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THE Z-TRANSFORM APPLIED TO A BIRTH-DEATH PROCESS HAVING VARYING BIRTH AND DEATH RATES

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ABSTRACT
The analysis of a birth-death process using the z-transform was recently reported for processes having fixed transition probabilities between states. The current report extends that analysis to processes having transition probabilities that can differ from state to state. It is then shown that the model can be used to study practical queuing and birth-death systems where the arrival, birth, service and death rates may differ from state to state.

Keywords: *Markov processes, birth-death models, frame synchronization, queuing systems, z-transform*

INTRODUCTION
Models used in physics and mathematics have turned out to be very novel tools for the investigation and solving of problems in other fields of study (Gaver et al., 1984; Rojdestvenski and Cottam, 2000; Drummond, 2004). Markov models in particular have been used in modelling protein sequences in biological systems (Krogh et al., 1994), in modelling molecular sequences in proteins (Felsenstein and Churchill, 1996), in gene identification (Salzberg et al., 1998), and in the study of epidemics (Hernandez-Suarez, 2002). The models have also been widely used in the analysis of communication systems. Examples include the performance degradation arising from imperfect power control in land based satellite systems (Vazquez-Catro and Perez-Fontan, 2002), the performance of high speed communication systems (Kundaeli, 1998), packet transmission in CDMA-based communication systems (Perez-Romero et al., 2003), receiver diversity in communication systems (Yang and Alouini, 2004), and block error processing for systems operating in fading environments (Hueda and Rodrigruez, 2004).

Among the markov processes that have been widely studied are the birth-death processes, which are used in the analysis of systems whose states involve changes in the size of some population. They have therefore been used to study population extinction times in biological systems (Brockwell, 1985; Tomiuk, 1994), population evolutions from their molecular phylogenies (Nee et al., 1994), and the evolution of genes in living things (Karev et al., 2004). They have also been used to characterize information storage and flow in computer systems (Kleinrock, 1975), the allocation of channels in networks supporting mobile computing (Lee et al., 1999) and the transmission performance of frame synchronized communication systems (Kundaeli, 2002).

In a previous report, the z-transform was used to analyze a birth-death process in which the birth and death transition probabilities were the same in all states (Kundaeli, 2008). In this report, we extend on the results of that report to analyze a birth-death process in which the birth and death transition probabilities can vary from state to state. The performance parameters are derived, and it is then shown how they vary with the process parameters.

System model and analysis

The transition diagram of the system is given in Fig. 1 with $N = 6$ states numbered $1,..n,...m,..\ N$, where the transition probabilities between the states are also shown.

Following the approach in (Kundaeli, 2008), state reduction techniques can be used to reduce the transition diagram to that of Fig. 2 where we have used the notations $M = m - n$, $K = n$ and $J = N - m + 1$.

The transfer function from state n to m is then given by

$$\Phi_{nm}(z) = \frac{F_{nm}(z)}{1 - F_{nn}(z)} \tag{1}$$

where

$$F_{nm}(z) = \frac{\prod_{r=0}^{M-1} P_{n+r}\, z^{M}}{T_{21}(M,z)} \tag{2}$$

$$F_{nn}(z) = F_{nn}(K,z) + F_{nn}(M,z) \tag{3}$$

$$F_{nn}(K,z) = \begin{cases} P_{nn} z, & n = 1 \\ P_{nn} z + \dfrac{Q_n P_{n-1} z^2 T_{22}(K,z)}{T_{21}(K,z)}, & n > 1 \end{cases} \tag{4}$$

$$F_{nn}(M,z) = \frac{P_n Q_{n+1} z^2 T_{22}(M,z)}{T_{21}(M,z)} \tag{5}$$

with

$$T_{21}(M,z) = \prod_{r=1}^{M-1}(1 - P_{n+r,n+r} z) + \sum_{k=1}^{M_1}(-1)^k \sum_{i=1}^{M-2k} P_{n+i} Q_{n+i+1} z^2 \sum_{j=i+2}^{M-2k+2} P_{n+j} Q_{n+j+1} z^2 \cdots \sum_{w=s+2}^{M-4} P_{n+w} Q_{n+w+1} z^2 \sum_{x=w+2}^{M-2} P_{n+x} Q_{n+x+1} z^2 \prod_{r=1, r \notin V}^{M-1} (1 - P_{n+r,n+r} z) \tag{6}$$

$$T_{22}(M,z) = \prod_{r=2}^{M-1}(1 - P_{n+r,n+r} z) + \sum_{k=1}^{M_2}(-1)^k \sum_{i=2}^{M-2k} P_{n+i} Q_{n+i+1} z^2 \sum_{j=i+2}^{M-2k+2} P_{n+j} Q_{n+j+1} z^2 \cdots \sum_{w=s+2}^{M-4} P_{n+w} Q_{n+w+1} z^2 \sum_{x=w+2}^{M-2} P_{n+x} Q_{n+x+1} z^2 \prod_{r=2, r \notin V}^{M-1} (1 - P_{n+r,n+r} z) \tag{7}$$

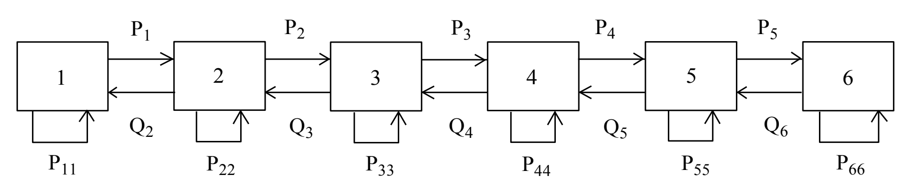

Fig. 1: The transition diagram of a birth-death process with $N = 6$.

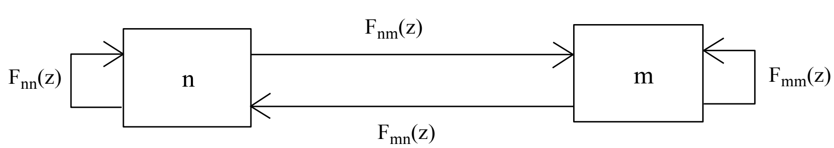

Fig. 2: The reduced transition diagram of Fig. 1.

$$T_{21}(K,z) = \prod_{r=1}^{K-1}(1 - P_{n-r,n-r}\, z) + \sum_{k=1}^{K_1}(-1)^k \sum_{i=1}^{K-2k} Q_{n-i} P_{n-i-1} z^2$$
$$\sum_{j=i+2}^{K-2k+2} Q_{n-j} P_{n-j-1} z^2 \ldots \sum_{w=s+2}^{K-4} Q_{n-w} P_{n-w-1} z^2$$
$$\sum_{x=w+2}^{K-2} Q_{n-x} P_{n-x-1} z^2 \prod_{r=1, r\notin V}^{K-1}(1 - P_{n-r,n-r}\, z) \quad (8)$$

$$T_{22}(K,z) = \prod_{r=2}^{K-1}(1 - P_{n-r,n-r}\, z) + \sum_{k=1}^{K_2}(-1)^k \sum_{i=2}^{K-2k} Q_{n-i} P_{n-i-1} z^2$$
$$\sum_{j=i+2}^{K-2k+2} Q_{n-j} P_{n-j-1} z^2 \ldots \sum_{w=s+2}^{K-4} Q_{n-w} P_{n-w-1} z^2$$
$$\sum_{x=w+2}^{K-2} Q_{n-x} P_{n-x-1} z^2 \prod_{r=2, r\notin V}^{K-1}(1 - P_{n-r,n-r}\, z) \quad (9)$$

$$T_{2i}(U,z) = \begin{cases} 0, & U < i \\ 1, & U = i \end{cases} \quad (10)$$

$$M_i = floor((M-i)/2) \quad (11)$$
$$K_i = floor((K-i)/2)$$

and

$$V = \{i, i+1, j, j+1, \ldots s, s+1, w, w+1, x, x+1\}$$

Likewise, the transfer function from state *m* to *n* is given by

$$\Phi_{mn}(z) = \frac{F_{mn}(z)}{1 - F_{mm}(z)} \quad (12)$$

where

$$F_{mn}(z) = \frac{\prod_{r=0}^{M-1} Q_{m-r}\, z^M}{\hat{T}_{21}(M,z)} \quad (13)$$

$$F_{mm}(z) = F_{mm}(J,z) + F_{mm}(M,z) \tag{14}$$

$$F_{mm}(J,z) = \begin{cases} P_{mm} z, \ m = N \\ P_{mm} z + \dfrac{P_m Q_{m+1} z^2 T_{22}(J,z)}{T_{21}(J,z)}, \ m < N \end{cases} \tag{15}$$

$$F_{mm}(M,z) = \frac{Q_m P_{m-1} z^2 \hat{T}_{22}(M,z)}{\hat{T}_{21}(M,z)} \tag{16}$$

with

$$T_{21}(J,z) = \prod_{r=1}^{J-1}(1 - P_{m+r,m+r}\, z) + \sum_{k=1}^{J_1}(-1)^k \sum_{i=1}^{J-2k} P_{m+i} Q_{m+i+1} z^2 \sum_{j=i+2}^{J-2k+2} P_{m+j} Q_{m+j+1} z^2 \cdots \sum_{w=s+2}^{J-4} P_{m+w} Q_{m+w+1} z^2 \sum_{x=w+2}^{J-2} P_{m+x} Q_{m+x+1} z^2 \prod_{r=1\, r\notin V}^{J-1}(1 - P_{m+r,m+r}\, z) \tag{17}$$

$$T_{22}(J,z) = \prod_{r=2}^{J-1}(1 - P_{m+r,m+r}\, z) + \sum_{k=1}^{J_2}(-1)^k \sum_{i=2}^{J-2k} P_{m+i} Q_{m+i+1} z^2 \sum_{j=i+2}^{J-2k+2} P_{m+j} Q_{m+j+1} z^2 \cdots \sum_{w=s+2}^{J-4} P_{m+w} Q_{m+w+1} z^2 \sum_{x=w+2}^{J-2} P_{m+x} Q_{m+x+1} z^2 \prod_{r=2\, r\notin V}^{J-1}(1 - P_{m+r,m+r}\, z) \tag{18}$$

$$\hat{T}_{21}(M,z) = \prod_{r=1}^{M-1}(1 - P_{m-r,m-r}\, z) + \sum_{k=1}^{M_1}(-1)^k \sum_{i=1}^{M-2k} Q_{m-i} P_{m-i-1} z^2 \sum_{j=i+2}^{M-2k+2} Q_{m-j} P_{m-j-1} z^2 \cdots \sum_{w=s+2}^{M-4} Q_{m-w} P_{m-w-1} z^2 \sum_{x=w+2}^{M-2} Q_{m-x} P_{m-x-1} z^2 \prod_{r=1\, r\notin V}^{M-1}(1 - P_{m-r,m-r}\, z) \tag{19}$$

$$\hat{T}_{22}(M,z)=\prod_{r=2}^{M-1}(1-P_{m-r,m-r}z)+\sum_{k=1}^{M_2}(-1)^k\sum_{i=2}^{M-2k}Q_{m-i}P_{m-i-1}z^2$$
$$\sum_{j=i+2}^{M-2k+2}Q_{m-j}P_{m-j-1}z^2\cdots\sum_{w=s+2}^{M-4}Q_{m-w}P_{m-w-1}z^2$$
$$\sum_{x=w+2}^{M-2}Q_{m-x}P_{m-x-1}z^2\prod_{r=2,r\notin V}^{M-1}(1-P_{m-r,m-r}z) \tag{20}$$

and the other parameters are as defined before. The transition time from state n to m can then be obtained as

$$L_{nm}=\frac{d}{dz}\left(\Phi_{nm}(z)\right)_{z=1} \tag{21}$$

If we apply (21) to (1) and employ some algebraic manipulations, we obtain

$$L_{nm}=\frac{\Delta_{nm}}{Y_n^2} \tag{22}$$

where

$$\Delta_{nm}=\prod_{r=0}^{M-1}P_{n+r}\Big[T_{21K}^2T_{21M}+(M-2)(P_n+Q_n)T_{21K}^2T_{21M}+(P_n+Q_n)T_{21K}^2(T_{21M}-T'_{21M})$$
$$-(M-3)Q_nP_{n-1}T_{21M}T_{21K}T_{22K}-(M-3)P_nQ_{n+1}T_{21K}^2T_{22M}$$
$$-P_nQ_{n+1}T_{21K}^2(T_{22M}-T'_{22M})+Q_nP_{n-1}T_{21M}(T_{21K}T'_{22K}-T'_{21K}T_{22K})$$
$$-Q_nP_{n-1}T_{21K}T_{22K}(T_{21M}-T'_{21M})\Big] \tag{23}$$

$$Y_n=(P_n+Q_n)T_{21M}T_{21K}-Q_nP_{n-1}T_{21M}T_{22K}-P_nQ_{n+1}T_{22M}T_{21K} \tag{24}$$

and we have used the notations

$$T_{2iU}=T_{2i}(U,z)|_{z=1},\ T'_{2iU}=\frac{d}{dz}\left(T_{2i}(U,z)\right)_{z=1} \tag{25}$$

Note that when n = 1 then Q_1 = 0 and we therefore obtain

$$\Delta_{1m}=\prod_{r=0}^{M-1}P_{1+r}\Big[T_{21M}+(M-1)P_1T_{21M}-P_1T'_{21M}$$
$$-(M-2)P_1Q_2T_{22M}+P_1Q_2T'_{22M}\Big] \tag{26}$$

and

$$Y_1 = P_1 T_{21M} - P_1 Q_2 T_{22M}. \tag{27}$$

The transition time from state *m* to *n* can be obtained in a similar manner as the one from state *n* to *m* as

$$L_{mn} = \frac{d}{dz}\left(\Phi mn(z)\right)\Big|_{z=1} = \frac{\Delta_{mn}}{Y_m^2} \tag{28}$$

where

$$\begin{aligned}\Delta_{mn} = \prod_{r=0}^{M-1} Q_{m-r} \Big[& T_{21J}^2 \hat{T}_{21M} + (M-2)(Q_m + P_m) T_{21J}^2 \hat{T}_{21M} + (Q_m + P_m) T_{21J}^2 (\hat{T}_{21M} - \hat{T}'_{21M}) \\ & - (M-3) P_m Q_{m+1} \hat{T}_{21M} T_{21J} T_{22J} - (M-3) Q_m P_{m-1} T_{21J}^2 \hat{T}_{22M} \\ & - Q_m P_{m-1} T_{21J}^2 (\hat{T}_{22M} - \hat{T}'_{22M}) + P_m Q_{m+1} \hat{T}_{21M} (T_{21J} T'_{22J} - T'_{21J} T_{22J}) \\ & - P_m Q_{m+1} T_{21J} T_{22J} (\hat{T}_{21M} - \hat{T}'_{21M}) \Big]\end{aligned} \tag{29}$$

$$Y_m = (P_m + Q_m) T_{23M} T_{21J} - P_m Q_{m+1} T_{23M} T_{22J} - Q_m P_{m-1} T_{24M} T_{21J} \tag{30}$$

and

$$\hat{T}_{2iU} = \hat{T}_{2i}(U,z)\Big|_{z=1}, \quad \hat{T}'_{2iU} = \frac{d}{dz}\left(\hat{T}_{2i}(U,z)\right)\Big|_{z=1}. \tag{31}$$

Note that when $m = N$ then $P_N = 0$ and we therefore obtain

$$\begin{aligned}\Delta_{Nn} = \prod_{r=0}^{M-1} Q_{N-r} \Big[& \hat{T}_{21M} + (M-1)\hat{T}_{21M} - \hat{T}'_{21M} \\ & - (M-2) Q_N P_{N-1} \hat{T}_{22M} + Q_N P_{N-1} \hat{T}'_{22M} \Big]\end{aligned} \tag{32}$$

and

$$Y_N = Q_N \hat{T}_{21M} - Q_N P_{N-1} \hat{T}_{22M}. \tag{33}$$

RESULTS AND DISCUSSION

The results of the analysis are given in Figs. 3 to 10 for a process having $N = 16$ states. In all of these results, the transition probabilities have been fixed at $P_1 = Q_1 = P_N = Q_N = 0.50$, $P_i = Q_i = 0.35$ and $P_{ii} = 0.30$ for $1 < i < N$ unless indicated otherwise. Also, *n* and *m* have been fixed at 6 and 11 respectively unless indicated otherwise. Fig. 3 then shows how L_{nm} varies with P_i for four cases: $P_n = P_i$ only varying, P_2 .. $P_n = P_i$ only varying, P_n .. $P_m = P_i$ only varying and P_2 .. $P_{N-1} = P_i$ varying. It can be seen that, generally, the time to transit from state *n* to *m* decreases with increasing P_i. This is expected because increasing probabilities of moving towards *m* favour the system transiting towards that state. It is also seen that when $P_i = 0.35$, which is the default transition probability in this analysis, then L_{nm} is the same for the four cases as expected. A similar trend is also seen in all the other results. It is also seen

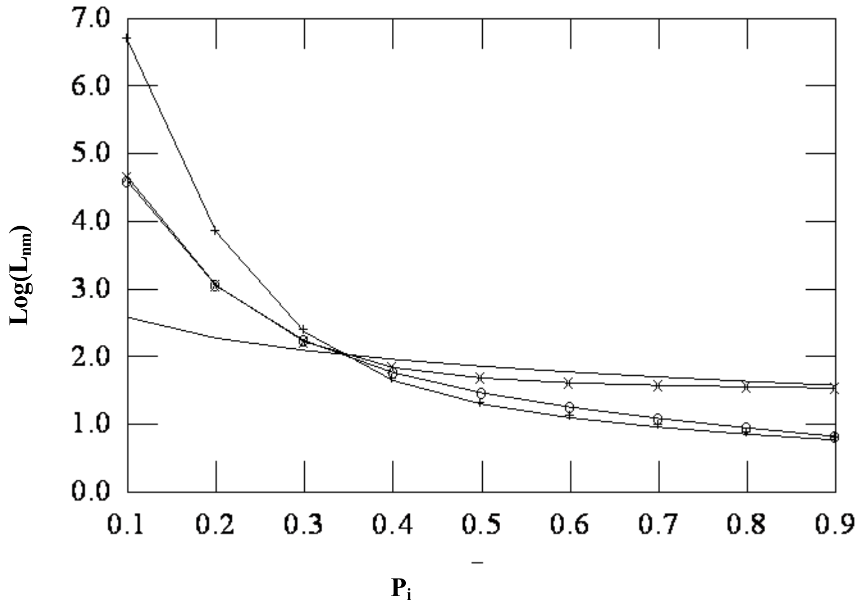

Fig. 3: Variation of L_{nm} with P_n (—), $P_2...P_n$ (-x-), $P_n...P_m$ (-o-), and $P_2...P_{N-1}$ (-+-) at $n = 6$, $m = 11$ and $N = 16$.

that the values of P_i between states n and m have a higher effect than those between states 1 and n. This is evident because the values of P_i between state n and m are more determinant in moving the system from state n to m than the values of P_i in states below n.

Fig. 4 shows the variation of L_{nm} with P_{ii}. As expected, L_{nm} increases with P_{ii} indicating the reluctance of the system to leave an attained state. Compared to Fig. 3, it is seen that P_{ii} has a lower effect on L_{nm} than P_i, and also the values of P_{ii} between states 2 and n have more effect than those between states n and m. The variation of L_{nm} with Q_i is given in Fig. 5 showing that L_{nm} increases with Q_i. This is expected because higher values of Q_i imply the tendency of the system to move towards state 1 rather than towards state m. Comparing Figs. 3, 4 and 5 it is seen that Q_i has the higher effect on L_{nm}. The higher effect of Q_i compared to P_{ii} arises from the fact that whereas P_{ii} plays the role of keeping the system in a particular state, Q_i has the effect of moving the system away from m. It is also seen that the values of Q_i between states n and m have a higher effect than those between states 1 and n.

The variation of L_{mn} with the transition probabilities are given in Figs. 6 to 8. The resemblance between Figs. 3 and 8 is due to the fact that P_i plays the same role on L_{nm} as Q_i does on L_{mn}. This is also the case with Figs. 4 and 7, where P_{ii} plays the same role on L_{nm} in Fig. 4 as it does on L_{mn} in Fig. 7, and in Figs. 5 and 6, where Q_i plays the same role on L_{nm} in Fig. 5 as P_i does on L_{mn} in Fig. 6.

Figs. 9 and 10 show the variation of L_{nm} and L_{mn} with both the initial state n and the final state m. In both cases the probabilities have been chosen such that both P_2 .. P_{N-1} and Q_2 .. Q_{N-1} increase proportionally with the state. Note that the plots of L_{nm} and L_{mn} have been combined in Figs. 9 and 10 to save plotting space. Consequently, comparison of L_{nm} in Fig. 9 should be with L_{mn} in Fig. 10, and also comparison of L_{mn} in Fig. 9 should be with L_{nm} in Fig. 10. It can then be seen that n and m play interchangeable roles in L_{nm} and L_{mn}. As seen in Fig. 9, both L_{nm} and L_{mn} decrease with n where m has been fixed to N, the highest state of the system. This is expected because as n ap-

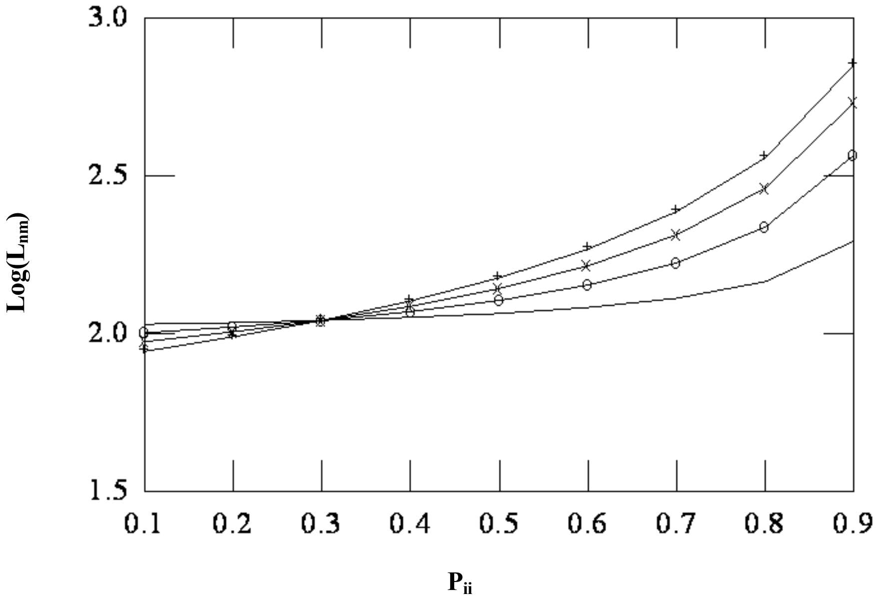

Fig. 4: Variation of L_{nm} with P_{nn} (—) , $P_2 ... P_{nn}$ (-x-), $P_{nn} ... P_m$ n(-o-), and $P_{22} ... P_{N-1,N-1}$ (-+-) at $n = 6$, $m = 11$ and $N = 16$.

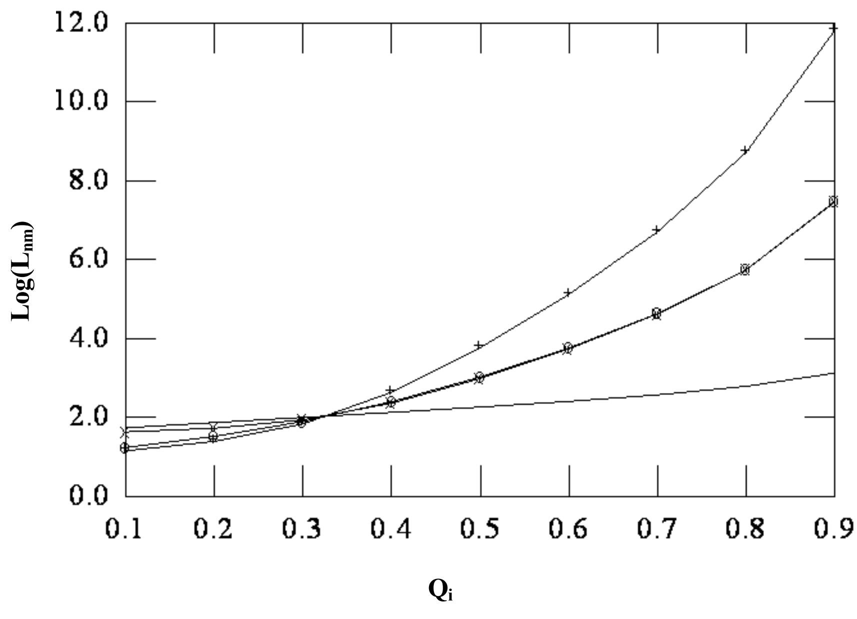

Fig. 5: Variation of L_{nm} with Q_n (—), $Q_2 ... Q_n$ (-x-), $Q_n ... Q_m$ (-o-) and $Q_2 ... Q_{N-1}$ (-+-) at $n = 6$, $m = 11$ and $N = 16$.

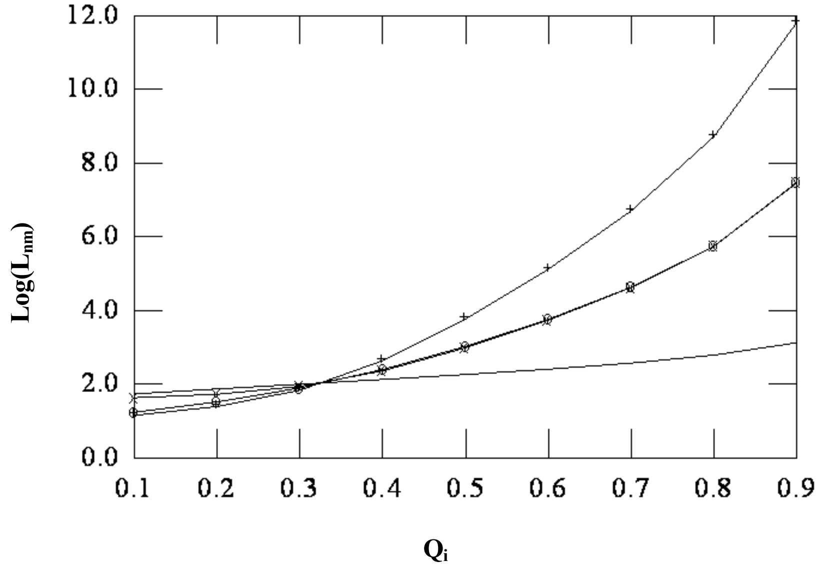

Fig. 6: **Variation of L_{mn} with P_n (—), $P_2...P_n$ (-x-), $P_n...P_m$ (-o-) and $P_2...P_{N-1}$ (-+-) at $n = 6$, $m = 11$ and $N = 16$**

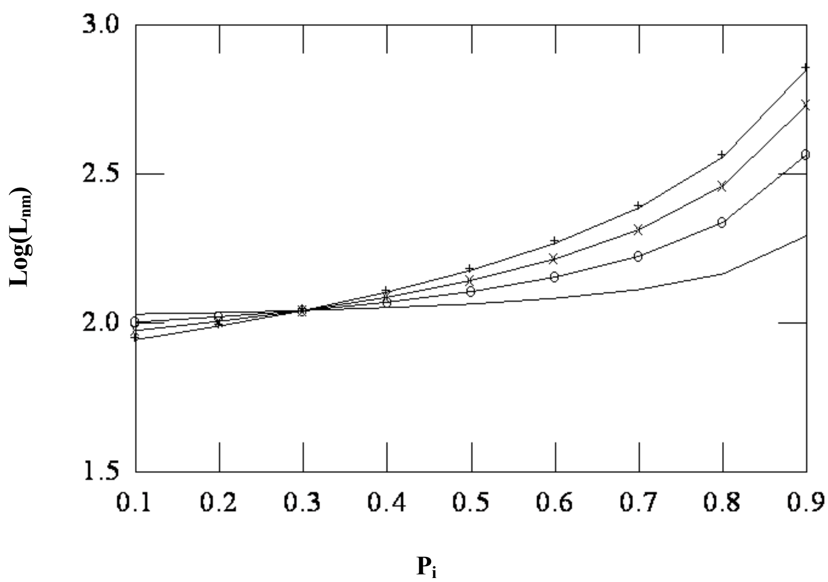

Fig. 7: **Variation of L_{mn} with P_{nn} (—), $P_{22}...P_{nn}$ (-x-), $P_{nn}...P_{mm}$ (-o-) and $P_{22}...P_{N-1,N-1}$ (-+-) at $n = 6$, $m = 11$ and $N = 16$.**

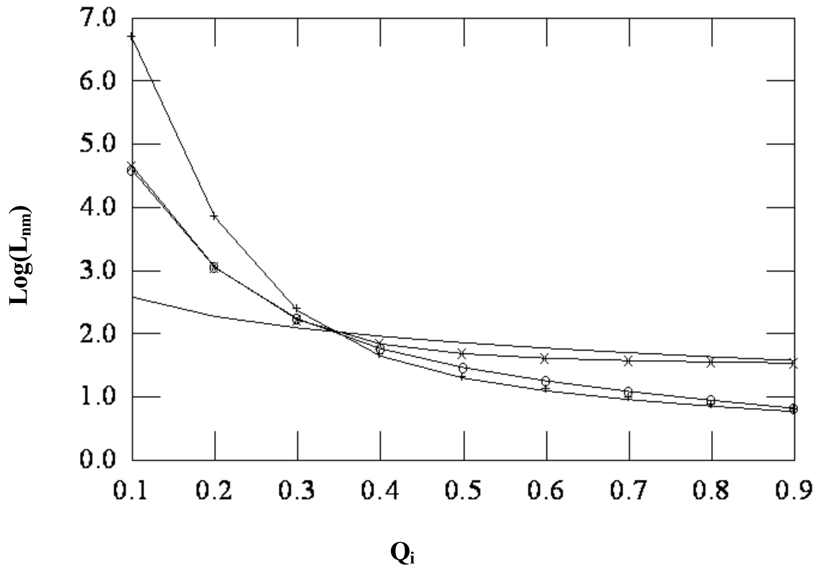

Fig. 8: Variation of L_{mn} with Q_n (—), $Q_2...Q_n$ (–x–), $Q_n...Q_m$ (–o–) and $Q_2...Q_{N-1}$ (–+–) at $n = 6$, $m = 11$ and $N = 16$.

proaches N, it becomes easier for the system to transit from state n to N (L_{nm}) as well as to transit from state N to n (L_{mn}). In Fig. 10 n has been fixed to 1 and the variation of L_{nm} and L_{mn} with m observed.

As expected, both L_{nm} and L_{mn} increase with m, the implication here being that as m approaches N, it becomes more difficult for the system to transit from state 1 to m (L_{nm}) as well as from state m to 1 (L_{mn}). Results for cases where the transition probabilities do not change from state to state have been presented in (Kundaeli, 2008) where a similar trend was observed. Moreover, and although not shown, the values of L_{nm} and L_{mn} in the current analysis were found to be lower than those presented in the previous report. This is expected because the transition probabilities (P_i and Q_i) in that report were fixed, whereas in this report they increase with the state.

As explained in (Kundaeli, 2008), the presented model can be used to study communication and queuing systems having birth-death like structures. The significance of the present analysis is the possibility to investigate real life population systems where birth and death rates are proportional to population size, and also conditions may exist that favour births to deaths and vice versa. This significance is also applicable to queuing systems having arrival and service rates that are state dependent. While the presented behaviour of the system has been obtained by using particular values of the transition probabilities and states, the model is flexible enough to enable alternative behaviours of such systems to be investigated by using different values for the system parameters.

CONCLUSION

A birth-death process in which the transition probabilities between states can vary with the state has been investigated. The behaviour of the system has been found to vary with the transition probabilities as expected. It has been shown for example that the transition time from a given state to another is more affected by the tendency of the system to move from the states rather than the tendency to remain in the states. The model offers flexibility to study different systems under different conditions. The model can for example

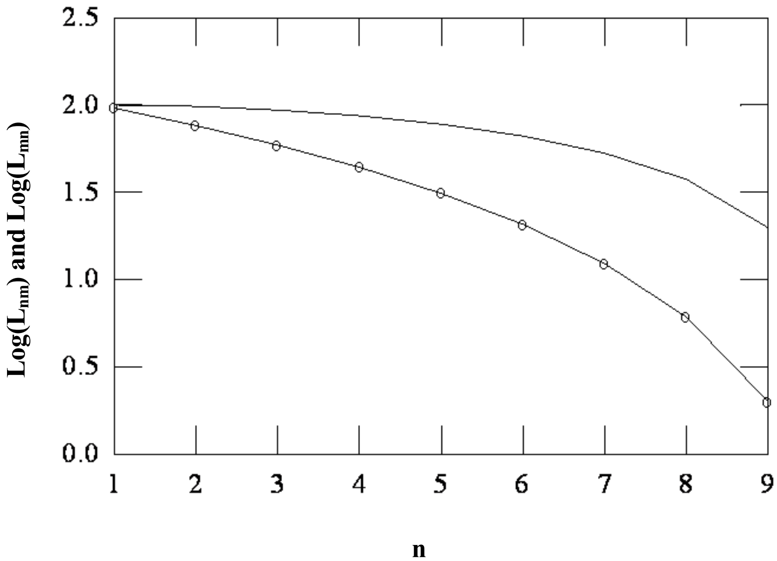

Fig. 9: Variation of L_{nm} and L_{mn} with n at $N = 16$ and increasing $P_2 \ldots P_{N-1}$ (—) and $Q_2 \ldots Q_{N-1}$ (-o-).

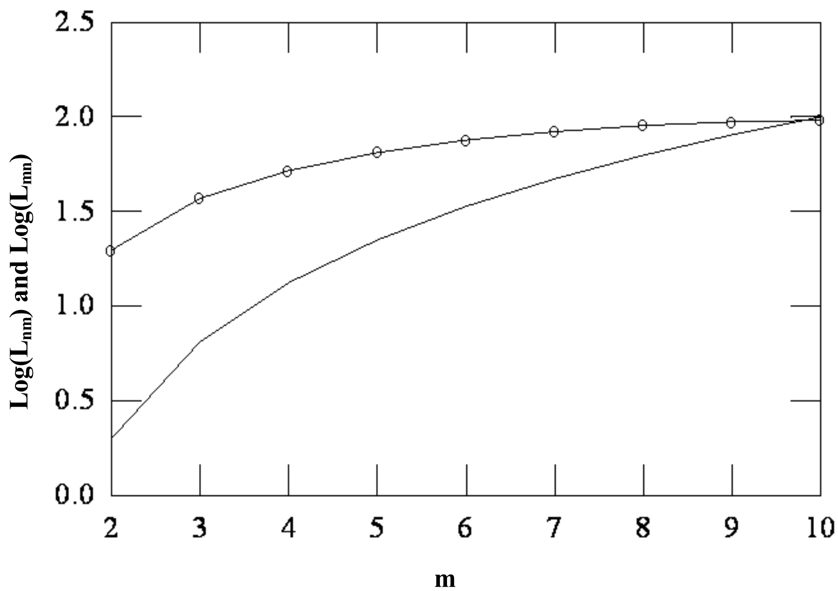

Fig. 10: Variation of L_{nm} and L_{mn} with m at $N = 16$ and increasing $P_2 \ldots P_{N-1}$ (—) and $Q_2 \ldots Q_{N-1}$ (-o-).

be used to study practical queuing and birth-death systems where the arrival, birth, service and death rates differ from state to state. Further research in this area will therefore focus on applications of the developed models to practical real life problems.

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