tilde{u} \in H^2(\Omega), \tilde{p} \in H^1(\Omega), \tilde{f}\ L^2(\Omega_\delta)$ such that :

1°) $\tilde{u} \in H^2(\Omega_\delta), \tilde{p} \cap H_0^1(\Omega_\delta)$

2°) $div\, \tilde{u} = 0$ in Ω_h

3°) $\left\| -\Delta \tilde{u} + \nabla \tilde{p} - \tilde{f} \right\| L^2(\Omega) \le c \left[\| u \| H^2(\Omega) + \| p \| H^1(\Omega) + \| f \| L^2(\Omega) \right]$

c=C(δ,Ω) a constant independent of u

Proof

Extensions $\tilde{u} \in H^2(\Omega_\delta), \tilde{p} \in H^1(\Omega_\delta), \tilde{f} \in L^2(\Omega_\delta)$ are classical and are done by localization and reflection as in [10] theorem 3.10 p. 80 [10] and 3°) is satisfied.

Let then $\tilde{u}, \tilde{p}, \tilde{f}$ be such extension satisfying 3°).

Set $D_\delta = \Omega_\delta \setminus \Omega$

We suppose without lost of generality that D_δ is as smooth as Ω and is connected.

(The following being possible on each connected component).

We have $div\, \tilde{u} \in H^1(\Omega_\delta)$

Since $div\, \tilde{u}\, 1_\Omega = div u = 0$ by some modification on exterior part of ∂D_δ we can obtain $div\, \tilde{u} \in H_0^1(D_\delta)$ with 3°) still satisfied. By theorem 2.4.1 and remark 2.4.2 there exists $w \in H_0^2(D_\delta)$ such divw = $div\, \tilde{u}$ in D_δ

and $\left\|\nabla^2 w\right\| L^2(D) \leq c(D_\delta).\left\|div\, \tilde{u}\right\| L^2(D) \quad c = C(D_\delta) > 0$

Set $\tilde{u}_1(x) = \begin{cases} u(x) & if \quad x \in \bar{\Omega} \\ \tilde{u}(x) + w(x) & if \quad x \in D_\delta \end{cases}$

Since $w \in H_0^2(D_\delta)$, we have $\tilde{u}(x) \in H^2(\Omega_\delta)$ and $div\, \tilde{u}_1 = 0$ with $\left\|\tilde{u}\right\| H^2(\Omega_\delta) \leq C(\delta, \Omega) \|u\| H^2(\Omega)$

Comment 2.4.4

1.) We suppose in all what follows that Ω is a bounded smooth domain of class $c^m\, [m \geq 2]$

Any polygonal approximation Ω_h of Ω satisfies :

(2.4.4.1) All boundary vectices of Ω_h belong to $\Gamma = \partial\Omega$

2.) For any regular triangulation $\Im_h$ of Ω_h, the number of vectices on is of order h^{-1} ($h = \max_{T \in \Im_h} diam T$)

3.) X_h and X_{oh} denoting the discret spaces approximating respectively $H^1(\Omega_h)$ and $H_0^1(\Omega_h)$ defined in 1.4

4.) We have $\dim X_h - \dim X_{oh} = 0\,(h^{-1})$

Hence in general d(u,Xh) is not of the same order as d(u, X_{oh}) When $u \notin H_0^1(\Omega)$

This fact is rigorously examined in the following sections.

Proposition 2.4.5

There exists a constant c>0 such that

$$\left(\int_{\Gamma_h} \left|\tilde{u}\right|^2 dx\right)^{1/2} \leq ch^{2(1-1/r)} \left\|\tilde{u}\right\| H^{1,r}(\Omega_\delta) \text{ for any } \tilde{u} \in H^{1,r}(\Omega_\delta) \cap H_0^1(\Omega)$$

Proof

We can write $\Omega_h \setminus \Omega = \bigcup_{i=1}^{N} S_i$ where each S_1 is formed by a face (side) of a boundary triangle and an arc belonging to $\partial\Omega$

We assume that each S_1 has the representation $S_1 = \left\{(x', x_n), x' \in \tilde{w}_1 \subset \Re^{n-1}, 0 \leq x_n \leq Z_1(x')\right\}$

Where W_1 is an open set of $\Re^{n-1}$ the arc on $\partial\Omega$ given by $\left\{ x_n = Z_1(x') \right\}_{X' \in W_1}$ and the side or face by $x_n = 0$,

$x' \in w_1$ with $|Z_1(x')| \leq ch^2$, c depending only on Ω

Let $v \in C^1(\Omega_\delta) \cap C_0(\Omega)$ and $0 \leq \alpha \leq Z_1(x')$ we have $v(x', \alpha) = \int_\alpha^{Z_1(x')} \frac{\partial v}{\partial x_n}(x', \xi) d\xi$ and by Hölder

inequality and the fact that $v(x', Z(x')) = 0$ for all $x' \in w_1$

$|v(x', \alpha)|^2 \leq |Z(x') - \alpha|^{2/q'} . \left[\int_\alpha^{Z(x')} |D_{x_n} v(x', \xi)|^{q_{d\xi}} \right]^{2/q}$ with $\frac{1}{q} + \frac{1}{q'} = 1$ Integrating on W_1 and applying one

more line Holder inequality, we obtain $\int_{W_1} |v(x', \alpha)|^2 dx' \leq ch^{4/q'} \left(\int_{W_1} dx' \int_0^{Z(x')} |D_{X_n} v(x', q)|^{q_{dx'}} \right)^2$ c

depending only on $\partial\Omega$

Summing on all S_1 and applying Jensen's inequality $\int_{\Gamma_h} |v|^2 ds \leq ch^{4/q'} \|u\|^2 H^{1,q}(\Omega_\delta)$ hence

$\left(\int_{\Gamma_h} |v|^2 dx \right)^{1/2} \leq ch^{2(1-1/q)} \|u\| H^{1,q}(\Omega_\delta)$

for all $q > 1$

using proprieties of X_h defined in 1.4 and dealing as in Lemma 2.4.5 we have

Lemma 2.4.6

There exists a constant c>0 such that

$$\int_{\Omega_h \backslash \Omega \cup \Omega \backslash \Omega_h} |\nabla u|^2 dx \leq ch^2 \|u\| H^2(\Omega_\delta)$$

$$\left(\int_{\Omega_h \backslash \Omega} |v|^2 dx \right)^2 \leq ch \|v\|_h$$

For all u $\in H^2(\Omega_\delta) \cap H_0^1(\Omega)$ c independent of u and h.

for all v $\in X_{oh}$

Comment 2.4.7

Let us note that for any polygonal approximation Ω_h of Ω (Ω sufficiently smooth) the error analysis is done for continuous solution posed on Ω_h which in case of nonconvex domain did not possess the regularity of the solution on Ω. The exact lost of accuracy is obtained via the estimates of: $\left\| \tilde{u} - \theta_h \right\| H^1(\Omega_h)$ θ_h being the exact solution on Ω_h and $\tilde{u}$ an extension of u ; the exact solution on Ω_h to Ω

The following first estimates can be obtain from the general existence theorem as expressed e.g. in [ ] :

(2.4.7.1) $\left\| \tilde{u} - \theta_h \right\| H^1(\Omega_h) \leq c \left\| u/_{\Gamma_h} \right\| H^{1/2}(\Gamma_h)$ $\Gamma_h = \partial\Omega_h$

The second member of (2.4.7.1) gives the main difference with the classical case.

It is also to note that $\left(\int_{\Gamma_h} |u|^2 ds \right)^{1/2} \neq \|u\| H^{1/2}(\Gamma_h)$ in general.

The estimates of $\|u\|H^{1/2}(\Gamma_h)$ is done via the concept of interpolation of Sobolev spaces as defined in Lions [36], [ ]. Each polygonal domain Ω_h is approximated by a set of smooth domain Ω_{nh}, $n \geq 1$ tending at least in $C^{0,1}$ to Ω_h uniformly in h.

We have in particular for $\tilde{u} \in H^2(\Omega_\delta)$

$$(2.4.7.2)\left\|\tilde{u}/\Gamma_h^n\right\| H^{1/2}(\Gamma_h^n) \to \left\|\tilde{u}/\Gamma_h\right\| H^{1/2}(\Gamma_h) \text{ uniformly in h.}$$

We recall here some proprieties of interpolation of Banach spaces.
Let X and Y be two Hilbert spaces with X continuously imbedded in Y.
It is proved that there exists a positive definite operator A densely defined in Y such tat
DA = X
For any $0\leq\theta\leq1$ ones defines the intermediate space$[X,Y]_\theta=D_A^{1-\theta}$

which is a Banach space under the norm

$$(2.4.7.3) \quad IIuII_{[X;Y]_\theta} = IIuII + IIA^{1-\theta}uII$$

We have (2.4.7.4)

$$X \subset [X;Y]_\theta \subset Y$$

With continuous injections.

we can easily deduce from (2.4.7.3) and (2.4.7.4) that :
(2.4.7.5)

$$IIuII_{[X;Y]_\theta} \leq cIIuII_X^{1-\theta}.IIuII_Y^\theta$$

c depending only on θ

(2.4.7.5) can also be deduced from the moment inequality :

$$IIA^\beta uII \leq cIIA^\alpha uII^{\frac{\gamma-\beta}{\gamma-\alpha}}.IIA^\gamma uII^{\frac{\beta-\alpha}{\gamma-\alpha}}$$

for $0\leq\alpha\leq\beta\leq\gamma$
In particular if Ω is a bounded domain of IR^n sufficiently smooth with boundary Γ
Lions[1] proved the following properties

$$(2.4.7.6)\left[H^s(\Omega), H^s(\Omega)\right] = H^{(1-\theta)s_1+\theta s_1}(\Omega)$$

$$(2.4.7.7)\left[H^s(\Gamma), H^s(\Gamma)\right] = H^{(1-\theta)s_1+\theta s_1}(\Gamma)$$

For piecewise smooth domain (2.4.7.6) and (2.4.7.7) have been established by Jensens [10] for the spaces $H_0^s(\Omega)$,for any $s\in IN$.
The definition of the spaces $H^S(\Gamma)$ for any piecewise domain can be found in Grisvard [11].

We have in particular for any bounded $u\in W^{m,p}(\Omega),uI\Gamma\in W^{m1-1/p}(\Gamma)$
With

$$IIuII_{W^{m/p,p}(\Gamma)} = \inf II\phi II_{W^{m,p}} \leq IIuII_{W^{m,p}(\Omega)}$$

$$\phi I\Gamma = uI\Gamma$$

We state now the main lemma in proving our main result of this work

Lemma2.4.8

Ω being a bounded domain of class C^3,Ω_δ a neighbourhood of Ω such that :
any polygonal approximation Ω such of Ω satisfying $\Omega_h \subset \Omega$
there exists a constant c>0,independent of h such that :

$$\|\tilde{u}|\Gamma_h\|_{H^{1/2}(\Gamma_h)} \leq c\|\tilde{u}\|^{1-\theta}{}_{H^m(\Omega_h)}.\|u|_{\Gamma_h}\|^{\theta}{}_{L^2(\Gamma_h)}$$

for any

$\tilde{u} \in H^m(\Omega_h)$;where $\theta = \dfrac{2m-2}{2m-1} \quad m \geq 1$

Proof

Since Ω is of class C^m,there exists for each polygonal approximation Ω_h of Ω satisfying $\Omega_h \subset \Omega$

A set of smooth domains Ω_h^n $n \in \mathbb{N}$ such that $\Omega_h^n \subset, \Omega_\delta$ for all $n \in \mathbb{N}$
Ω_h^n tend to Ω uniformly in h as n tend to infinity in $C^{0,1}$ therefore we have for sufficient large n
(2.4.8.1)

$$\|\tilde{u}|_{\Gamma_h}\|_{H^{1/2}(\Gamma_h)} \leq c\|\tilde{u}\|_{H^{1/2}(\Gamma_h{}^n)} \quad \text{for all } \tilde{u} \in H^m(\Omega_\partial)$$

c independent of h and u.

By (2.4.7.7) and (2.4.8.1) we have

$$\|\tilde{u}|_{\Gamma_h^n}\|_{H^{1/2}(\Gamma^n{}_h)} \leq c\|\tilde{u}\|^{1-\theta}{}_{H^{1/2}(\Omega_\partial)}.\|\tilde{u}|\Gamma^n{}_h\|^{\theta}{}_{L^2(\Gamma_h^n)} \leq \|\tilde{u}\|^{1-\theta}{}_{H^{1/2}(\Omega_\partial)}.\|\tilde{u}|\Gamma^n{}_h\|^{\theta}{}_{L^2(\Gamma_h)} \quad \text{for all}$$

$\tilde{u} \in H^m(\Omega_\partial)$

The last inequality is obtained by (2.4.7.2) and (2.4.7.8) with $s_2 = 0$, $s_1 = m1/2$ and $\theta = (2m-2)/(m-1)$.

The following two theorems give the error estimates in the energy norm both for polygonal approximation of smooth domain and for domains with finite many corners.
The error estimates $\|p-p_h\|_{L2(\Omega h)}$ for the pressure is deduced from that of the velocity by the classical duality argument that yields
$\|p-p_h\|_{L2(\Omega h)} \leq ch\|u-u_h\|_h$
:

Theorem2.4.9

Let Ω be a bounded domain of class C^m,Ω_h a set of polygonal approximations of Ω satisfying (2.4.2) and (2.4.4.1)
If $(u,p) \in \left[H^{m+1}(\Omega)\right]^2 x H^m(\Omega)/IR$ is the solution of the stokes problem (P) on Ω

(u_h,p_h) the corresponding discret solution on Ω_h then there exist a constant c>0 independent of h such that :

$$\|u-u_h\|_{h,\Omega_h} \leq c\left\lfloor h^m \|u\|_{H^{mm+}(\Omega)}\right\rfloor + \|u\|^{1-\theta}_{H^{m+1}(\Omega)}.h^{2\theta(-1/r)}$$

(2.4..9.1) for all $1<r<+\infty$

$$with \quad \theta = \frac{2m}{2m+1}$$

Comment 2.4.9.2

(a) The first term of the second member of (2.4.9.) estimates the regularity of the solution on Ω and the second effect of the domain approximation.
(b) For m=1,we obtain for the P^1/P^0 nonconforming approximation that no lost of accuracy is observed as already proved for the linear conforming approximation for scalar elliptic problems.
(c) for m=2 we obtain the order (8/5)-ε $\varepsilon>0$ which is better than the 3/2 till now admitted as the best order.

Proof of theorem 2.4.9

Let (u,p) be the solution of the stokes equation (P)

We have :

$$L_h(u,p,v_h)=\int_{\Omega_h}(-\Delta u+\nabla p-f)v_h dx=0 \quad for \quad all \quad v_h \in X_h$$

Applying the Green formula on each triangle $T\varepsilon\Im_T$ we obtain

$$L_h(u,p,v_h)=a_h(u,v_h)-\sum_{T\in\Im_h}\int_{\partial T}\frac{\partial u}{\partial n}.v_h ds-\int_T p.div(v_h)dx+\int_{\partial T}p.(v_h.\vec{n})ds-\int_{\Omega_h}f.v_h dx$$

where $\vec{n}$ is the exterior normal to T.

(2.4.9.3) can be written into the form :(2.4.9.3')

$$L_h(u,p,v_h)=a_h(u,v_h)\ \sum\int\ -\int_T p.div(v_h)dx+B_h(u,p,v_h) \quad where B_h(u,p,v_h)=\int_{\partial T}p.(v_h.\vec{n})ds-\int_{\Omega_h}f.v_h dx-\sum_{T\in\Im_h}\left(\int_{\partial T}\frac{\partial u}{\partial n}.v_h ds\right)$$

(

We have for all $v_h \in X_h$

$IIu-u_hII_h \leq IIu-v_hII_h+IIu_h-v_hII_h$ and

$IIu-v_h\ II^2=a_h(u_h-u,u_h-v_h)+a_h(u,u_h-v_h)$.

From which follows :

$$IIu-u_hII_h \leq 2\inf_{v_h\in X_h} IIu-v_hII_h+\sup_{w_h\in V_h}\frac{Ia_h(u-u_h,w_h)I}{IIw_hII_h}$$

Noting that $L_h(u,p,v_h)=0$ for all $v_h \in X_h$ we deduce from (2.4.9.3') that

$$Ia_h(u-u_h,v_h)I \leq I\sum_{T\in\Im_h}\int_T p.div(v_h)dxI+IB_h(u,p,v_h)I \quad for \quad all \quad v_h \in X_h$$

Let θ_h be the solution of :

$$\begin{cases}\{-\Delta\theta_h+\nabla q_h=0 \ in \ \Omega_h\\ div(\theta_h)=0 \quad in \quad \Omega_h\\ \theta_h I\Gamma_h=uI\ \Gamma_h\end{cases}$$

By theorem 1.3 and remark 1.4 in [13] there exists a unique

$(\theta_h, q_h) \in H^m(\Omega_h) x H^{m-}(\Omega_h) / IR$ such that $II\theta_h II_{H\ (\Omega_h)} \leq cIIuI_{\Gamma_h} II_{H^{/2}(\Gamma_h)}$

c independent of h and u

and $II\theta_h II_{H^m(\Omega_h)} \leq cIIuI_{\Gamma_h} II_{H^{m-/2}(\Gamma)} \leq cIIuII_{H^m(\Omega)}$

Ω_h being convex.
It follows that
$u - \theta_h \in H^m(\Omega_h) \cap H_0^1(\Omega) \cap V$

Using assumptions (H1),(H2) and (H3) we deduce that

$$(2.4.9.4) \quad \inf_{v_h \in X_h} IIu - \theta_h - v_h II_h \leq ch^{m-1} IIu - \theta_h II_{H^m(\Omega_h)}$$

We have :
(2.4.9.5) $IIu - v_h II_h \leq IIu - v_h - \theta_h II_h + II\theta_h II_h$

From lemma (2.4.8) and relations (2.4.9.4) and (2.4.9.5) we deduce

$$\inf_{v_h \in X_{0h}} IIu - v_h II_h \leq \left[c\left(h^{m-1} IIuII_{H^m} + h^{2\theta(-/r)} IIuII_{H^m}^{-\theta} \right) \right]$$

for any r>1

For conforming approximation X_h we have $X_h \subset H^1(\Omega_h)$ and $B_h(u,p,v_h)=0$ for all

$V_h \in X_h$ the result follows immediately by noting that :

$$I \sum_{T \in \Im_h} \int_T p.div(v_h) dxI \leq ch^{m-1} IIpII_{H^{m-1}(\Omega_h)} . IIv_h II_h \quad for \quad v_h \in V_h$$

For the linear nonconforming case the result follows from lemma 2.1.2.

Corollary 2.4.10

If Ω is a bounded convex domain of class C^3 for a P2/P1 conforming approximation we have for any polygonal approximation of Ω satisfying (2.4.2), (2.4.4.1).

$IIu - u_h II_{H^1(\Omega_h)_h} \leq ch^{8/5-\varepsilon} IIuII_{H^3(\Omega)}$

c independent of h and u for any ε>0.

Proof

This is an immediate consequence of theorem (2.4.9) with m = 3 and θ = 4/5 by noting that

$H^3(\Omega) \to H^{1,r}(\Omega)$ for all $1 \leq r < +\infty$

By Sobolev imbedding theorems □.

We state now a counterpart of theorem 2.4.9 for the case of nonconvex domains and for domain with corners.

The results here are just proved for linear nonconforming case.

This restriction is justified by the fact that any polygonal approximation of Ω possesses some corners with angle greater than π and the discret solution is in reality approximates the solution of (P) posed on, Ω_h which unfortunately did not have the H^2 regularity.

Theorem 2.4.11

Suppose Ω is a bounded nonconvex domain of class C^2 ; (u,p) the solution of the stokes problem (P) on Ω with second member $f \in L^2(\Omega)$

Let $(\widetilde{u}; \widetilde{p}; \widetilde{f})$ be the extensions of (u,p,f) given in theorem 2.4.3.

Let $(u_h ;p_h)$ be a discret solution of (p) on Ω_h .a polygonal approximation of Ω satisfying (2.4.4.1) and (2.4.2) then

$$1) \|\widetilde{u} - u\|_{h;\Omega_h} \leq ch\|u\|_{H^2(\Omega)}$$

if the triangulation of Ω_h satisfies assumption A1
c a constant independent of h and u

$$2) \|\widetilde{u} - u_h\|_{h;\Omega_h} \leq ch^{1-\alpha} \|u\|_{H^2(\Omega)}$$

$$\alpha = \min(1 - \pi / w_i)$$

$$i = 1; \ldots\ldots\ldots n$$

$$w_i > \pi \quad for \quad each \quad i$$

w_i being the opening angle of corner a_i
N being the total number of corners of Ω_h

Remark 2.4.12

Theorem 2.4.11 states that the classical order h for the velocity is obtained only under some refinements constraints in the neighbourhood of corners.
An order of $h^{1/2}$ is obtained at least for arbitrary regular triangulation.

Proof of Theorem 2.4.11

Let

$$(2.4.13) \quad L_h(\widetilde{u}; \widetilde{p}; \widetilde{f}; v_h) = \int_\Omega (-\Delta\widetilde{u} + \nabla\widetilde{u} - \widetilde{f}).v_h dx \quad for \quad all \quad v_h \in X_{0h} \subset H_0^1(\Omega)$$

By Green formula on each triangle $T \in \Im_h$ we have
(2.4.13')

$$L_h(\widetilde{u}; \widetilde{p}; \widetilde{f}; v_h) = a_h(\widetilde{u}; v_h) - \sum_{T \in \Im_h} \left(\int_{\partial T} \frac{\partial \widetilde{u}}{\partial n}.v_h ds - \int_T \widetilde{p}.div(v_h)dx + \int_T p.(v_h.\vec{n})ds.. \right) - \int_{\Omega_h} \widetilde{f}..v_h dx$$

$$= a_h(\widetilde{u}; v_h) + B_h(\widetilde{u}; \widetilde{p}; v_h) + \sum_{T \in \Im_h} - \int_T \widetilde{p} div(v_h)dx$$

noting that
repeating word by word the line in theorem 2.4.10 we have : 2.4.14
Since satisfies 2.4.4.1) we can write where a triangle with a side on
Recalling that the elements in Xoh are continuous at midpoint of triangle sides and vanishing at boundary midpoints we have on each S1. An
being constant on the midside of boundary side of T1.
Integrating and summing on all S1 we obtain (2.4.16)
From theorem (2.4.3), relations (2.4.13), (2.4.16), it follows The second part of theorem follows as consequence of theorem (2.4.4),
A corresponding result for polygonal domain with finite number of corners can be obtained as in theorem (2.4.10 and theorem (2.2.4) and it reads.

Theorem 2.4.17

Let Ω be a polygonal domain with a finite number of corners a1....an
1)If the triangulation of Ω satisfies A1 then there exists a constant c>0 such that

$$\|u - u_h\|_h \leq ch\left[\sum_{i=1}^{n} \|u_1\|_{H_{a1}^{2;\alpha}} + \|w\|_{H^2(\Omega)}\right] \quad \text{c independent of h and u}$$

1) If $\Im_h$ only regular we have

$$\|u-u_h\|_h \leq ch^{1-\alpha}\left[\sum_{i=1}^{n}\|u_1\|_{H_{a1}^{2;\alpha}} + \|w\|_{H^2(\Omega)}\right]$$

The solution u of (P) given by:

$$u=\sum_{i=1}^{n}u_1+w \quad with \quad u_1 \in H_{a1}^{2;\alpha}(\Omega); \quad w \in H^2(\Omega)$$

BIBLIOGRAPHY

[1] Adams,R.A :Sobolev spaces. Academic Press, New York, San-Francisco-London (1975)

[2]Axellson,O.,Barker V.A : Finite element solutions of boundary value problems. London, Academic Press (1984).

[3]Borchers,W ;Sohr,H : On the equations Rotv=g, divu=f with zero boundary conditions. Hokkaido Mathematical Journal. Vol XIX no 1(1990)

[4]Ciarlet P.G: The finite element method for elliptic problems. North Holland (1978).

[5]Crouzeix,M.,Raviart ,P.A : Conforming and non conforming finite element approximations for solving the stokes equations. RAIRO pp33-76(1976).

[6] Feistauer, M ; Najar, K :Finite element approximation of a problem with a nonlinear Newton boundary condition. Num Math. 78(1998) pp403-425

[7]Feistauer, M ;Najar, K ; Sobotikova, V :Error estimates for finite element approximation of a problem with non linear Newton boundary condition.
Num Funct. Anal and optiiz 20(9&10)pp 835-851(2000).

[8]Gilbarg, D ;Trudinger, MS :Elliptic partial differential equations of second order. Springer-Verlag, Heidelberg (1983) second edition.

[9]Kellog,K ; Osborn,J :A regularity result for the stokes problem in a convex polygon.
J of funct anal 21,(1976) p397-431.

[10] Necas, J. :Les méthodes directes en théorie des équations elliptiques.
Masson et Cie (1967)

[11]Grisvard, P : Elliptic problems in nonsmooth domains.
Monographs and studies in mathematics. Pitman, Boston (1985)

[12] Strang, G .Fix, An analysis of the finite element method. Princehall, inc.
Englhood, N.J (1973)

[13] Tchoua P. Finite element analysis of the Stokes problem in domains with corners (unpublished Thesis) in revised form(2000)