THE Z-TRANSFORM APPLIED TO BIRTH-DEATH MARKOV PROCESSES

H. N. Kundaeli,
Department of Electronics and Communication,
University of Dar-es-Salaam,
P.O. Box 35194,
Dar-es-Salaam, Tanzania.

ABSTRACT
Birth-death Markov models have been widely used in the study of natural and physical processes. The analysis of such processes, however, is mostly performed using time series analysis. In this report, a finite state birth-death Markov process is analyzed using the z-transform approach. The performance metrics of the system and their variation with the system parameters are then derived and presented.

Keywords: *Markov processes, birth-death models, frame synchronization, queuing systems, z-transform.*

INTRODUCTION

The study of physical systems has relied heavily on the approach of building models, and using the models to both analyze and design the systems. Some of such models initially developed to study problems in mathematics and physics have turned out to be very novel tools for investigating and solving problems in other fields of study (Rojdestvenski and Cottam, 2000; Drummond, 2004). One of such models is the birth-death markov model, which is used to characterize processes involving some kind of population. Since many processes can be analyzed using this model, it has been widely used to study the dynamics of natural and physical systems. Among the areas in which birth-death models have been employed are communication systems, computer data storage, and biological systems. In communication systems the models have been used to investigate packet transmission in CDMA-based communication (Perez-Romero *et al*., 2003), diversity in systems employing receiver diversity (Yang and Alouini, 2004), block error processing for systems operating in fading environments (Hueda and Rodrigruez, 2004), and synchronization in high speed communication systems (Kundaeli, 1998; 2002). In computer systems the models have been used to characterize the storage and flow of information (Kleinrock, 1975) and the allocation of channels in networks supporting mobile computing (Lee *et al*., 1999). In biological systems the models have been used to study population extinction times (Tomiuk and Loeschcke, 1994) and the evolution of genes (Karev *et al*., 2004). Birth-death models have and can therefore be used to solve a diversity of problems.

The approach normally used to study processes exhibiting Markovian characteristics is time series analysis. The derivation of the performance metrics, however, can also be accomplished by using the z-transform, which is quite applicable to cases where the systems under investigation exhibit discrete‑time dependence (Kleinrock, 1975). In this report, the z‑transform is used to investigate the performance of a process exhibiting birth‑death Markovian properties. The performance parameters of the process are derived, and it is then shown how the derived scheme can be used to characterize various real life processes.

SYSTEM ANALYSIS

The transition diagram of the system analyzed in this report is given in Fig. 1, having N+1 states numbered 0 to N with the transition probabilities between the states also shown. The transition diagram can be reduced to that of Fig. 2 using state reduction techniques found in Howard (1971). In making the derivations in this report however, the same notations used in earlier reports by the author (e.g. Kundaeli, 2002) have been maintained in order to ensure consistency. We then introduce the following extra parameters: $K = n$, $M = m - n$ and $J = N - m$ with $0 \le n < m \le N$ to obtain the partial transfer functions in Fig. 2 as

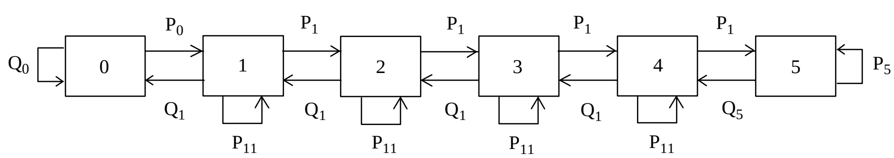

Fig. 1: The transition diagram of a finite state birth-death process with N = 5

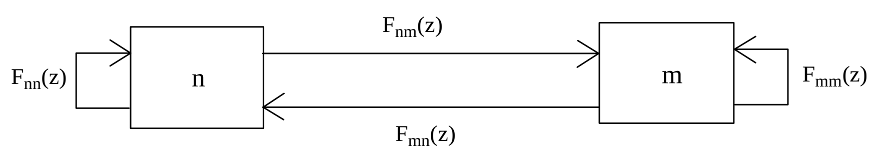

Fig. 2: The reduced transition diagram of Fig. 1.

$$F_{nm}(z) = \begin{cases} \dfrac{P_0 P_1^{M-1} z^M}{T_{21}(M,z)}, & n = 0 \\ \dfrac{P_1^M z^M}{T_{21}(M,z)}, & n > 0 \end{cases} \quad (1)$$

$$F_{mn}(z) = \begin{cases} \dfrac{Q_N Q_1^{M-1} z^M}{T_{21}(M,z)}, & m = N \\ \dfrac{Q_1^M z^M}{T_{21}(M,z)}, & m < N \end{cases} \quad (3)$$

$$F_{nn}(z) = \begin{cases} F_{nn}(K,z), & M \le 1 \\ F_{nn}(K,z) + F_{nn}(M,z), & M > 1 \end{cases} \quad (2)$$

$$F_{mm}(z) = \begin{cases} F_{mm}(J,z), & M \leq 1 \\ F_{mm}(J,z) + F_{mm}(M,z), & M > 1 \end{cases} \tag{4}$$

where

$$F_{nn}(K,z) = \begin{cases} Q_0 z, K=0, \\ \dfrac{Q_1 P_0 z^2}{(1-Q_0 z)} + P_{11} z, K=1 \\ \dfrac{Q_1 P_1 z^2 \left[(1-Q_0 z) T_{22}(K,z) - Q_1 P_0 z^2 T_{23}(K,z)\right]}{(1-Q_0 z) T_{21}(K,z) - Q_1 P_0 z^2 T_{22}(K,z)} + P_{11} z, K>1 \end{cases} \tag{5}$$

$$F_{nn}(M, z) = \begin{cases} \dfrac{P_0 Q_1 z^2 T_{22}(M, z)}{T_{21}(M, z)}, & n = 0 \\ \dfrac{P_1 Q_1 z^2 T_{22}(M, z)}{T_{21}(M, z)}, & n \neq 0 \end{cases} \tag{6}$$

$$F_{mm}(J,z) = \begin{cases} P_N z, J=0 \\ \dfrac{P_1 Q_N z^2}{(1-P_N z)} + P_{11} z, J=1 \\ \dfrac{P_1 Q_1 z^2 \left[(1-P_N z) T_{22}(J,z) - P_1 Q_N z^2 T_{23}(J,z)\right]}{(1-P_N z) T_{21}(J,z) - P_1 Q_N z^2 T_{22}(J,z)} + P_{11} z, J>1 \end{cases} \tag{7}$$

$$F_{mm}(M,z) = \begin{cases} \dfrac{Q_N P_1 z^2 T_{22}(M,z)}{T_{21}(M,z)}, & m = N \\ \dfrac{Q_1 P_1 z^2 T_{22}(M,z)}{T_{21}(M,z)}, & m \neq N \end{cases} \tag{8}$$

$$T_{2i}(U,z) = \sum_{k=0}^{U_i} (-1)^k A(U,k,i) (1-P_{11} z)^{U-2k-i} \left(P_1 Q_1 z^2\right)^k \tag{9}$$

$$A(U,k,i) = \begin{cases} \dfrac{(U-k-i)!}{(U-2k-i)!\,k!}, & U \geq i \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

and $U_i = \text{floor}\{(U-i)/2\}$.

Using (1) and (2), the transfer function from state n to m is given by

$$\Phi_{nm}(z) = \frac{F_{nm}(z)}{1 - F_{nn}(z)} \tag{11}$$

from which we obtain the transition time from state n to m as

$$L_{nm} = \frac{d}{dz}\left(\Phi_{nm}\right)\Big|_{z=1}. \tag{12}$$

We then use the following notations in the ensuing derivations

$$T_{ijU} = T_{ij}(U,z)\Big|_{z=1}, \quad T'_{ijU} = \frac{d}{dz}\left(T_{ij}(U,z)\right)\Bigg|_{z=1} \tag{13}$$

and consider two cases: $n = 0$, $M > 1$ and $n > 1$, $M > 1$ because the other cases can be obtained from them.
When $n = 0$ and $M > 1$

$$F_{nm}(z) = \frac{P_0 P_1^{M-1} z^M}{T_{21}(M,z)} \tag{14}$$

and $$F_{nn}(z) = Q_0 z + \frac{P_0 Q_1 z^2 T_{41}(M,z)}{T_{21}(M,z)}. \tag{15}$$

Therefore

$$\Phi_{nm}(z) = \frac{P_0 P_1^{M-1} z^M}{(1-Q_0 z)T_{21}(M,z) - P_0 Q_1 z^2 T_{22}(M,z)}. \tag{16}$$

If we apply (12) to (16) and perform some algebraic manipulations we obtain L_{nm} as

$$L_{nm} = \frac{T_{21M} + P_0 G_M(P_1)}{P_0 P_1^{M-1}}. \tag{17}$$

where

$$\begin{aligned} G_M(P_1) = {} & (1 - P_{11})^{M-2} \\ & + \sum_{k=1}^{S-1} (-1)^k (M-1)(1-P_{11})^{M-k-1} P_1^k \left[\sum_{s=0}^{kc} U_s + \sum_{s=0}^{kc^*} V_s - \sum_{s=0}^{kc} W_s \right] \\ & + \sum_{k=S}^{2S-2} (-1)^k (M-1)(1-P_{11})^{M-k-1} P_1^k \left[\sum_{s=k-S+1}^{kc} U_s + \sum_{s=k-S}^{kc^*} V_s - \sum_{s=k-S+1}^{kc} W_s \right] \\ & - \sum_{k=1}^{S-1} (-1)^k (1-P_{11})^{M-k-2} P_1^k \left[\sum_{s=0}^{kc} U_s^* + \sum_{s=0}^{kc^*} V_s^* - \sum_{s=0}^{kc} W_s^* \right] \\ & - \sum_{k=S}^{2S-2} (-1)^k (1-P_{11})^{M-k-2} P_1^k \left[\sum_{s=k-S+1}^{kc} U_s^* + \sum_{s=k-S}^{kc^*} V_s^* - \sum_{s=k-S+1}^{kc} W_s^* \right] \end{aligned} \tag{18}$$

when M is even and given as $M = 2S$ and

$$
\begin{aligned}
G_M(P_1) = {} & (1-P_{11})^{M-2} + S\,P_1^{M-2} \\
& + \sum_{k=1}^{S-1} (-1)^k (M-1)(1-P_{11})^{M-k-1} P_1^k \left[\sum_{s=0}^{kc} U_s + \sum_{s=0}^{kc*} V_s - \sum_{s=0}^{kc} W_s \right] \\
& + \sum_{k=S}^{2S-2} (-1)^k (M-1)(1-P_{11})^{M-k-1} P_1^k \left[\sum_{s=k-S+1}^{kc} U_s + \sum_{s=k-S}^{kc*} V_s - \sum_{s=k-S}^{kc} W_s \right] \\
& - \sum_{k=1}^{S-1} (-1)^k (1-P_{11})^{M-k-2} P_1^k \left[\sum_{s=0}^{kc} U_s^* + \sum_{s=0}^{kc*} V_k^* - \sum_{s=0}^{kc} W_s^* \right] \\
& - \sum_{k=S}^{2S-2} (-1)^k (1-P_{11})^{M-k-2} P_1^k \left[\sum_{s=k-S+1}^{kc} U_s^* + \sum_{s=k-S}^{kc*} V_s^* - \sum_{s=k-S}^{kc} W_s^* \right]
\end{aligned}
\qquad (19)
$$

when M is odd and given as $M = 2S + 1$ with

$$
\begin{aligned}
U_s &= \frac{(M-k+s-2)!}{(M-2k+2s-2)!\,(k-2s)!\,s!} \\
V_s &= \frac{(M-k+s-1)!}{(M-2k+2s)!\,(k-2s-1)!\,s!} \\
W_s &= \frac{(M-k+s-1)!}{(M-2k+2s-1)!\,(k-2s)!\,s!}
\end{aligned}
\qquad (20)
$$

and

$$
\begin{aligned}
U_s^* &= \frac{(M-k+s-2)!}{(M-2k+2s-3)!\,(k-2s)!\,s!} \\
V_s^* &= \frac{(M-k+s-1)!}{(M-2k+2s-1)!\,(k-2s-1)!\,s!} \\
W_s^* &= \frac{(M-k+s-1)!}{(M-2k+2s-2)!\,(k-2s)!\,s!}
\end{aligned}
\qquad (21)
$$

k_c = floor{k/2} and k_c^* = floor{(k-1)/2}.

The elaborate algebraic manipulations have been omitted in deriving (18) and (19) but interested readers can contact the authors for details. Note that when $M = 1$, $G_M(P_1) = 0$ and therefore

$$L_{nm} = \frac{1}{P_0} \tag{22}$$

When $n > 1$ and $M > 1$ we obtain

$$F_{nm}(z) = \frac{P_1^M z^M}{T_{21}(M,z)} \tag{23}$$

and

$$F_{nn}(z) = \frac{Q_1 P_1 z^2 \left[(1-Q_0 z) T_{22}(K,z) - Q_1 P_0 z^2 T_{23}(K,z)\right]}{\left\{ \begin{array}{c} (1-Q_0 z) T_{21}(K,z) - Q_1 P_0 z^2 T_{22}(K,z) + P_{11}(z) \\ + \dfrac{P_1 Q_1 z^2 T_{22}(M,z)}{T_{21}(M,z)} \end{array} \right\}} \tag{24}$$

thus giving

$$\phi_{nm}(z) = \frac{P_1^M z^M \left[(1-Q_0 z) T_{21}(K,z) - Q_1 P_0 z^2 T_{22}(K,z)\right]}{\left\{ \begin{array}{l} (1-P_{11} z) T_{21}(M,z) \left[(1-Q_o z) T_{21}(K,z) - Q_1 P_o z^2 T_{22}(K,z)\right] \\ \quad - Q_1 P_1 z^2 \left[(1-Q_0 z) T_{22}(K,z) - Q_1 P_0 z^2 T_{23}(K,z)\right] T_{21}(M,z) \\ \quad - P_1 Q_1 z^2 T_{22}(M,z) \left[(1-Q_0 z) T_{21}(K,z) - Q_1 P_o z^2 T_{22}(K,z)\right] \end{array} \right\}}. \tag{25}$$

Again, using (12) and applying algebraic manipulations to (25) we obtain

$$L_{nm} = \frac{P_0 T_{21M} P_1^{K-1} + P_0 P_1^K G_M(P_1) + P_0 Q_1 T_{21M} \left[G_K(P_1) - P_1 H_K(P_1)\right] + Q_1^K T_{21M}}{P_0 P_1^{M+K-1}} \tag{26}$$

where $H_U(x) = G_{U-1}(x)$.
Note that when $n = 1$ and $M > 1$ then

$$L_{nm} = \frac{T_{21M}\left[P_0 + Q_1\right] + P_0 P_1 G_M(P_1)}{P_0 P_1^M} \tag{27}$$

when $n > 1$ and $M = 1$ then

$$L_{nm} = \frac{P_0 P_1^{K-1} + Q_1^K + P_0 Q_1 \left[G_K(P_1) - P_1 H_K(P_1)\right]}{P_0 P_1^K} \tag{28}$$

and when $n = M = 1$ then

$$L_{nm} = \frac{1 + P_o - P_1 - P_{11}}{P_o P_1}. \tag{29}$$

The transition times from state m to n can be obtained in a similar manner to those for n to m. Therefore, when $m = N$ and $M > 1$

$$\Phi_{mn}(z) = \frac{Q_N Q_1^{M-1} z^M}{(1 - P_N z) T_{21}(M, z) - Q_N P_1 z^2 T_{22}(M, z)} \tag{30}$$

and therefore

$$L_{mn} = \frac{T_{21M} + Q_N G_M(Q_1)}{Q_N Q_1^{M-1}}. \tag{31}$$

Likewise, when $m < N$ and $M > 1$ we obtain

$$\Phi_{mn}(z) = \frac{Q_1^M z^M \left[(1 - P_N z) T_{21}(K,z) - P_1 Q_N z^2 T_{22}(K,z)\right]}{\left\{\begin{array}{l} (1 - P_{11} z) T_{21}(M,z) \left[(1 - P_N z) T_{21}(K,z) - P_1 Q_N z^2 T_{22}(K,z)\right] \\ - P_1 Q_1 z^2 \left[(1 - P_N z) T_{22}(K,z) - P_1 Q_N z^2 T_{23}(K,z)\right] T_{21}(M,z) \\ - Q_1 P_1 z^2 T_{22}(M,z) \left[(1 - P_N z) T_{21}(K,z) - P_1 Q_N z^2 T_{22}(K,z)\right] \end{array}\right\}} \tag{32}$$

which gives

$$L_{mn} = \frac{Q_N T_{21M} Q_1^{J-1} + Q_N Q_1^J G_M(Q_1) + Q_N P_1 T_{21M} \left[G_J(Q_1) - Q_1 H_J(Q_1)\right] + P_1^J T_{21M}}{Q_N Q_1^{M+J-1}} \tag{33}$$

From the above results, when $m = N$-1 and $M = 1$ then

$$L_{mn} = \frac{1 + Q_N - Q_1 - P_{11}}{Q_N Q_1} \tag{34}$$

and when $m = N$ and $M > 1$

$$L_{mn} = \frac{T_{21M} + Q_N G_M(Q_1)}{Q_N Q_1^{M-1}}. \tag{35}$$

RESULTS AND DISCUSSION

The results of the analysis are given in Figs. 3 to 10. In these results, it is assumed that transitions between states take place at regular intervals denoted by T, and the transition times *Lnm* and *Lmn* are then given as multiples of T. Also, unless indicated otherwise, the parameters have been fixed at $P_0 = Q_N = 0.5$, $P_1 = P_{11} = 0.33$, $N = 10$, $n = 2$ and $m = 6$. Fig. 3 shows how the transition time from state 2 to 6 varies with the transition probability (P_0) in state 0. As expected, the transition time is very high at low values of P_0 indicating the high reluctance of the system to leave state 0. As P_0 increases, however, the transition time decreases as expected. In Fig. 4 it is seen that the transition time from state 2 to 6 decreases very sharply as P_1 increases, implying that the high value of P_1 forces the system to move to state 6 faster. It is also seen that P_1 has a higher effect on the transition time than P_0. Fig. 5 shows that the transition time increases with P_{11}. This implies that the system has a higher tendency to stay in any state as P_{11} increases. Fig. 6 shows the transition time as a function of *n* when *m*, the state to which the system is supposed to transit to, is fixed. It is seen that the transition time does not decrease sharply as *n* approaches *m* as would be expected. This can be attributed to the fact that the system spends appreciable time looping in the states below *n*, and this increases the transition time. Fig. 7 shows that the transition time increases as *m*, the state to which the system is to transit to, increases. It is also seen that this curve takes on a shape that is opposite to that of Fig. 6. Fig. 8 shows how the transition time from a higher state *m* to a lower one *n* varies with Q_N, the transition probability in state *N*. The shape of this curve resembles the one in Fig. 3 as expected. It is also seen in Fig. 9 that the transition time from state *m* to *n* decreases as *n* approaches *m*. This is expected because the distance between *m* and *n* decreases with *n*. Finally, Fig. 10 shows as expected that the transition time from state *m* to a fixed state *n* increases as *m* increases. The behaviour of the system has therefore been well illustrated by the presented plots. Whereas these results represent some sample behaviours of such

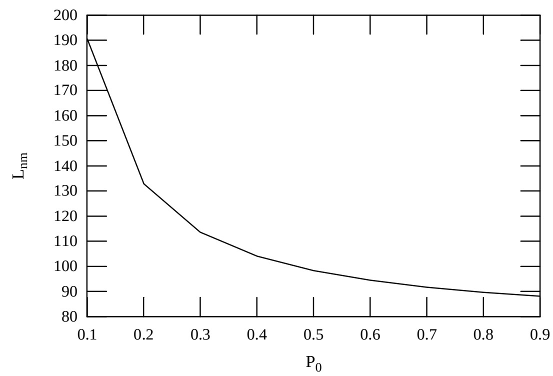

Fig. 3: The variation of Lnm with P_0 at $P_1 = P_{11} = 0.3$, n = 2 and m = 6.

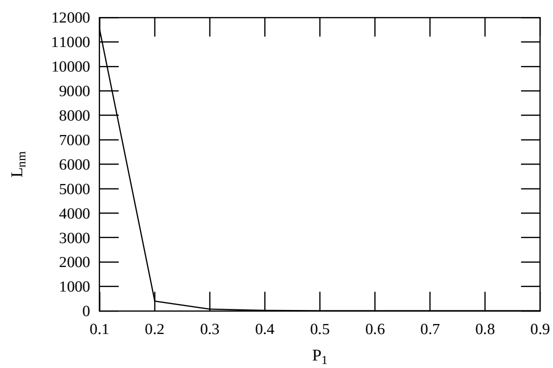

Fig. 4: The variation of Lnm with P_1 at $P_{11} = (1-P_1)/2$, $P_0 = 0.5$, n = 2 and m = 6.

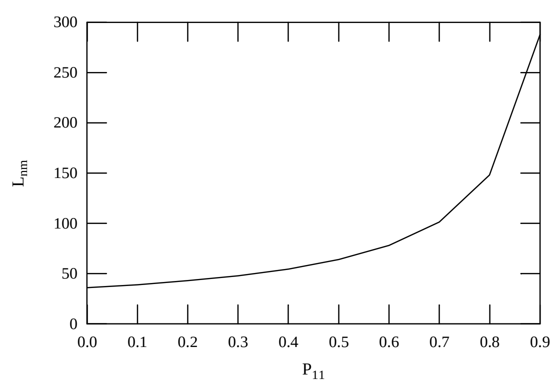

Fig. 5: The variation of Lnm with P_{11} at $P_1 = (1-P_{11})/2$, $P_0 = 0.5$, n = 2 and m = 6.

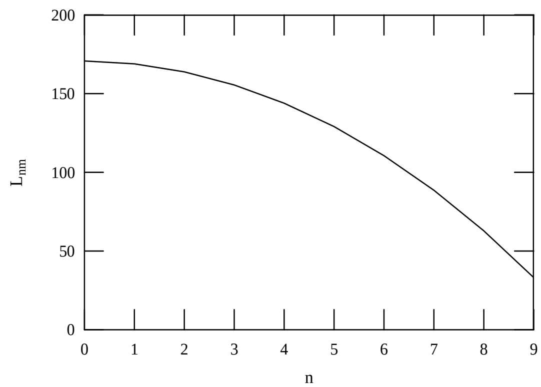

Fig. 6: The variation of Lnm with n at P_{11} = P_1 = 0.33, P_0 = 0.5 and m = N = 10.

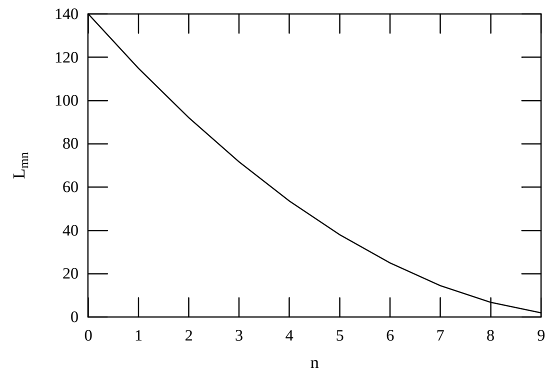

Fig. 9: The variation of Lmn with n at P_{11} = Q_1 = 0.33, Q_N = 0.5, and m = N = 10.

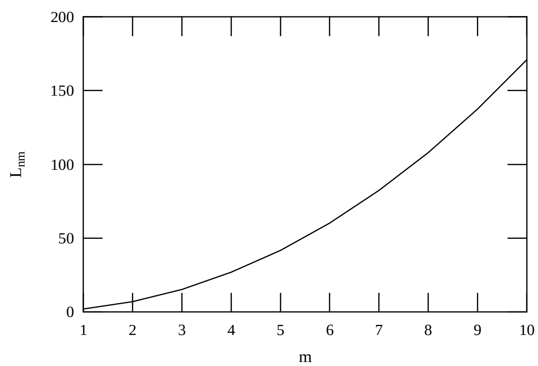

Fig. 7: The variation of Lnm with m at P_{11} = P_1 = 0.33, P_0 = 0.5, n = 0 and N = 10.

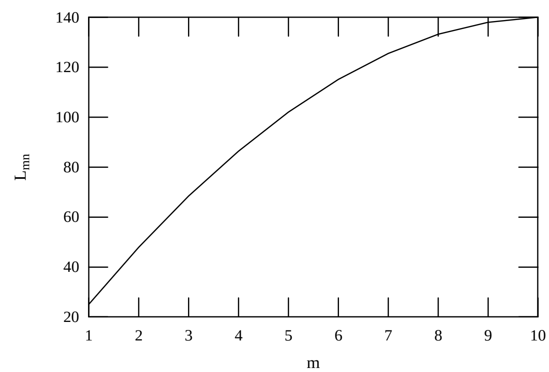

Fig. 10: The variation of Lmn with m at P_{11} = P_1 = 0.33, Q_N = 0.5, n = 0 and N = 10.

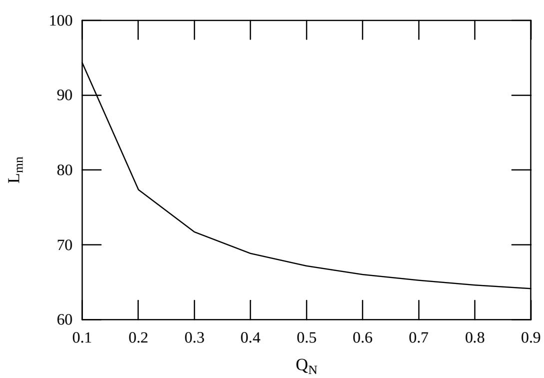

Fig. 8: The variation of Lmn with Q_N at P_{11} = Q_1 = 0.33, n = 2, m = 6 and N = 10.

a system, other behaviours can be obtained by using different parameters.

As mentioned earlier, some communication systems can be represented by the birth-death model. For example, in some communication systems a bidirectional counter is employed to implement the synchronization algorithm. In such cases, the synchronization states of the system are represented by the states of the counter, and the transition probabilities between the states of the counter represent the probabilities of receiving either corrupted or uncorrupted synchronization information. The transition times *Lnm* and *Lmn* then represent the time it takes the system to gain or lose synchronization respectively. The model can also be employed in queuing systems. In this report, however, the model is applicable if the single queue single server model is employed. In such cases, the durations needed for the number of users to change from *n* to *m* or *m* to *n* are *Lnm* and *Lmn* respectively. Finally, many biological systems can be represented by the birth‑death model. In such cases, the states used in this report represent the population of the biological system. Since the birth and death rates in a biological system are not necessarily constant, they need to be normalised to the population in each state, thus making the model investigated in this paper applicable. The time needed for the population to change from *n* to *m* is then *Lnm* whereas the time needed for the population to change from *m* to *n* is *Lmn*.

CONCLUSION

The analysis of a birth‑death process in which the birth and death transition probabilities are fixed has been investigated. It has been shown that the obtained results represent the expected behaviour of the system, and the investigated model can also be used for practical systems. It can for example be used to investigate frame synchronization systems that employ bidirectional counters to store the state and status of synchronization, queuing systems in which the arrival and service rates are constant, and the population dynamics of biological birth‑death systems. Further research in this area will consider cases where the transition probabilities between the states are not constant.

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