GROSS PRODUCT GROWTH: A MULTIPLICATIVE FACTOR MODEL

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Published online: 7 March 2018

ABSTRACT

The goal of this study is to propose a differential equation which relates a scalar economic indicator to the rate of change in renewable production factors. Based on this equation, I endeavour to build a multiplicative factor model of gross product growth and suggest an indicator that determines stagnation level and economic advance.

Below the behavior of gross product change is addressed, by means of the vector analysis. Our analysis proves that the rate of gross product change equals the dynamic divergence integral of the product of an economy's peak output and the rate of change throughout all production factors' volumes. It is found that the rate of production factors volumes' change is equal to the dynamic divergence integral of the rate of factor change throughout the whole volume of factors. On the basis of the derived integral-differential equation, the sought indicator is obtained. Adequacy of the developed indicator is demonstrated through analysis of several European economies.

The resulting model is a differential free-boundary problem. Its solution allows to make predictions about the gross product growth. The indicator, proposed in the study, constitutes a new tool for analysis of economic growth.

Keywords: stagnation indicator, production factors, dynamic divergence, growth model.

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doi: http://dx.doi.org/10.4314/jfas.v10i6s.152
INTRODUCTION

Trends of industrial performance and various types of technical progress are traditionally analyzed using various production functions based on the methods and procedures of the theory of production factors. Such production functions were first built as single-factor functions by Johan Gustaf Knut Wick sell for capital gain that is conditioned upon wage level and percentage, and by Justus Freiherr von Liebig who defined the law of limiting factor (see Heady and Dillon, 1961: 667). Later on, E.O. Heady and J.L. Dillon (1961: 667) proposed a generalization of these models. C.W. Cobb and P.H. Douglas (1928) grounded and developed multiplicative production function for gross product calculation using the two production factors: labour force and physical capital: these advances were based on their analysis of statistical data on North American industry from 1899 - 1922. To describe the achieved results, the authors built a function that nowadays bears their names.

The further development of production functions approach was provided in R. G. D. Allen’s works (1960, 1968). The function he proposes is designed for modelling processes, in which a sharp increase of one of the factors affects the final product negatively.

The classical type of a mixed production function is represented by the function termed “constant elasticity substitution of factors” (CES) and its equivalents (see Arrow, Chenery, Minhas, and Solow, 1961, and Brown, 1966). The function is used when there is no information about the degree of interchangeability of the production factors but it is supposed that substitution of a factor will not significantly change the development level.

The function proposed by R.M. Solow (1956) is unusual in that the percentage change of production factor replacement limit, which is obtained by changing any of the factors by 1 percent, does not depend on the initial level of these factors.

The scholars also proposed lots of combinations of the functions we discussed here. For instance, R. Sato (1967) multiplied the CES function to the Cobb-Douglas function, which allowed him to introduce a multiplicative two-level function [8].

The very first attempt at measuring technologic progress (TP) was made by Jan Tinbergen (1962). His exponential multiplicative model took various manifestations of the TP into account. It can be characterized as a model of economic development with autonomous TP.

Nowadays, when the gross product changes largely depend on the technological advances, this results in an extensive use of production functions proposed by K.J. Arrow, R.M. Solow, M.

The most salient modern models of endogenous growth affected by the human capital were developed by T.W. Schultz (1962), W. Fellner (1969) and N.G. Mankiw, D. Romer, and D. Weil (1992).

Using these functions, we can develop indicators and measures which indicate the gross product growth and allow analyzing the health of an economy. I propose a new indicator that relates the rate of change of renewable production factors and the marginal productivity of economy to the change of gross product.

METHODOLOGY OF THE STUDY

Our study lies in the domain of econometrics – a system of statistical methods that are used to test economic data. Econometrics allows making economic analysis more efficient by shortening of study periods, wider coverage of factors that impact the economic performance of companies, and also due to formulating and solving new multidimensional problems of economic analysis, which cannot be achieved by traditional approximation methods.

To be able to apply mathematical methods to economic analysis, we have to meet a number of requirements:

- to apply the system approach to studies on national economy, to take into account multiple important correlations that are revealed in the process of interaction between economic agents;
- to build econometric models that reflect the quantitative parameters of economic processes required for the analysis;
- to improve and build up the economic information system.

The above requirements do not exhaust the inventory of econometrics, but they are always necessary for building an economic model.

Work on multi-dimensional objects requires application of vector analysis. Its methods can be extrapolated vectors. The contemporary mathematics of vector analysis does not allow for a time lag that impacts economic phenomena significantly. To bridge this gap, I provide the two statements proved below. They define my multiplicative model.
The model is elaborated within the set of assumptions proposed by W. Levine, R. Easterly (2001):
- economic growth is not necessarily stable over time; there are periods of booming growth, recession and depression, i.e. the growth is subject to variations which, in the totality of their causes, are syntheses of an economy's inner conditions and external factors;
- economic change relies on three main production factors (physical capital, labour resources and human capital);
- production factors are independent of each other;
- production factors can grow simultaneously but not necessarily in a uniform manner; the rate of renewal of each production factor (precisely it, and not the factors themselves) depends on the impact made by other factors and on investment;
- the key external source of production factors development is the amount of investment in them;
- the number of economic agents possessing the approximately similar volume of production factors is relatively large because the economy is a continuum;
- the government plays a decisive role in periods of fast acceleration or recession of economic development, by means of regulating derivative capital investment and implementing a policy of cooling or warming the economy.

Among the natural variables that impact the industrial productivity, environmental resources and climate should be mentioned; we do not consider them here. For my model I used only renewable factors.

RESULTS OF THE STUDY

Let us now consider a continuous scalar economic indicator \( \phi(L,K,H,t) \) and a vector economic indicator \( \bar{F}(r,t) = (F_L(L,K,H,t), F_K(L,K,H,t), F_H(L,K,H,t)) \) that are dependent on the volume of labour resources \( L \), physical capital \( K \), human capital \( H \) and time \( t \).

In the space \( \Omega \), we shall analyze a certain volume of production factors \( w \). If we choose in \( w \) a point with the coordinates \( r = (L,K,H) \); then, the \( \frac{d\overline{r}}{dt} = \overline{v}(r,t) \) value is the rate of factor renewal at this point of space.

It should be noted here that the analysis is not necessary restricted by the three production factors: in principle, the model is compatible with any number of factors. Introducing additional
factors will transform the situation to a multi-dimensional problem and make the explanation less clear but shall not change the key statements and formulae.

The value of rate is determined by the equation $|\mathbf{v}| = \sqrt{(v_x)^2 + (v_y)^2 + (v_z)^2}$. In order to determine the vector field of production factor renewal rates (the space of economy states), one can require to specify the renewal rate within a defined period of time, or to specify the direction of the rate vector direction while ignoring its value. Thus, it is required to build a class of curves, such as the tangents to them at each point of the production factor space will coincide, at a given point of time, with the direction of the rate of production factor renewal at the same point. For the field of production factor renewal rates, such curves are termeduch curves are called 'lines of development' of the factors; in the analysis of an arbitrary vector field of an economic indicator, the curve is identified as the line of development of this indicator.

The line of development of production factors is a line that, for this precise instant $t$, exhibit the property as follows: the vector $\mathbf{v}$ of production factor rate, calculated for each point along this line, is directed tangentially to this line. An economic agent that is situated on this line of development does not have the component of the production factor development rate working crosswise to it and therefore cannot traverse it.

Let us choose in space $\Omega$ a closed contour $C$ encircling surface $S_1$. In this surface we shall take initial conditions for economic agents and form a line of development of the rate of production factor change for these agents right up to surface $S_2$ (Fig. 1).

![Fig.1. Development corridor](image-url)
We will obtain a certain volume $w$ with surface $\Gamma$ and call the whole configuration ‘the corridor of economic development’. Development lines cannot cross its lateral boundary.

**Statement 1.** Let us suppose that the components of production factor change rate are dependent on the impact of all production factors. Now, for any continuous development corridor that contains first-order continuous derivatives, and also at the initial and final moments of time of vector economic indicator $\vec{F}(\vec{r}, t) = (F_L(L, K, H, t), F_K(L, K, H, t), F_H(L, K, H, t))$ that is dependent on production factors, we have the formula:

$$\int_{\Gamma} \vec{F} \cdot d\vec{S} = \int_{w} DIV(\vec{F})dV,$$

(1)

Equation (1), which uses the concept of dynamic divergence ($DIV$) for time-dependent values, shall take the form, described in (Kuznetsov, 2011: 46-51):

$$\int_{\Gamma} \vec{F} \cdot d\vec{S} = \int_{w} DIV(\vec{F})dV.$$

**Deduction.** Let us consider the behaviour of a continuously changing economic indicator $\vec{F}(\vec{r}, t) = (F_L(\vec{r}, t), F_K(\vec{r}, t), F_H(\vec{r}, t))$ (this can be the production factor renewal rate or the vector of production factors' marginal efficiencies) in the production factor space for period of time $\Delta t$. Now let us analyze a certain set of production factors for which it is possible to obtain the economic indicator with value $\vec{F}(\vec{r}, t)$ (a particular case is an is quant curve for a production function). From the set of factors, let us choose surface $S_1$ in the production factor space, such that at each point of it the vector of the economic indicator $\vec{F}(\vec{r}, t)$ is perpendicular to it (a particular case is an isocline for a production function). Now let us trace the development of indicator $\vec{F}(\vec{r}, t)$ from surface $S_1$ over time $\Delta t$ along development lines right up to a certain surface $S_2$. As an illustration, let us consider vector $\vec{F}(\vec{r}, t)$ instead of vector $\vec{v}(\vec{r}, t)$ on Figure 1.

Now we shall find the divergence of vector $\vec{F}(\vec{r}, t)$ in shrinking of a chosen volume of production factors to a point, based on a representation for space in a time-invariant case, which corresponds to the classical definition (see, for instance, Krasnov, 1978: 44-46):

$$DIV(\vec{F}(\vec{r}, t)) = \lim_{n \to 0} \frac{\int_{w} \vec{F}(\vec{r}, t) \cdot d\vec{S}}{\int_{w} dV},$$

where $\vec{F}(\vec{r}, t) \cdot d\vec{S}$ is a scalar product of the two vectors.
At the following stage, the resulting figure will be divided into infinitely small figures having elementary volumes \( w_i \), in which economic indicator \( F(\vec{r}, t) \) is constant. These elementary volumes have end surfaces \( S^1_i \) and \( S^2_i \) that are perpendicular to vector \( \vec{F}(\vec{r}, t) \), which follows from constancy \( \vec{F}(\vec{r}, t) \). The surface of elementary volume \( \Gamma_i \), according to the set up, consists of three components \( \Gamma_i = S^1_i + S^2_i + S_{side} \), where \( S^1_i \) and \( S^2_i \) are end surfaces of the chosen elementary volume and \( S_{side} \) - the side surface of the chosen volume. The requirement of coincidence of the side surface with economic development lines leads to the fact that integrals of the perpendicular vectors scalar product go to zero:

\[
\int_{\Gamma_i} \vec{F}(\vec{r}, t_i) \cdot d\vec{S} = \int_{S^1_i} \vec{F}(\vec{r}, t_i + \Delta t) \cdot d\vec{S} - \int_{S^1_i} \vec{F}(\vec{r}, t_i) \cdot d\vec{S} = \\
-\int_{S^1_i} \vec{F}(\vec{r}, t_i) \cdot d\vec{S} + \int_{S^1_i} \vec{F}(\vec{r}, t_i + \Delta t) \cdot d\vec{S} + \int_{S^2_i} \Delta \vec{F}(\vec{r}, t_i) \cdot d\vec{S} + \int_{S^2_i} \Delta \vec{F}(\vec{r}, t_i) \cdot d\vec{S},
\]

where \( \Delta \vec{F}(\vec{r}, t_i + \Delta t) = \vec{F}(\vec{r}, t_i + \Delta t) - \vec{F}(\vec{r}, t_i) \cdot \Delta \vec{F}(\vec{r}, t_i) = \vec{F}(\vec{r}, t_i) - \vec{F}(\vec{r}, t_i) \) and \( \Delta \vec{F} = (\Delta L_1, \Delta K, \Delta H_i) \).

In relation (2) at instant \( t_i + \Delta t \) surface \( S^2_i \) will be traversed by vector flux \( \vec{F}(\vec{r}, t_i + \Delta t) \) that entered the volume under consideration at instant \( t_i \) through surface \( S^1_i \) in volume \( \vec{F}(\vec{r}, t_i) \).

If the development corridor does not contain any agents of change of the economic indicator, then the sum of the first two integrals shall be equal to zero. The change of production factors in space is determined by vector equation \( \Delta \vec{r} = \vec{v} \Delta t_i \).

Now we shall multiply both sides of the equation by the rate vector and find:

\[
\Delta t_i = \frac{\Delta \vec{r} \cdot \vec{v}}{||\vec{v}||^2}.
\]

From the definition of rate circulation along a certain curve \( M \):

\[
\int_M v_i dl = \int_M v_x dL + v_y dK + v_z dH
\]

Where \( v_i \) is projection \( \vec{v} \) upon a tangent to curve \( M \).

From the last equation we get the following one:

\[
\frac{\partial \vec{F}(\vec{r}, t_i + \Delta t_i)}{\partial t} \Delta t_i + o(\Delta t_i)^2 = \int_M \frac{v_i}{||\vec{v}||^2} \frac{\partial \vec{F}(\vec{r}, t_i + \Delta t_i)}{\partial t} dl
\]
The third and fourth integrals (2) can be transformed into volume integrals. For this purpose, the integrands are to be decomposed into a Taylor series with a Peano remainder in the vicinity of points \( t \) and \( r_i \), respectively. From latter equations, it follows that:

\[
\Delta_i F(\vec{r}_i + \Delta \vec{r}_i, t_i) = F(\vec{r}_i + \Delta \vec{r}_i, t_i) + \frac{\partial F(\vec{r}_i + \Delta \vec{r}_i, t_i)}{\partial t} \Delta t_i + o(\Delta t_i)^2 = F_i + o(\Delta t_i)^2,
\]

\[
\Delta_i F(\vec{r}_i, t_i) = F(\vec{r}_i + \Delta \vec{r}_i, t_i) - F(\vec{r}_i, t_i) = \frac{\partial F(\vec{r}_i, t_i)}{\partial \vec{r}_i} \Delta \vec{r}_i + o(\Delta \vec{r}_i)^2,
\]

where

\[
F_i = \left( \int_{S_i} \frac{v_L}{|v|^2} \frac{\partial F_L}{\partial t} \, dS_i + \frac{v_K}{|v|^2} \frac{\partial F_K}{\partial t} \, dS_i + \frac{v_H}{|v|^2} \frac{\partial F_H}{\partial t} \, dS_i \right)
\]

\( \Delta r_i = \vec{n}_2 \cdot (\Delta L_i, \Delta K_i, \Delta H_i) \), \( o(\Delta t_i)^2 \), \( o(\Delta \vec{r}_i)^2 \) are small quantities of the second order; \( M \) is an arbitrary line of economic development within the chosen volume of production factors from instant \( t_i \) till instant \( t_i + \Delta t_i \).

If we integrate the resulting equations over \( S_i \) we get the following:

\[
\int_{S_i} \Delta_i F(\vec{r}_i + \Delta \vec{r}_i, t_i) \cdot dS = \int_{S_i} F_i \cdot dS + o(\Delta t_i)^2
\]

\[
\int_{S_i} \Delta_i F(\vec{r}_i, t_i) \cdot dS = \int_{S_i} \frac{\partial F(\vec{r}_i, t_i)}{\partial \vec{r}_i} \Delta \vec{r}_i \cdot dS + o(\Delta \vec{r}_i)^2
\]

Since \( v = \text{const} \) in the elementary volumes under consideration and since the following equation is valid:

\[
\int_{S_i} \frac{\partial F}{\partial r_i} \Delta r_i \cdot dS = \int_{S_i} \frac{\partial F}{\partial r_i} \vec{n}_2 \cdot dV + o(\Delta r_i)^2 = \int_{S_i} \left( \frac{\partial F}{\partial L} + \frac{\partial F}{\partial K} + \frac{\partial F}{\partial H} \right) \vec{n}_2 \cdot dV + o(\Delta r_i)^2,
\]

then we finally get relation (2) in the form as follows:

\[
\int_{S_i} F_i(\vec{r}_i, t_i) \cdot dS = \int_{S_i} F_i \cdot dS + o(\Delta t_i)^2 + \int_{S_i} \frac{\partial F}{\partial \vec{r}_i} \Delta \vec{r}_i \cdot dS + o(\Delta \vec{r}_i)^2 =
\]

\[
= \int_{S_i} \left( \frac{v_L}{|v|^2} \frac{\partial F_L}{\partial t} + \frac{v_K}{|v|^2} \frac{\partial F_K}{\partial t} + \frac{v_H}{|v|^2} \frac{\partial F_H}{\partial t} + \frac{\partial F_L}{\partial L} + \frac{\partial F_K}{\partial K} + \frac{\partial F_H}{\partial H} \right) dV +
\]

\[
+ \int_{S_i \sim S(i)} \vec{n}_2 \cdot \left( F_i + \frac{\partial F}{\partial \vec{r}_i} \right) dS + o(\Delta t_i)^2 + o(\Delta \vec{r}_i)^2,
\]
where $S(l)$ is the area of development corridor's cross-section in the production factors space, formed by a closed contour, such that $w_i = S(l) \cdot M$; quantity $S_i^2 - S(l)_{i}^2$ of the second order of smallness over $l$; $n_2$ is a normal to $S_{i}^2$, $d\vec{S} = \vec{n}_2 dS$.

The direction of economic indicator $\vec{F}(\vec{r}, t_i)$ cannot intersect the development corridor; therefore the integral over $\Gamma_t$ is equal to integral over $S_{i}^2$.

Now let us sum together the last equation over all elementary volumes and take into account the fact that on conjugate boundaries the integrals shall shrink due to oppositeness of the normals to the surfaces, while on the sides they will be equal to zero. Then, we consider the limit with the number of partitions tending to zero with volumes also tending to zero, i.e. $\Delta t \to 0$ and $\Delta r \to 0$, and take into account that the last integral in the right-hand side, with contraction of chosen volume $(\Delta t \to 0)$, has the order of smallness higher than that of a differential over $w$. As a result, we get an equation for the whole figure:

$$\int_{\Gamma} \vec{F}(\vec{r}, t) \cdot d\vec{S} = \int_{w} \left( \frac{\partial F_L}{\partial t} + \frac{\partial F_K}{\partial t} + \frac{\partial F_L}{\partial L} + \frac{\partial F_K}{\partial K} + \frac{\partial F_L}{\partial H} \right) dV.$$ 

Since this relation is valid for an arbitrary volume $w$, it must also be identical for the whole economy.

The resulting assertion is valid for any vector indicator. $\vec{F}(\vec{r}, t)$ can be interpreted as the vector of production factor renewal rate or the vector of factors' marginal efficiencies.

A note. Dynamic convergence of an economic indicator is uniquely determined and does not depend on the choice of the coordinate system.

Indeed, suppose that there is a point in the space of production factors displaying two different divergences: $D_1 = DIV(\vec{F}(\vec{r}, t))$ and $D_2 = DIV(\vec{F}(\vec{r}, t))$. Then, as it follows from the above assertion:

$$\int_{w} (D_1 - D_2) dV = \int_{\Gamma} \vec{F}(\vec{r}, t) \cdot d\vec{S} - \int_{\Gamma} \vec{F}(\vec{r}, t) \cdot d\vec{S} = 0.$$ 

Since the divergences are different, $D_1 - D_2 \neq 0$, and this in equation shall persist in the vicinity of our point. If we call this vicinity an integration range, we get the following:

$$\int_{w} (D_1 - D_2) dV \neq 0,$$
which is in contradiction to our equation. Thus, in all the points of the production factor space these divergences do coincide.

**Inference.** If we have a valid equation $DIV(\bar{F}(\bar{r},t)) = 0$, then the flux of economic indicator $\bar{F}(\bar{r},t)$ through any tranverse cross-section of the development corridor is constant.

According to the development corridor definition, the flux through side surfaces is zero, while from the established assertion it follows that:

$$\int_{S_1} F(\bar{r},t) \cdot d\bar{s} = \int_{S_2} F(\bar{r},t) \cdot d\bar{s}.$$  

From this inference we can conclude that dynamic convergence is a measure of vector indicator change.

**Statement 2.** Suppose that we have a scalar function $\varphi(\bar{r},t)$ that describes the behaviour of an economic indicator. Let us consider a limited time-dependent volume of production factors $w$. Suppose that function $\varphi(\bar{r},t)$ is differentiable in the volume under consideration and on its boundaries. Suppose that volume $w$ is transformable from volume $w_0 = w(t_0)$, ($t_0$— initial instant) by constant changes which result from transformation of elements $w_0$ along the lines determined by the production factor renewal rate vector $\bar{v}$.

In this case, the following identity relation is valid:

$$\frac{d}{dt} \int_{w} \varphi(\bar{r}(t),t) dV = \int_{w} DIV(\varphi(\bar{r})) dV$$  \hspace{1cm} (3)

where $DIV(\bar{F}(\bar{r},t)) = \frac{v_L}{|\bar{v}|} \frac{\partial F_L}{\partial t} + \frac{v_K}{|\bar{v}|} \frac{\partial F_K}{\partial t} + \frac{v_H}{|\bar{v}|} \frac{\partial F_H}{\partial t} + \frac{\partial F_L}{\partial L} + \frac{\partial F_K}{\partial K} + \frac{\partial F_H}{\partial H}$  [27, c. 46-51].

**Deduction.** Let us now calculate a derivative over the moving volume $w$ in the space $\Omega$:

$$\frac{dA(\bar{r},t)}{dt} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial L} v_L + \frac{\partial A}{\partial K} v_K + \frac{\partial A}{\partial H} v_H,$$

where $A(\bar{r},t) = \int_{w} \varphi(\bar{r}(t),t) dV$.

The first term in the right-hand side can be represented as:

$$\frac{\partial}{\partial t} \int_{w} \varphi(\bar{r}(t),t) dV = \lim_{\Delta t \to 0} \frac{\int_{w} \varphi(\bar{r},t+\Delta t) dV - \int_{w} \varphi(\bar{r},t) dV}{\Delta t} =$$
Volume \( w \) can be always divided into infinitely small volumes \( w_i \), for which the rate of production factor change is constant. The change of production factors in space is determined by vector equation \( \Delta \vec{\mathbf{r}} = \vec{v} \Delta t \).

Now we shall scalarly multiply both sides of the equation by rate vector \( \vec{v} \) and find:

\[
\Delta t_i = \frac{\Delta \vec{\mathbf{r}}_i \cdot \vec{v}}{||\vec{v}||^2}.
\]

For each volume \( w_i \), the following equation is valid:

\[
\int_{w_i \cap \Delta t} \left[ \phi(\vec{r}, t + \Delta t) - \phi(\vec{r}, t) \right] dV + \int_{w_i \cap \Delta t} \phi(\vec{r}, t) dV = \int_{w_i \cap \Delta t} \vec{v} \cdot \vec{v} \frac{\partial}{\partial t} \phi(\vec{r}(t), t) dV \left( \phi(\vec{r}, t) \right) dV = \int_{w_i \cap \Delta t} \frac{\partial}{\partial t} \phi(\vec{r}(t), t) dV.
\]

Now let us sum across all volumes and consider the limit with the number of partitions tending to zero with volumes that (also) tend to zero at \( \Delta t \to 0 \). Given that the volume \( w - w_0 \) tends to zero, we get the following:

\[
\frac{\partial}{\partial t} \int w \phi(\vec{r}(t), t) dV = \int w \vec{v} \cdot \vec{v} \frac{\partial}{\partial t} \phi(\vec{r}(t), t) dV = \int w \frac{\partial}{\partial t} \phi(\vec{r}(t), t) dV.
\]

Now we shall analyze the scalar product of the vector of production factor change rate \( \vec{v} \) to \( \text{grad}(A) \). Let us suppose that \( p \) is one of the production factors \( \vec{r} = (L, K, H) \):

\[
v_p \frac{\partial}{\partial p} \int w \phi(\vec{r}(t), t) dV = \lim_{\Delta p \to 0} \frac{v_p \int w \phi(\vec{r} + \Delta \vec{r}, t) dV - v_p \int w \phi(\vec{r}, t) dV}{\Delta p} =
\]

\[
v_p \int w \left[ \phi(\vec{r} + \Delta \vec{r}, t) - \phi(\vec{r}, t) \right] dV + v_p \int w \phi(\vec{r}, t) dV
\]

\[
= \lim_{\Delta p \to 0} \frac{v_p \int w \phi(\vec{r} + \Delta \vec{r}, t) dV - v_p \int w \phi(\vec{r}, t) dV}{\Delta p} + \lim_{\Delta p \to 0} \frac{v_p \int w \phi(\vec{r}, t) dV}{\Delta p}
\]

It is possible to divide the volume \( w - w_0 \) into small volumes where the rate of production factor change is constant, and then to place them under the integral sign \( v_p \). Then we sum across all the volumes and go over to the limit in \( \Delta p \to 0 \)

\[
\lim_{\Delta p \to 0} \frac{v_p \int w \phi(\vec{r} + \Delta \vec{r}, t) dV - v_p \int w \phi(\vec{r}, t) dV}{\Delta p} + \lim_{\Delta p \to 0} \frac{v_p \int w \phi(\vec{r}, t) dV}{\Delta p} = \int \frac{\partial}{\partial p} (v_p \phi(\vec{r}, t)) dV.
\]
Since the time increment tends to zero $\Delta t \to 0$, then $\Delta \bar{F} = \nabla \Delta t \to 0$. Provided that the volume $w - w_0$ tends to zero, we find out that the last limit is equal to zero. As a result, we have the following equality:

$$\frac{d}{dt} \int_w \phi(\bar{r}, t) dV = \int_w \nabla \cdot \nabla \phi(\bar{r}, t) dV + \int_w \frac{\partial}{\partial L} (v_L \phi(\bar{r}, t)) dV +$$

$$+ \int_w \frac{\partial}{\partial H} (v_H \phi(\bar{r}, t)) dV = \int_w \text{DIV}(\phi) dV .$$

Equality (3) is one of the representations of the classic flux continuity equality, described by Kochin (1965: 158-159], where the dynamic component is taken into account.

Suppose that in formula (3), $\phi(r, t) = 1$. This gives us the following:

$$\frac{d}{dt} \int_w dV = \frac{d w}{dt} = \int_w \text{DIV}(\nabla) dV . \quad (4)$$

The latter equality shows that the rate of production factor volume change (instantaneous absolute increment) is equal to the dynamic divergence integral of the rate of factor change throughout the whole volume of factors involved in production. As mentioned before, divergence of an indicator is the measure of change of this indicator. The cumulative measure of production factors' change rate is equal to the absolute increment of production factors at every instant.

Let us now consider another application of our statement. Let us suppose that function $\phi(r, t) = \frac{dY}{dw} = q$ is the marginal productivity of the economy. From equation (3) we get:

$$\frac{dY}{dt} = \int_w \text{DIV}(q \nabla) dV . \quad (5)$$

The equation says that the rate of gross product change is equal to the dynamic divergence integral over the product of economy's peak output and the rate of factor change throughout all volumes of production factors. The cumulative measure of the product of economy's peak output and the rate of production factor change rate is equal to the absolute increment of gross product at every instant.

Now we will consider the stage of stagnation, i.e. $Y(r, t) = \text{const}$, then (5) takes the form of:

$$\frac{dR_\ell}{dt} + \text{DIV}(\nabla) = 0 . \quad (6)$$

where $R_\ell = \ln|\phi|$ is the logarithmic marginal productivity of the economy.
The equation demonstrates the behaviour of logarithmic performance of the economy to depend on the rate of production factor renewal, in the case if the economy is in the stage of stagnation. Equations (4) - (6) are considered within the development corridor (Fig. 1). The focus in problems, related to this figure, is finding the rate of change of production factors, the rate of change of the gross product and the efficiency of economy at an instant \( t_0 \). For limiting and initial conditions in each of the problems we can take the known variables: the rate of production factors' renewal and the output of economy at some instant. The instant is chosen when all values of the production factors are known, i.e. on the cross-section of the development corridor at the instant \( t_0 \). It follows from the definition of the development corridor that flux through lateral faces is equal to zero, i.e. \( \frac{\partial \bar{v}}{\partial n} = 0 \) and \( \frac{\partial q}{\partial n} = 0 \), where \( n \) is the outer normal to the lateral face of the development corridor.

Let us now introduce the notion of stagnation indicator of an economy:

\[
Z = \frac{1}{q} \frac{dq}{dt} + DIV(\bar{v})
\]

The indicator is derived from equation (5) and is calculated for a time interval that must tend to zero. The longer is the time interval, the less accurate is the indicator. When the indicator is tending to zero, it allows us to conclude that the economy enters a stagnation process. Positive values point to economic growth, and negative ones to recession.

Let us now consider the behaviour of the indicator, proposed above, as illustrated by data from some European economies. The statistics were retrieved from the Euro stat web-site (Obshchaya i regional'naya statistika, 2016).

The crisis of 2008 led to the drop in the GDP of France by 2.6 percent in 2009. The beginning of the GDP decrease in 2007 is shown by the indicator, which shows negative values, too. In 2010 - 2011 the GDP grew slightly (this fact is reflected in the positive values of the indicator), and in 2012 stagnation set in, as shown by the stagnation indicator. In 2015 we see economic recovery; not surprisingly this period is characterized by positive values of the indicator (Fig. 2).
The UK belongs to five healthiest industrial economies of the world. The financial crisis of 2008 destroyed the finance sector of the economy. The crisis brought down residential property prices, triggered consumer debt accumulation and paved way for a global economic decline, which pushed the economy to recession in late 2008. Negative indicator values, beginning from 2007, do correspond to the actual economic context. In 2010, the UK government worked out and implemented certain measures aimed at stimulating economic growth: nationalization of some banks, temporary tax relaxation, increasing spending on capital projects. In the same year, the budget deficit decreased to 11 percent of GDP; in 2015, it was only 1 percent. The economic upswing in 2010 is reflected in the behaviour of the indicator, as well as its subsequent downfall. In late 2011, the government announced new policies of harsh austerity: VAT was pushed from 17.5 up to 20 percent. It was already in 2013 that GDP growth was equal to 1.4 percent. In the second half of 2013, due to recovery of the residential property market, and growth in consumer spending, GDP began to bounce back, which is proved by the suggested indicator (Fig. 3).
The inconsistent behaviour of the indicator in 2012 can be accounted for by the fact that the statistics were retrieved once each year, which brings about a very rough finite-difference pattern of derivatives in the calculation of the indicator.

German economy has showed a convincingly stable and competitive behaviour during the crisis times. The reforms initiated by G. Schroeder's government were what ensured relative sustainability of the economy in the course of the crisis. Thanks to these measures, they did not let the employment level collapse in the 2008 - 2009 crises. If German GDP in 2009 shrank by 5.1 percent, which is demonstrated by the indicator, then already in 2010 it recovered by 3.7 percent and by another 3.4 percent in 2011. The subsequent two years were the time of stagnation, which our indicator only confirms. Since 2014 we observe a slight growth, at the average, by 1.5 percent annually; again, this trend is confirmed by the indicator's behaviour (Fig. 4).
Of all the European economies I have considered, Italy was most affected by the crisis. In the period of economic crisis that took seven quarters, the Italian economy shrank by 6.8 percent (this fact corresponds to negative or close to zero behaviour of the indicator. The public debt of Italy in 2010 amounted to 116 percent of GDP. Thanks to austerity policies introduced in December 2010, Italy demonstrated positive economic figures but later stagnation came. The values of the indicator correspond well to this behaviour. Starting from 2015, Italy demonstrates some growth again, which is shown by a positive value of the indicator (Fig. 5).
DISCUSSION

The assertions, proven in the study, require only some very general conditions to be valid. The first assertion is valid when the vector value $\vec{F}(\vec{r}, t)$ is the vector of production factor growth, or the vector of peak output, or the vector of investing in production factors. By the term 'investment in labour resources', any funding of social programs (exclusive of pension schemes) is ment, while 'investment in human capital' means various financing of intellectual development of a person and his/her professional qualification/skills.

The second statement is valid for a rather wide range of time-varying scalar economic values that are dependent on renewable production factors. The function $\varphi(\vec{r}, t)$ can be interpreted as volume of consumption, or volume of investment, or the absolute growth of production factors.

Equations (4)-(6) allow formulating integral-differential free-boundary problems. Their solution will provide us with tools for predicting changes in production factors and gross product.

The resulting statements are not only valid for macroeconomic parameters, but they can also be applied to modelling micro-economic phenomena.

The proposed indicator $Z$ can be used as an additional instrument for forecasting the behaviour of national economies. For a more accurate calculation of the parameters, monthly or at least quarterly statistics of the behaviour of production factors. This requirement results from the fact that the determination of this indicator involves differentials, whose accurate calculation depends on the time step.

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How to cite this article: