

RESEARCH PAPER

**FORECASTING WITH NONLINEAR TIME SERIES MODEL:  
A MONTE-CARLO BOOTSTRAP APPROACH**

**N. Ekhosuehi and S. E. Omosigho**

*Department of Mathematics, University of Benin, Benin City, Nigeria.  
Corresponding author: ekhosu@yahoo.com*

**ABSTRACT**

*In this paper, we propose a new method of forecasting with nonlinear time series model using Monte-Carlo Bootstrap method. This new method gives better result in terms of forecast root mean squared error (RMSE) when compared with the traditional Bootstrap method and Monte-Carlo method of forecasting using a special nonlinear time series model, called logistic smooth transition autoregressive (LSTAR) model. We illustrate this new method using some simulation experiments.*

**Keywords:** *Bootstrap, Forecasting, LSTAR, Monte-Carlo, Monte-Carlo Bootstrap*

**INTRODUCTION**

One of the major reasons for building a time series model is for the purpose of forecasting. Basically, there are two kinds of time series models for forecasting, viz: the linear time series model and the nonlinear time series model. Point forecast using linear time series model is without difficulty because forecasts can be generated recursively up to any step greater than one. For nonlinear time series model, point forecast for step one can be done easily like in the linear case but forecast for a step greater than or equal to two is no longer easy to handle. In practice, however, the exact method of forecasting is difficult to compute. This is because

it involves an integrable function whose dimension increases as forecast horizon increases and therefore, usually replaced by simulation methods. The simulation methods include: the Monte-Carlo and the Bootstrap techniques. Details of these techniques are found in Granger and Terasvirta (1993), Franses *et. al* (2000), Franses and Van Dijk (2000), Lundbergh and Teräsvirta (2002). Besides, there is yet another method of forecasting called the Naïve method. This method is not commonly used because it produces biased results; see Lin and Granger (1994) for exposition. Many authors have made tremendous effort in forecasting using nonlinear model. Some

claimed that forecast using linear models are superior to their nonlinear models fitted to the same data. Stock and Watson (1999a), Marcelino (2004), Terui and van Dijk (2002) and a host of others also used a combination of forecast. Their findings is that combining the forecasts from several methods on average performed better than simply relying on forecasts from individual models such as neural networks, LSTAR or autoregressions.

As noted by Franses (1998), a typical method of evaluating forecast is to keep  $m$  observations to evaluate forecasts from models which are fitted to the first  $n$  observations. We use this idea to illustrate the usefulness of the proposed Monte-Carlo bootstrap method of forecasting with nonlinear time series model by comparing the RMSE with the traditional bootstrap and Monte-Carlo method of forecasting. We use the logistic smooth transition autoregressive (LSTAR) model as a case study.

We first consider a linear model called the AR (p) model of order  $p$  which satisfies the following linear equation for some integer  $p \geq 1$ ,

$$y_t = \phi' w_t + \varepsilon_t \quad (1)$$

where  $w_t = (1, y_{t-1}, y_{t-2}, \dots, y_{t-p})'$  and

$\phi = (\alpha_0, \alpha_1, \dots, \alpha_p)'$  are real parameters such that the zeroes of  $\phi(z) = 1 - \alpha_1 z - \dots - \alpha_p z^p$

lie outside the unit disk and  $\varepsilon_t$  is normally distributed with mean 0 and variance  $\sigma_\varepsilon^2$ .

Then, the smooth transition autoregressive (STAR) model for a univariate time series  $y_t$  which is observed at  $t = 1, 2, \dots, n$  is given by:

$$y_t = \phi_1' w_t + \phi_2' w_t G(s_t; \gamma, c) + \varepsilon_t \quad (2)$$

where  $\phi_i = (\alpha_{i,0}, \alpha_{i,1}, \dots, \alpha_{i,p})'$ ,  $i = 1, 2$ ,  $\gamma > 0$ ,  $w_t$  and  $\varepsilon_t$  are as defined in equation (1).

$G(s_t; \gamma, c)$  is called the transition function that

is bounded between 0 and 1. The transition function could either be of the logistic type or the exponential type. The variable  $S_t$  is called the transition variable which could be assumed to be a lagged endogenous variable that is  $s_t = y_{t-d}$  for certain integer  $d > 0$  or a linear trend ( $s_t = t$ ), which gives rise to smoothly changing parameters. The STAR model could be seen as a regime switching model that allows for two regimes associated with the extreme values of the transition function,  $G(s_t; \gamma, c) = 0$  and  $G(s_t; \gamma, c) = 1$ , where the transition from one regime to the other is smooth. For a survey on STAR model, see Teräsvirta (1994), Franses and Van Dijk (2000), Van Dijk *et.al* (2002) and a host of others.

If the transition function is of the logistic type, we have

$$G(\gamma, c; s_t) = \frac{1}{(1 + \exp\{-\gamma(s_t - c)\})} \quad (3)$$

and if it is of the exponential type, we have

$$G(\gamma, c; s_t) = (1 - \exp\{-\gamma(s_t - c)^2\}) \quad (4)$$

## MATERIALS AND METHODS

The methods of forecasting with nonlinear model discussed in this Section are based on the work of Granger and Teräsvirta (1993), Franses and Van Dijk (2000), Lundbergh and Teräsvirta (2002). Now, let us rewrite the STAR given in equation (2) as:

$$y_t = g(w_t; \psi) + \varepsilon_t \quad (5)$$

where  $g(w_t; \psi) = \phi_1' w_t + (\theta' w_t) G(\gamma, c; s_t)$  and

$G(\gamma, c; s_t)$  is a logistic transition function. A

one-step ahead forecast for this model in equation (5) is given by:

$$y_{t+1}^f = E(y_{t+1} / I_t) = g(w_{t+1}; \psi). \quad (6)$$

Equation (6) is called an unbiased forecast of made at time  $t$  given the past information up till that time. The relevant information is contained in

$$w_{t+1} = (1, y_t, y_{t-1}, \dots, y_{t-(p-1)})'$$

Now when the forecast period is greater than one, forecast becomes difficult to handle. We use two-step ahead forecast to demonstrate this difficulties. An analytic expression for two-step ahead forecast  $y_{t+2}^f$  is given by the expression

$$y_{t+2}^f = E\{[g(w_{t+2}^f; \psi) + \varepsilon_{t+2}] / I_t\} = E\{g(w_{t+2}^f; \psi) / I_t\} \quad (7)$$

where  $w_{t+2}^f = (1, y_{t+1}^f + \varepsilon_{t+1}, y_t, \dots, y_{t-(p-1)})'$

There are four basic methods of handling equation (7). They are as follows:

**Naïve method**

This method of forecast considers the direct use of the skeleton of the model, which can be written as:

$$y_{t+2}^f = g(w_{t+2}^f; \psi) \quad (8)$$

where the presence of  $\varepsilon_{t+2}$  is ignored.

**The exact method**

The exact method is defined by making use of the direct analytic expression of the mathematical expectation of a function of a random variable. This is given by:

$$y_{t+2}^f = E\{g(w_{t+2}^f; \psi) / I_t\} = \int_{-\infty}^{\infty} g(w_{t+2}^f; \psi) d\phi(z) dz \quad (9)$$

where  $\phi(z)$  is the cumulative distribution function of  $\varepsilon_{t+1}$ . Obtaining this forecast requires numerical integration and the integral will have to be determined at each time point.

**Monte-Carlo method**

The Monte Carlo method of forecasting is a simulation method used in replacement of the exact method. It is given by the expression:

$$y_{t+2}^{fm} = (1 / M) \sum_{m=1}^M g(w_{t+2,m}^f; \psi) \quad (10)$$

where each of the  $M$  values of  $\varepsilon_{t+1}$  in  $w_{t+2,m}^f$  is drawn independently from the assumed error distribution in equation (2). The forecast made

Using equation (10) said to be asymptotically unbiased as  $M \rightarrow \infty$  (By the weak law of large numbers).

**Bootstrap method**

An equivalent simulation method of forecasting instead of using the Monte-Carlo method is called the bootstrap method. This is given by equation (11):

$$y_{t+2}^{fb} = (1 / B) \sum_{b=1}^B g(w_{t+2,b}^f; \psi) \quad (11)$$

where each of the  $B$  values of  $\varepsilon_{t+1}$  in  $w_{t+2,b}^f$  is drawn independently from the set of the residuals of the estimated model in equation (2) with replacement were  $B$  is usually taken to be very large.

**Monte-Carlo bootstrap method**

We propose a new method of forecasting with the STAR model which we call the Monte-Carlo bootstrap method. It is a combination of the Monte-Carlo and bootstrap method. It is in fact, a repeated bootstrap based on equation (11). The method is given by the following:

$$y_{t+2/i}^{fmb} = (1 / M) \sum_{i=1}^M \left( \frac{1}{n(\varepsilon)} \sum_{b=1}^{n(\varepsilon)} g_i(w_{t+2,b}^f; \psi) \right) \quad (12)$$

where each of the  $n(\varepsilon)$  values of  $\varepsilon_{t+1}$  in  $w_{t+2,b}^f$  is drawn independently from the set of the residuals of the estimated model equation (2) with replacement  $M$  number of times. The difference between equation (11) and equation(12) is that the bootstrap for equation (11) terminates after a single large bootstrap sample is drawn from the residual with replacement, while in equation(12), the bootstrap is done  $M$  number of times having sample size of the length of the residuals generated from the estimated model in equation (2). It is a Monte-Carlo experiment using the bootstrap sample mean which converges to the true expected value (by the weak law of large numbers).

**Measuring Forecast Accuracy**

As noted by Franses (1998), a typical method of evaluating forecast is to keep  $m$  observations to evaluate forecasts from models which are fitted to the first  $n$  observations. Another

method is to check whether 95% of the forecasts lie within the 95% forecast interval. Typically, evaluation can be based on forecast root mean squared error (RMSE) or mean absolute percentage error (MAPE) which are given by:

$$RMSE(h) = \sqrt{\frac{1}{m} \sum_{h=1}^m (\hat{y}_{t+h} - y_{t+h})^2} \quad (13)$$

$$MAPE = (1/m) \left[ \sum_{h=1}^m \left| \frac{\hat{y}_{t+h} - y_{t+h}}{y_{t+h}} \right| \right] \quad (14)$$

where  $\hat{y}_{t+h}$  is the forecast made at time  $t$  for a specific horizon  $h$  and  $y_{t+h}$  the expected future value of the time series. Diebold and Mariano (1995) also proposed a method of testing the significant difference of two forecasting models say A and B. Their test statistic is given by:

$$S = (m/4)^{-1/2} \left[ \sum_{h=1}^m d_j - (m/2) \right] \quad (15)$$

where  $d_j$  is a random variable which equal 1 when  $RMSE$  of model A exceeds  $RMSE$  of model B and zero otherwise.

For simplicity, in this study, we set  $m = 10$ , so that the expected future value is the  $m$  values kept for out of sample performance. In this case we make use of the three forecasting methods given in equation (10), equation (11) and equation (12) and compare their results using  $RMSE$  given in equation (13) only.

## SIMULATION EXPERIMENTS AND RESULTS

We consider the following data generating process (DGP) using the logistic smooth transition autoregressive (LSTAR) model. Model (I) is an LSTAR(1) model while Model (II) is an LSTAR(2) model that was also used in Teravirsta (1994).

$$\text{Model (I)} \quad y_t = 0.8y_{t-1} - 0.8y_{t-1}(1 + \exp(-10y_{t-1}))^{-1} + \varepsilon_t, \quad y_0 = 0$$

$$\text{Model (II)} \quad y_t = 1.8y_{t-1} - 1.06y_{t-2} + (0.02 - 0.9y_{t-1} + 0.795y_{t-2})G(y_{t-1}) + \varepsilon_t, \quad y_0 = 0$$

$$\text{where } G(y_{t-1}) = (1 + \exp\{-20(y_{t-1} - 0.02)\})^{-1}$$

The sequence of error  $\varepsilon_t$  which is normally distributed with mean 0 and variance 1, was generated using the random number generator in MATLAB 7.5.0. We generated  $(300 + n)$  sample sizes using Models (I) and (II). Only the last  $n$  observations are kept, while the first 300 are discarded to minimize initialization effect. Among the  $n$  generated artificial time series observations, the first  $(n - m)$  observations were used for modelling while the remaining  $m$  observations are kept for out-of-sample performance. The modelling cycle which involves testing for non-linearity, model specification and evaluation such as in Luukkonen *et al.* (1988a), Luukkonen *et al.* (1988b), Teräsvirta (1994), Van Dijk *et al.* (2002) was executed using a Time series software package called JMULTI. After estimation of the model, it is then used for forecasting. This stage of computing point forecast in non-linear LSTAR model was written and executed in MATLAB 7.5.0 since JMULTI does not provide forecast option in their package. The forecasting horizon ( $h$ ) is taken from  $h=1$  to  $m$ , so as to agree with the number of observations kept for the out-of-sample performance. Both the methods of forecasting used in this paper and the program implementation using MATLAB 7.5.0 was achieved successfully using the idea of Efron and Tibshirani (1998) and Martinez and Martinez (2002). This process is replicated many times. We however, present some results which are displayed using Figs. 1, 2, 3, 4 and Table 1.

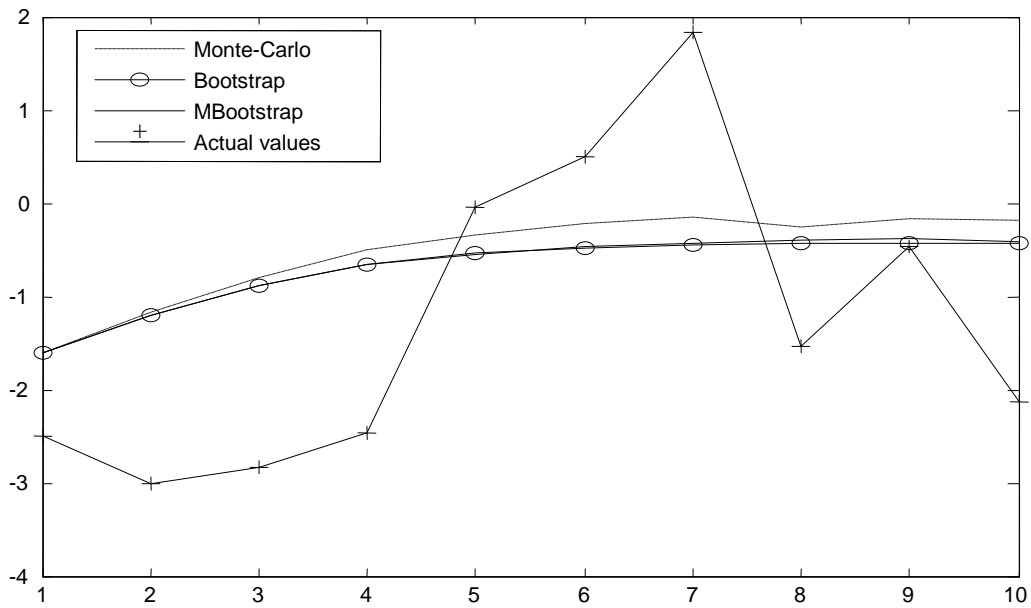


Fig.1: Point forecast made using the three forecasting methods for sample size  $n-m = 100, m = 10$  using Model (I)

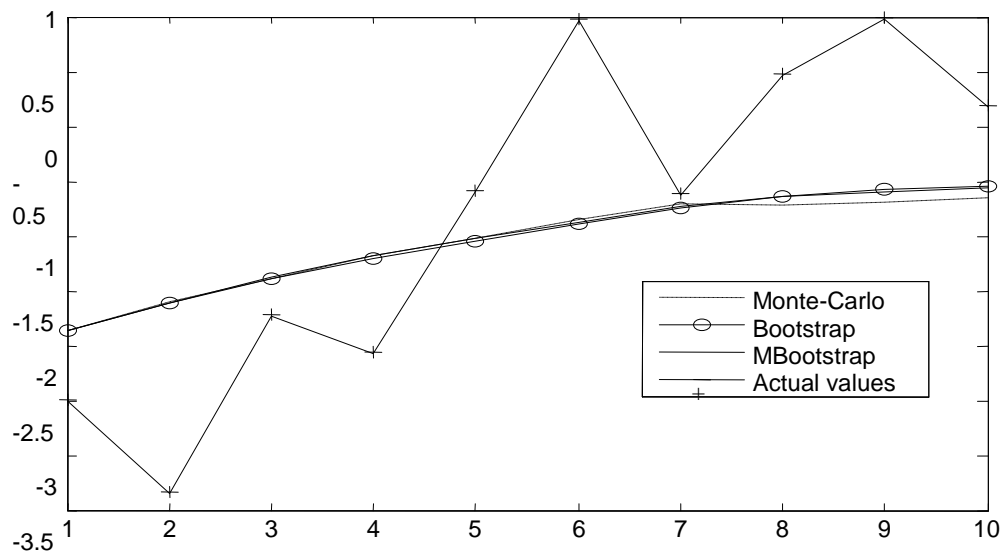


Fig. 2: Point forecast made using the three forecasting methods for sample size  $n-m = 200, m = 10$  using Model (I)

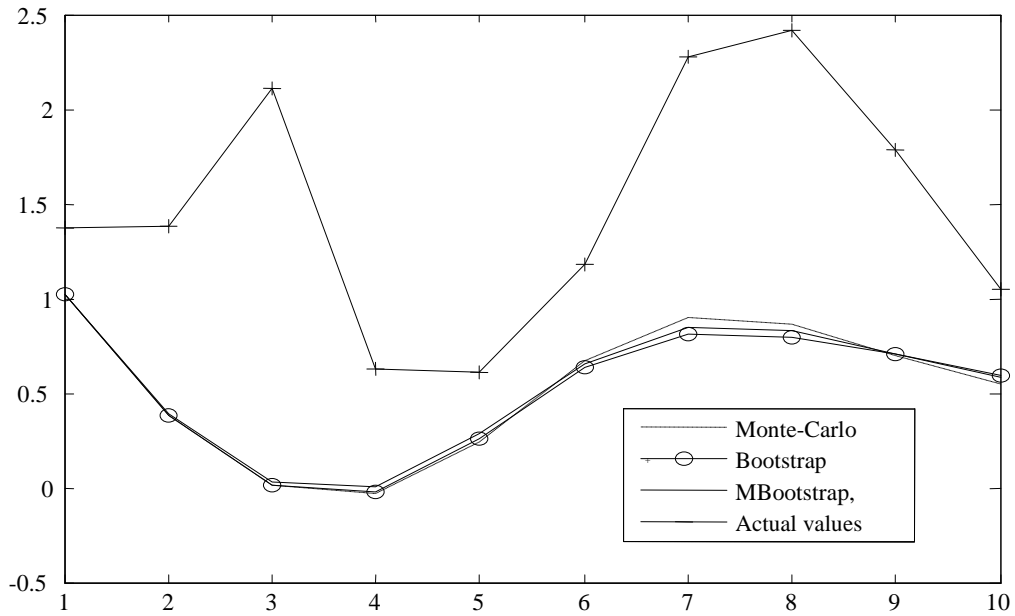


Fig. 3: Point forecast made using the three forecasting methods for sample size  $n-m = 100$ ,  $m = 10$  using Model (II)

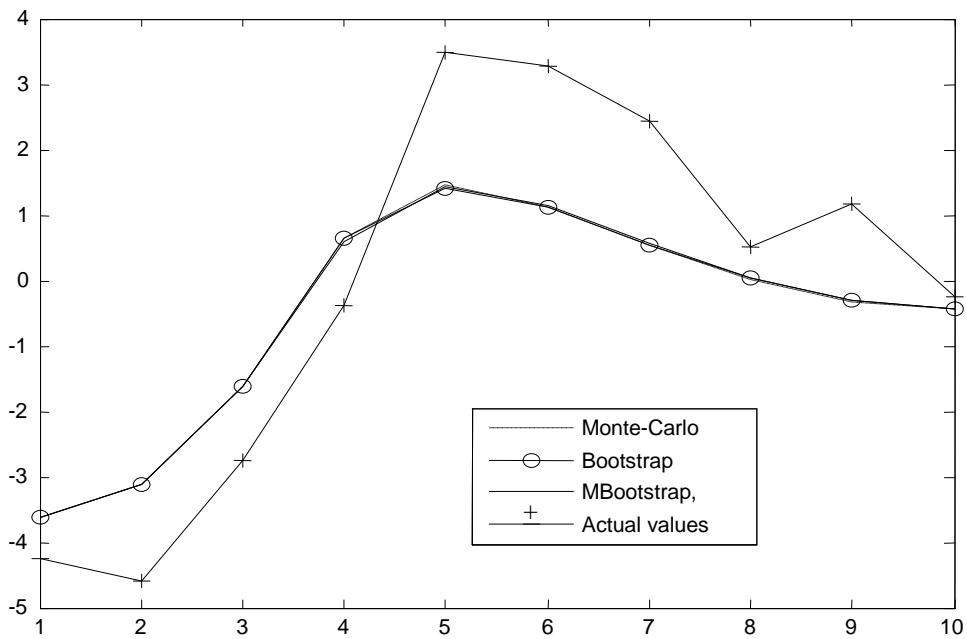


Fig. 4: point forecast made using the three forecasting methods for sample size  $n-m = 200$ ,  $m=10$  using Model (II)

**Table 1: RMSE obtained from forecast using estimated Model (I) and Model (II) for the Monte-Carlo, bootstrap and Monte-Carlo bootstrap method of forecasting for sample sizes  $n = 100$  and  $n = 200$** 

Forecast method	$n = 100$		$n = 200$
		Model (I)	
Monte-Carlo	1.6195		1.1914
Bootstrap	1.5504		1.1654
Monte-Carlo bootstrap	1.5438		1.1638
		Model (II)	
Monte-Carlo	1.1952		1.4805
Bootstrap	1.1710		1.4844
Monte-Carlo Bootstrap	1.1591		1.4686

Figs. 1 and 2 show the graph of point forecast generated using Model (I) when the three forecasting methods are employed for sample sizes  $n = 100$  and  $200$  respectively. Similarly, Fig. 2 and Fig. 3 also show the graph of point forecast generated using Model (II) for sample sizes  $n = 100$  and  $200$  respectively. In Table 1, the study present the RMSE obtained using the Monte-Carlo, bootstrap and the Monte-Carlo bootstrap methods of forecasting. The RMSE is used to compare these three methods.

In Fig. 1, the Bootstrap method and the Monte-Carlo Bootstrap (MBootstrap) are undistinguishable from step 1 to step 10 forecast while the Monte-Carlo forecast for the first three steps agree favourably with the other two methods but with discrepancies from steps 4 to 8. In Fig. 2, the three methods agree equally except the Monte-Carlo after step 7. Similarly, Fig. 3 and Fig. 4 represent point forecast generated from Model II using the three methods for sample sizes  $n = 100$  and  $200$ . Fig. 3 show that the three methods are the same with slight difference from the Monte-Carlo at steps 7 and 8. In Fig. 4, the three methods are undistinguishable.

Table 1 displays the RMSE obtained using the

three methods of forecasting using sample sizes  $n = 100$  and  $200$ . From the Table 1, it is clearly evident that the Monte-Carlo Bootstrap method has the smallest RMSE and hence it is considered the best among the three methods.

#### CONCLUSION

In this paper, it is clearly evident that the new Monte-Carlo Bootstrap method of nonlinear forecasting out performs the traditional method of forecasting using the Monte-Carlo or Bootstrap method. A measure of performance is the RMSE in an out of sample performance. The superiority of our newly proposed method of nonlinear forecasting was based on simulation studies using an LSTAR(1) and LSTAR(2) models respectively.

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