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THE CENTRAL SUBGROUP OF THE NONABELIAN TENSOR SQUARE OF BIEBERBACH GROUP WITH POINT GROUP $C_2 \times C_2$

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ABSTRACT

A Bieberbach group with point group $C_2 \times C_2$ is a free torsion crystallographic group. A central subgroup of a nonabelian tensor square of a group G, denoted by $\nabla(G)$ is a normal subgroup generated by generator $g \otimes g$ for all $g \in G$ and essentially depends on the abelianization of the group. In this paper, the formula of the central subgroup of the nonabelian tensor square of one Bieberbach group with point group $C_2 \times C_2$, of lowest dimension 3, denoted by $S_3(3)$ is generalized up to n dimension. The consistent polycyclic presentation, the derived subgroup and the abelianization of group this group of n dimension are first determined. By using these presentations, the central subgroup of the nonabelian tensor square of this group of n dimension is constructed. The findings of this research can be further applied to compute the homological functors of this group.

Keywords: Bieberbach group; central subgroup; nonabelian tensor square.

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1. INTRODUCTION

1.1. Introduction

A Bieberbach group is a free torsion crystallographic group. This group is an extension of free abelian group L of finite rank by a finite group P which satisfy the short exact sequence $1 \longrightarrow L \xrightarrow{\varphi} G \xrightarrow{\phi} P \longrightarrow 1$ such that the quotient group $G/\varphi(L) \cong P$ isomorphic to group P. Here L is called as the lattice group and P is a point group. The dimension of G is also known as the rank of L. In this case, G is called as a Bieberbach group with point group P.

Many properties of this group can be explored where one of the properties is its central subgroup of the nonabelian tensor square, $\nabla(G)$. The nonabelian tensor square, $G \otimes G$ of a group G is generated by the symbols $g \otimes h$, for all $g, h \in G$, subject to relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h) \text{ and } g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h') \qquad [1]$$

for all $g, g', h, h' \in G$, where ${}^{g}g' = gg'g^{-1}$. The nonabelian tensor square is a specialization of the more general nonabelian tensor product introduced by Brown and Loday [1].

The computations of $\nabla(G)$ of some Bieberbach groups with certain point groups can be found in previous studies. Masri [2] has constructed the abelianization and the $\nabla(G)$ of Bieberbach groups with cyclic point group of order 2. The results of the $\nabla(G)$ of the groups were then used to compute the nonabelian tensor square of the groups. The studies of the $\nabla(G)$ of some Bieberbach groups with dihedral point group can be found in Mohd Idrus et al. [3]. She used the central subgroup of the nonabelian tensor square of the group in order to determine the presentation of the nonabelian tensor square of the group. Also recently, Tan *et al.* [4] and Masri *et al.* [5] have explored the formula of the $\nabla(G)$ of the Bieberbach group with symmetric point group of certain dimension.

The subgroup $\nabla(G)$ is normal and is generated by $g \otimes g$ for all g in G. Blyth, Fumagalli and Morigi [6] have showed that there is a relationship between the structure of $\nabla(G)$ and the abelianization of the group, G^{ab} given by the following proposition.

Proposition 1 [6]

Let G be a group such that G^{ab} is finitely generated. Assume that G^{ab} is the direct product of the cyclic groups $\langle x_i G' \rangle$, for $i = 1, \ldots, s$ and set $E(G)$ to be $\langle [x_i, x_j^{\varphi}] \mid i < j \rangle [G, G'^{\varphi}]$. Then $\nabla(G)$ is generated by the elements of the set $\{[x_i, x_i^{\varphi}], [x_i, x_j^{\varphi}][x_j, x_i^{\varphi}] \mid 1 \le i < j \le s\}$.

In this paper, our main interest is the Bieberbach group of lowest dimension 3 with elementary abelian 2-group point group, $C_2 \times C_2$, denoted as $S_3(3)$. The presentation of $\nabla(S_3(3))$ which has been determined in Abdul Ladi *et al*. [7] will be generalized up to dimension *n*. The consistent polycyclic presentation of the group $S_3(3)$ has been constructed in Abdul Ladi *et al*. [7] as in the following :

$$S_3(3) = \left\langle a_0, a_1, l_1, l_2, l_3 \;\middle|\; \begin{array}{l} a_0^2 = l_1^{-1}, a_1^2 = l_3^{-1}, {}^{a_0}a_1 = a_1 l_1^{-1}, \\ {}^{a_0}l_1 = l_1, {}^{a_0}l_2 = l_2^{-1}, {}^{a_0}l_3 = l_3, \\ {}^{a_1}l_1 = l_1^{-1}, {}^{a_1}l_2 = l_2, {}^{a_1}l_3 = l_3, \\ {}^{l_1}l_2 = l_2, {}^{l_1}l_3 = l_3, {}^{l_2}l_3 = l_3 \end{array} \right\rangle (1) .$$

1.2 Preliminaries

Some basic definitions and structural results related to this study are presented in this section. The consistent polycyclic presentations of group $S_3(n)$ is constructed based on the following two definitions of the polycyclic presentation of group and the consistent polycyclic presentation of group [8]. First, the definition of the polycyclic presentation is given as follows:

Definition 1 [8]

Let F_n be a free group on generators $g_1, \ldots, g_n$ and R be a set of relations of group G. The relations of a polycyclic presentation of F_n / R have the form:

$$g_i^{e_i} = g_{i+1}^{x_{i,i+1}} \ldots g_n^{x_{i,n}} \qquad \text{for } i \in I,$$

$$g_j^{-1} g_i g_j = g_{j+1}^{y_{i,j,j+1}} \ldots g_n^{y_{i,j,n}} \qquad \text{for } j < i,$$

$$g_j g_i g_j^{-1} = g_{j+1}^{z_{i,j,j+1}} \dots g_n^{z_{i,j,n}} \qquad \text{for } j < i,\ j \notin I.$$

for some $I \subseteq \{1,\dots,n\}$, certain exponents $e^i \in \square$, for $i \in I$, and $x_{i,j}$, $y_{i,j,k}, z_{i,j,k} \in \square$, for all i, j and k.

Definition 2 [8]

Let G be a group generated by $g_1,\dots,g_n$ and the consistency relations in G can be determined using the following consistency relations.

$$g_k(g_j g_i) = (g_k g_j) g_i \qquad \text{for } k > j > i,$$

$$(g_j^{e_j}) g_i = g_j^{e_{j-1}}(g_j g_i) \qquad \text{for } j > i, j \in I,$$

$$g_j(g_i^{e_i}) = (g_j g_i) g_i^{e_{i-1}} \qquad \text{for } j > i, i \in I,$$

$$(g_i^{e_i}) g_i = g_i (g_i^{e_i}) \qquad \text{for } i \in I,$$

$$g_j = (g_j g_i^{-1}) g_i \qquad \text{for } j > i, i \notin I$$

for some $I \subseteq \{1,\dots,n\}$, for certain exponents $e^i \in \square$, $i \in I$. Therefore, the consistent polycyclic presentation of $S_3(n)$ can be determined by using Definition 1 and 2.

The consistency of polycyclic presentation of group $S_3(n)$ need to be determined in order to use the computational method of polycyclic groups [9]. Next, the definition of the abelianization of group is given as follows.

Definition 3

The abelianization of a group G , G^{ab} is the quotient of group G by its derived subgroup, G'

In 1991, Rocco [10] has initiated in investigating the group $\nu(G)$ which is defined as in the following.

Definition 4

Let G be a group with presentation $\langle G \mid R \rangle$ and let G^{φ} be an isomorphic copy of G via the mapping $\varphi : g \to g^{\varphi}$ for all $g \in G$. The group $\nu(G)$ is defined to be

$$\nu(G) = \left\langle G, G^{\varphi} \mid R, R^{\varphi}, {}^{x}[g,h^{\varphi}] = [{}^{x}g, ({}^{x}h)^{\varphi}] = {}^{x^{\varphi}}[g,h^{\varphi}], \forall x,g,h \in G \right\rangle.$$

Next theorem shows that $G \otimes G$ is isomorphic to a subgroup $[G, G^{\varphi}]$ of $\nu(G)$.

Theorem 1 ([10],[11])

Let G be a group. The map $\sigma : G \otimes G \to [G, G^{\varphi}] \triangleleft \nu(G)$ defined by $\sigma(g \otimes h) = [g, h^{\varphi}]$ for all g, h in G is an isomorphism.

With this theorem, all the tensor computations can be translated into the commutator computation within the subgroup $[G, G^{\varphi}]$ of $\nu(G)$.

In this paper, the subgroup $[G, G^{\varphi}]$ of $\nu(G)$ will be used to compute the presentation of the central subgroup of the nonabelian tensor square of group $S_3(n)$, denoted by $\nabla(S_3(n))$. Next, a list of commutator identities in $\nu(G)$ is given as follows. Let x, y and z be elements of group G. Then, for the left conjugation, ${}^{x}y = xyx^{-1}$ and the list of commutators are presented as in the following:

$$[xy, z] = {}^{x}[y, z] \cdot [x, z] \tag{2}$$

$$[x, yz] = [x, y] \cdot {}^{y}[x, z] \tag{3}$$

$$[x^{-1}, y] = [x^{-1}, [x, y]^{-1}] \cdot [x, y]^{-1} \tag{4}$$

$$[x, y^{-1}] = [y^{-1}, [x, y]^{-1}] \cdot [x, y]^{-1} \tag{5}$$

$$[x^{-1}, y^{-1}] = [x^{-1}, [y^{-1}, [x, y]]] \cdot [y^{-1}, [x, y]] \cdot [x^{-1}, [x, y]] \cdot [x, y] \tag{6}$$

$${}^{z}[x, y] = [{}^{z}x, {}^{z}y] \tag{7}$$

Proposition 2 [2]

Let G be any Bieberbach group of dimension n with point group P and lattice group L. Let $B = G \times F_m^{ab}$ where F_m^{ab} be a free abelian group of rank m. Then B is a Bieberbach group of dimension $n+m$ with point group P.

The derived subgroup $S_3(3)$, $S_3(3)'$, the abelianization of $S_3(3)$, $S_3(3)^{ab}$ and the central subgroup of the nonabelian tensor square of $S_3(3)$, $\nabla(S_3(3))$ have been determined as follows.

Proposition 3 [7]

The group $S_3(3)$ has derived subgroup, $S_3(3)' = \langle l_1^{-2}, l_2^{-2} \rangle$ and the abelianization of $S_3(3)$ is generated by cosets $l_1 S_3(3)'$ of order 2, $l_2 S_3(3)'$ of order 2 and $l_3 S_3(3)'$ of infinite order. In symbols,

$$S_3(3)^{ab} \cong \langle l_1 S_3(3)', l_2 S_3(3)', l_3 S_3(3)' \rangle \cong C_2^{\,2} \times C_0.$$

Proposition 4 [7]

The subgroup $\nabla(S_3(3))$ is generated by generators $[l_1, l_1^{\varphi}]$ and $[l_2, l_2^{\varphi}]$ of order 4, generator $[l_3, l_3^{\varphi}]$ of infinite order, generators $[l_1, l_2^{\varphi}][l_2, l_1^{\varphi}]$, $[l_1, l_3^{\varphi}][l_3, l_1^{\varphi}]$, and $[l_2, l_3^{\varphi}][l_3, l_2^{\varphi}]$, of order 2. In symbols,

$$\begin{aligned}\nabla(S_3(3)) &= \langle [l_1, l_1^{\varphi}], [l_2, l_2^{\varphi}], [l_3, l_3^{\varphi}], [l_1, l_2^{\varphi}][l_2, l_1^{\varphi}], [l_1, l_3^{\varphi}][l_3, l_1^{\varphi}], [l_2, l_3^{\varphi}][l_3, l_2^{\varphi}] \rangle \\ &\cong C_2^{\,3} \times C_4^{\,2} \times C_0.\end{aligned}$$

The following propositions are some basic identities used in this paper.

Proposition 5 [6]

Let G be any group. Then the following hold:

(i) If $g_1 \in G'$ or $g_2 \in G'$, then $[g_1, g_2^{\varphi}]^{-1} = [g_2, g_1^{\varphi}]$.

(ii) $[Z(G), (G')^{\varphi}] = 1$.

(iii) If A and B are two subgroups of G with $B \leq G'$, then $[A, B^{\varphi}] = [B, A^{\varphi}]$. In particular, $[G, G'^{\varphi}] = [G', G^{\varphi}]$.

Proposition 6 ([2], [9])

Let g and h be elements of G such that $[g, h] = 1$. Then, in $\nu(G)$,

(i) $[g^n, h^{\varphi}] = [g, h^{\varphi}]^n = [g, (h^{\varphi})^n]$ for all integers n;

(ii) $[g^n, (h^m)^{\varphi}][h^m, (g^n)^{\varphi}] = ([g, h^{\varphi}][h, g^{\varphi}])^{nm}$ for all integers n, m ;

(iii) $[g, h^{\varphi}]$ is in the centre of $\nu(G)$.

Proposition 7 [2]

Let G and H be groups and let $g \in G$. Suppose ϕ is a homomorphism from G onto H. If $\phi(g)$ has finite order then $|\phi(g)|$ divides $|g|$. Otherwise the order of $\phi(g)$ equals to order of g.

Proposition 8 [12]

Let A, B and C be abelian groups. The properties of the ordinary tensor product of two abelian groups are given as in the following.

(i) $C_0 \otimes A \cong A$,

(ii) $C_0 \otimes C_0 \cong C_0$,

(iii) $C_n \otimes C_m \cong C_{\gcd(n,m)}$, for $n, m \in \mathbb{N}$, and

(iv) $A \otimes (B \times C) = (A \otimes B) \times (A \otimes C)$.

Proposition 9 [1]

Let G and H be groups such that there is an epimorphism $\varepsilon : G \to H$. Then there exists an epimorphism

$$\alpha : G \otimes G \to H \otimes H$$

defined by $\alpha(g \otimes h) = \varepsilon(g) \otimes \varepsilon(h)$.

2. RESULTS AND DISCUSSION

In this section, the central subgroup of the nonabelian tensor square of $S_3(3)$, will be generalized up to n dimension. The generalization of polycyclic presentation of $S_3(3)$ of dimension $n, S_3(n)$ is constructed first.

Lemma 1

The polycyclic presentation of $S_3(n)$,

$$S_3(n)=\left\langle a_0,a_1,l_1,\ldots,l_n \;\middle|\; \begin{array}{l} a_0^2=l_1^{-1}, a_1^2=l_3^{-1}, {}^{a_0}a_1=a_1l_1^{-1}, \\ {}^{a_0}l_1=l_1, {}^{a_0}l_2=l_2^{-1}, {}^{a_0}l_3=l_3, \\ {}^{a_0}l_k=l_k, {}^{a_1}l_1=l_1^{-1}, {}^{a_1}l_2=l_2, {}^{a_1}l_3=l_3, \\ {}^{a_1}l_k=l_k, {}^{l_i}l_j=l_j, {}^{l_i^{-1}}l_j=l_j \end{array} \right\rangle \quad (8)$$

is consistent for $1 \le i < j \le n$ and $4 \le k \le n$.

Proof. By Proposition 2, $S_3(n)=S_3(3)\times F_{n-3}^{ab}$ for $n \ge 3$ where $S_3(3)$ has the consistent polycyclic presentation as in (1) and F_{n-3}^{ab} is a free abelian group of rank $n-3$ which is generated by $l_4, l_5, \ldots l_n$. Then, l_k commutes with all elements in $S_3(3)$, which gives ${}^{a_0}l_k=l_k, {}^{a_1}l_k=l_k, {}^{l_1}l_k=l_k, {}^{l_2}l_k=l_k$ and ${}^{l_3}l_k=l_k$ for all $4 \le k \le n$. Therefore, $S_3(n)$ has a polycyclic presentation as in (8).

The polycyclic presentation of $S_3(3)$ in (1) has been shown to be consistent in Abdul Ladi *et al*. (2016). Then, by Definition 2, and since ${}^{a_0}l_k=l_k, {}^{a_1}l_k=l_k, {}^{l_i}l_k=l_k$ $\forall i=1,2,3,$ the remaining consistency relations $l_k(l_3a_0)=(l_kl_3)a_0$, $l_k(l_3a_1)=(l_kl_3)a_1$, are also hold since a_0, a_1 commutes with l_3 and l_k. Then, it is showed that $l_k(l_3l_1)=(l_kl_3)l_1$, $l_k(l_3l_2)=(l_kl_3)l_2$, $l_k(l_2l_1)=(l_kl_2)l_1$, since l_1, l_2, l_3 and l_k. are commute with each other based on relation in (8). Since a_0 commutes with l_1 and a_1 commutes with l_2, then $l_k(l_1a_0)=(l_kl_1)a_0$, and $l_k(l_2a_1)=(l_kl_2)a_1$. Next, $l_k(l_2a_0)=l_k\left(a_0l_2^{-1}\right)=a_0l_2^{-1}l_k$ and

$\left(l_k l_2\right) a_0 = (l_2 l_k) a_0 = l_2 a_0 l_k = a_0 l_2^{-1} l_k$, $l_k\left(l_1 a_1\right) = l_k\left(a_1 l_1^{-1}\right) = a_1 l_1^{-1} l_k$ and $\left(l_k l_1\right) a_1 = (l_1 l_k) a_1 = l_1 a_1 l_k = a_1 l_1^{-1} l_k$, $l_k\left(a_1 a_0\right) = \left(l_k a_1\right) a_0$, $l_k\left(a_0^{\,2}\right) = l_k\left(l_1^{-1}\right) = l_1^{-1} l_k$ and $(l_k a_0) a_0 = (a_0 l_k) a_0 = a_0 a_0 l_k = l_1^{-1} l_k$, $l_k\left(a_1^{\,2}\right) = l_k\left(l_3^{-1}\right) = l_3^{-1} l_k$ and $(l_k a_1) a_1 = (a_1 l_k) a_1 = a_1 a_1 l_k = l_3^{-1} l_k$, $l_k = \left(l_k l_3^{-1}\right) l_3$, $l_k = \left(l_k l_2^{-1}\right) l_2$ and $l_k = \left(l_k l_1^{-1}\right) l_1$. Thus, the polycyclic presentation of $S_3(n)$ is consistent. □

Next, the generalization of the derived subgroup and the abelianization of the group $S_3(3)$ of dimension n is given as in the following lemma.

Lemma 2

The derived subgroup of $S_3(n)$, $S_3(n)' = \left\langle l_1^{-2},\, l_2^{-2}\right\rangle$ and the abelianization of $S_3(n)$,

$$S_3(n)^{ab} \cong \left\langle l_1 S_3(n)', l_2 S_3(n)', l_3 S_3(n)', l_k S_3(n)'\right\rangle$$

$$\cong C_2^{\,2} \times C_0^{\,n-2}$$

for $4 \le k \le n$.

Proof. From relation (8), since a_0 commutes with l_1, l_3, l_k and a_1 commutes with l_2, l_3, l_k for all $4 \le k \le n$, then $[a_0, a_1] = l_1$, $[a_0, l_2] = [a_1, l_2] = l_2^{-2}$ and $[a_1, l_1] = l_1^{-2}$. However, $l_1 = (l_1^{-2})^2$. Thus, $S_3(n)' = \left\langle\, l_1^{-2},\, l_2^{-2}\right\rangle$.

By Definition 3, the abelianization of group $S_3(n)$, $S_3(n)^{ab}$ is generated by $a_0 S_3(n)'$, $a_1 S_3(n)'$, $l_1 S_3(n)'$, $l_2 S_3(n)'$, $l_3 S_3(n)'$ and $l_k S_3(n)'$ for $4 \le k \le n$. By Proposition 3, the independent generators of $S_3(n)^{ab}$ are $l_1 S_3(3)'$, $l_2 S_3(3)'$ and $l_3 S_3(3)'$. By using similar arguments, we showed that $l_1 S_3(n)'$, $l_2 S_3(n)'$, $l_3 S_3(n)'$ and $l_k S_3(n)'$ are also the independent generators of $S_3(n)^{ab}$. Hence, by Definition 3,

$$S_3(n)^{ab} = \left\langle l_1 S_3(n)', l_2 S_3(n)', l_3 S_3(n)', l_k S_3(n)'\right\rangle$$

for $4 \le k \le n$.

By Proposition 3, it is shown that the generators in $S_3(n)^{ab}$ such as $l_1 S_3(3)'$ has order 2,

$l_2S_3(3)'$ has order 2 and $l_3S_3(3)'$ has infinite order. Next, the orders of cosets $l_1S_3(n)'$, $l_2S_3(n)'$, $l_3S_3(n)'$ and $l_kS_3(n)'$ are determined. By relations in (8), since $a_0{}^2 = l_1^{-1}$, then $a_0{}^4 = l_1^{-2}$. Hence, it is shown that $l_1S_3(n)'$ has order 2 since $l_1^2 \in S_3(n)'$. Since $l_2^{-2} \in S_3(n)'$, then $l_2^2 \in S_3(n)'$. It follows that $l_2S_3(n)'$ has order 2.

Suppose that the order of $l_kS_3(n)'$ is finite, then there must be $l_3^r \in S_3(n)'$. However, this is not true since there is no l_3^r in $S_3(n)'$. Therefore, $l_3S_3(n)'$ has infinite order. By using similar arguments, $l_kS_3(n)'$ is shown to have an infinite order, since there is no l_3^r in $S_3(n)'$ for any integer r. Since $4 \le k \le n$, then there are $n-3$ cosets in term of $l_kS_3(n)'$. Therefore,

$$\begin{aligned} S_3(n)^{ab} &= \left\langle l_1S_3(n)', l_2S_3(n)', l_3S_3(n)', l_kS_3(n)' \right\rangle \\ &\cong C_2 \times C_2 \times C_0 \times C_0{}^{n-3} \\ &= C_2{}^2 \times C_0{}^{1+n-3} \\ &= C_2{}^2 \times C_0{}^{n-2}. \end{aligned}$$

□

Next, the generalization of $\nabla(S_3(3))$ of dimension n is given in the following theorem.

Theorem 2

The subgroup of $\nabla(S_3(n))$ is given as in the following :

$$\begin{aligned} \nabla(S_3(n)) = \Big\langle & [l_1,l_1^{\varphi}],[l_2,l_2^{\varphi}],[l_3,l_3^{\varphi}],[l_k,l_k^{\varphi}],[l_1,l_2^{\varphi}][l_2,l_1^{\varphi}], [l_1,l_3^{\varphi}][l_3,l_1^{\varphi}],[l_1,l_k^{\varphi}][l_k,l_1^{\varphi}], \\ & [l_2,l_3^{\varphi}][l_3,l_2^{\varphi}],[l_2,l_k^{\varphi}][l_k,l_2^{\varphi}],[l_3,l_k^{\varphi}][l_k,l_3^{\varphi}],[l_i,l_j^{\varphi}][l_j,l_i^{\varphi}] \Big\rangle \\ \cong\ & C_2{}^{n-3} \times C_4{}^2 \times C_0{}^{\frac{(n-1)(n-2)}{2}} \end{aligned}$$

for $k = 4,5,\ldots,n$ and $4 \le i < j \le n$.

Proof. By Lemma 2, $S_3(n)^{ab}$ is generated by the cosets $l_1S_3(n)'$, $l_2S_3(n)'$, $l_3S_3(n)'$ and

$l_k S_3(n)'$. Then, by Proposition 1, $\nabla(S_3(n)) = \langle [l_1,l_1^{\varphi}],[l_2,l_2^{\varphi}],[l_3,l_3^{\varphi}],[l_k,l_k^{\varphi}],$ $[l_1,l_2^{\varphi}][l_2,l_1^{\varphi}],$ $[l_1,l_3^{\varphi}][l_3,l_1^{\varphi}],$ $[l_1,l_k^{\varphi}][l_k,l_1^{\varphi}],$ $[l_2,l_3^{\varphi}][l_3,l_2^{\varphi}],$ $[l_2,l_k^{\varphi}][l_k,l_2^{\varphi}],$ $[l_3,l_k^{\varphi}][l_k,l_3^{\varphi}],$ $[l_i,l_j^{\varphi}][l_j,l_i^{\varphi}]\rangle$ for $k = 4,5,\ldots,n$ and $4 \le i < j \le n$.

By Proposition 4, both $[l_1,l_1^{\varphi}]$ and $[l_2,l_2^{\varphi}]$ have order 4, $[l_1,l_2^{\varphi}][l_2,l_1^{\varphi}],$ $[l_1,l_3^{\varphi}][l_3,l_1^{\varphi}],$ $[l_2,l_3^{\varphi}][l_3,l_2^{\varphi}]$ have order 2 and $[l_3,l_3^{\varphi}]$ has infinite order which are the same generators as $\nabla(S_3(n))$. Next, the order of the remaining generators will be determined. By Proposition 5(i) and Proposition 6(ii), then $([l_1,l_k^{\varphi}][l_k,l_1^{\varphi}])^2 = [l_1^2,l_k^{\varphi}][l_k,l_1^{2\varphi}] = [l_1^2,l_k^{\varphi}][l_1^2,l_k^{\varphi}]^{-1} = 1$. It is showed that $[l_1,l_k^{\varphi}][l_k,l_1^{\varphi}]$ has order 2. By using similar arguments, $[l_2,l_k^{\varphi}][l_k,l_2^{\varphi}]$ has order 2.

Next, we want to show that $[l_3,l_k^{\varphi}][l_k,l_3^{\varphi}]$ has infinite order. Suppose that the order of $[l_3,l_k^{\varphi}][l_k,l_3^{\varphi}]$ is finite. Then, for any integer r,s, it is shown that $[l_3^r,l_k^{s\varphi}][l_k^s,l_3^{r\varphi}] = ([l_3,l_k^{\varphi}][l_k,l_3^{\varphi}])^{rs} = 1$. Thus, $[l_k^s,l_3^{r\varphi}] = [l_3^r,l_k^{s\varphi}]^{-1}$. However, by the relations of $S_3(n)$, neither l_3^r or l_k^s is in $S_3(n)'$. Hence this is not true that the order of $[l_3,l_k^{\varphi}][l_k,l_3^{\varphi}]$ is finite. Therefore $[l_3,l_k^{\varphi}][l_k,l_3^{\varphi}]$ has infinite order. By using similar arguments, it is shown that $[l_k,l_k^{\varphi}]$ and $[l_i,l_j^{\varphi}][l_j,l_i^{\varphi}]$ also has infinite order.

Since $k = 4,5,\ldots n$ and $4 \le i < j \le n$, then there is $n-3$ generators in terms of $[l_k,l_k^{\varphi}],$ $[l_1,l_k^{\varphi}][l_k,l_1^{\varphi}],$ $[l_2,l_k^{\varphi}][l_k,l_2^{\varphi}],$ $[l_3,l_k^{\varphi}][l_k,l_3^{\varphi}]$ and $\frac{(n-3)(n-4)}{2}$ generators in terms of $[l_i,l_j^{\varphi}][l_j,l_i^{\varphi}]$. Therefore,

$$\begin{aligned}\nabla(S_3(n)) &\cong C_4 \times C_4 \times C_0 \times C_0^{\,n-3} \times C_2 \times C_2 \times C_2 \times C_2^{\,n-3} \times C_2^{\,n-3} \times C_0^{\,n-3} \times C_0^{\frac{(n-3)(n-4)}{2}} \\ &= C_2^{\,3+(n-3)+(n-3)} \times C_4^{\,2} \times C_0^{\,1+(n-3)+(n-3)+\frac{(n-3)(n-4)}{2}} \\ &= C_2^{\,2n-3} \times C_4^{\,2} \times C_0^{\frac{(n-1)(n-2)}{2}}. \qquad \square\end{aligned}$$

3. CONCLUSION

In this paper, the generalization of the central subgroup of the nonabelian tensor square of the Bieberbach group $S_3(n)$ with point group $C_2 \times C_2$ is constructed up after the generalizations of the polycyclic presentation and the abelianization are determined. These results can further be used to find other useful properties of $S_3(n)$ such as the nonabelian tensor square of the group.

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