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Soft Symmetric Difference-Intersection Product of Groups

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ABSTRACT

Soft set theory has emerged as a comprehensive mathematical apparatus for the systematic representation and analysis of uncertainty, particularly in the environments governed by parametric variability. At the heart of this theoretical construct lie soft set operations and products, which collectively furnish powerful tools for the formulation and resolution of complex problems characterized by parameter-driven indeterminacy. The present work initiates with a meticulous and formal investigation of the symmetric difference operation of soft sets. It is rigorously established that the collection of the soft sets with a fixed parameter set, under this binary operation, forms an abelian group, thereby endowing the structure with foundational algebraic coherence. Subsequently, we introduce an original product, termed the soft symmetric difference–intersection product, defined on soft sets whose associated parameter sets are endowed with a group structure. A thorough axiomatic and structural investigation of this product is undertaken, with particular emphasis on its compatibility with existing notions of soft subsethood and equality. It is further demonstrated that the algebraic structure comprising the set of soft sets with a fixed parameter set, together with the symmetric difference of soft sets and the proposed product, satisfies the axiomatic requirements of both a ring and a hemiring. The theoretical implications of these findings are twofold: on the one hand, they significantly advance the algebraic underpinnings of soft set theory; on the other, they open new vistas for the formulation of a nascent soft group theory, potentially parallel to its classical counterpart. Given that the formal development of soft algebraic frameworks is intrinsically contingent upon rigorously defined operations and product constructions, the present study constitutes a substantial and foundational contribution to the algebraic maturation of soft set theory.

Keywords: Soft sets, Soft subsets, Soft equalities, Symmetric difference of soft sets, Soft symmetric difference-intersection product.

1. INTRODUCTION

A broad spectrum of mathematical frameworks has been proposed by numerous scholars to model and analyze phenomena characterized by inherent uncertainty, vagueness, and imprecision across diverse disciplines, including engineering, economics, social sciences, and healthcare. However, as highlighted by Molodtsov (1999), these frameworks exhibit intrinsic limitations. For instance, fuzzy set theory by Zadeh (1965) frequently encounters difficulties related to the subjective specification of membership functions, while probability theory

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presupposes the feasibility of repeated trials to ascertain mean values, thereby constraining its applicability in settings where such assumptions are untenable.

In response to these deficiencies, Molodtsov (1999) introduced soft set theory as a novel and flexible mathematical paradigm. By relaxing the stringent structural requirements imposed by traditional models, soft set theory offers a more adaptable framework, particularly suitable for problems involving parameter-dependent uncertainty. Its applicability has been demonstrated across multiple domains, including probability theory, game theory, and operations research. Over the past two decades, there has been a marked increase in research devoted to the formal foundations of soft set theory. Maji et al. (2003) initiated the axiomatic development of the theory by defining fundamental notions such as soft subsets and supersets, soft equality, and elementary operations including union, intersection, AND-product, and OR-product. This foundational work was subsequently refined by Pei and Miao (2005) who investigated the correspondence between soft sets and information systems, and reformulated intersection and subset operations within this context. Ali et al. (2009) further expanded the operational framework by introducing restricted union, restricted intersection, restricted difference, and extended intersection operations of soft sets. These developments were followed by a series of studies by Yang (2008); Feng et al. (2010); Jiang et al. (2010); Ali et al. (2011); Neog and Sut (2011); Fu (2011); Ge and Yang (2011); Singh and Onyeozili (2012a, 2012b, 2012c, 2012d); Zhu and Wen (2013); and Sen (2014) focusing on the algebraic properties of soft set operations, which addressed prior conceptual inconsistencies and advanced new structural methodologies.

Significant strides have been made in the formalization and expansion of soft set operations. The studies of Eren (2019); Stojanovic (2021); Sezgin et al. (2023); Sezgin and Dagtoros (2023); Sezgin and Çalışıcı (2024); Sezgin and Yavuz (2024); Sezgin and Sarıalioğlu, (2024); and Sezgin and Şenyiğit (2025) have helped in documenting the introduction and algebraic characterization of numerous novel operations. Central to this progression are refined formulations of soft equality and soft subsethood. Maji et al. (2003) laid the groundwork for the concept of soft subsets, which was subsequently generalized by Pei and Miao (2005) and Feng et al. (2010). Qin and Hong (2010) contributed to this line of inquiry by defining soft congruence and new forms of soft equality. Jun and Yang (2011) extended the distributive laws of Maji et al. (2003) by incorporating broader classes of soft subsets and proposing the concept of J-soft equality. Building upon these ideas, Liu et al. (2012) introduced soft L-subsets and L-equal relations, uncovering structural nuances that demonstrated the non-universality of classical distributive laws within the context of generalized soft equalities. Feng and Yongming (2013) advanced this trajectory by systematically classifying soft subsets and analyzing the algebraic behavior of operations such as the AND- and OR-products, originally introduced by Maji et al. (2003), within the framework of soft L-subsets. Their work resolved several previously open problems regarding associativity, commutativity, and distributivity, and further established that L-equal relations satisfy congruence properties in free soft algebras, wherein the resulting quotient structures form commutative semigroups. For further advancements in the theory of soft equalities, including generalized soft equality, soft lattice structures, relaxed parameter constraints, and various extended notions such as g-soft, gf-soft, and T-soft equalities (Abbas et al., 2014; Abbas et al., 2017; Alshami, 2019, 2020). To enhance the applicability of soft set theory in practical contexts, Çağman and Enginoğlu (2010) have revisited the definitions and operational constructs proposed by Maji et al. (2003). Building upon this, Sezgin et al. (2025a) have conducted an in-depth algebraic investigation of the AND-product under various equality frameworks, including L-equality, J-equality, M-equality, and soft F-subsets. Their analysis provided a comprehensive account of algebraic properties such as idempotency, commutativity, and associativity.

Parallel developments have emerged in the context of different types of soft product constructions. The concept of the soft union product was independently formulated for rings by Sezer (2012), for semigroups by Sezgin (2016), and for groups by Muştuoğlu et al. (2016), giving rise to the corresponding algebraic theories of soft union rings, semigroups, and groups, respectively. Analogously, the soft intersection product was defined for groups by Kaygısız (2012), for semigroups by Sezer et al. (2015), and for rings by Sezgin et al. (2017), wherein the presence or absence of identity elements and inverses yielded notable structural divergences for these products.

Within this research, we, firstly, focus on the symmetric difference operation of soft sets introduced by Çağman and Enginoğlu (2010). We rigorously demonstrate that the collection of soft sets with a fixed parameter set, equipped with the symmetric difference operation, forms an abelian group. Expanding upon this structure, we introduce a novel product for soft sets, termed the soft symmetric difference–intersection product, designed for soft sets whose parameter set is a group. This product is formulated within the definitional framework established by Çağman and Enginoğlu (2010), and is subjected to a detailed algebraic analysis, encompassing various soft subset classifications and generalized notions of equality. Our findings reveal that the algebraic system comprising soft sets with a fixed parameter set, when endowed with the symmetric difference operation of soft sets and the proposed product operation, constitutes both a ring and a hemiring. These characterizations elucidate not only the inherent algebraic properties of the proposed constructions but also delineate their potential to extend classical theories and contribute to the resolution of longstanding open problems. Within this framework, the present study does not merely fill existing theoretical gaps but also lays the groundwork for the emergence of a novel subfield in soft group theory, founded upon the introduced product.

The remainder of the manuscript is organized as follows: Section 2 provides a concise exposition of the foundational concepts relevant to soft set theory and hemiring. Section 3 investigates more on the symmetric difference operation of soft sets, and Section 4 introduces the soft symmetric difference–intersection product and undertakes a comprehensive algebraic analysis relative to different types of soft subset and equality. The concluding section summarizes the main results and outlines avenues for future research.

2. PRELIMINARIES

This section is dedicated to the systematic re-examination of selected foundational definitions and algebraic constructs that underpin the theoretical architecture of the developments elaborated in the subsequent section. While the concept of a soft set was originally introduced by Molodtsov (1999), the accompanying definitional apparatus and operational framework underwent a significant conceptual refinement in the work of Çağman and Enginoğlu (2010), aimed at enhancing both formal precision and practical applicability. The present investigation adopts this revised formulation as its foundational scaffold, and all ensuing analyses and constructions are articulated strictly within this enriched theoretical paradigm.

Definition 2.1. Let E be a parameter set, U be a universal set, $P(U)$ be the power set of U, and $\mathcal{H} \subseteq E$. Then, the soft set $f_{\mathcal{H}}$ over U is a function such that $f_{\mathcal{H}}: E \to P(U)$, where for all $w \notin \mathcal{H}$, $f_{\mathcal{H}}(w) = \emptyset$. That is,

$$f_{\mathcal{H}} = \{(w, f_{\mathcal{H}}(w)): w \in E\}$$

From now on, the soft set over U is abbreviated by SS (Molodtsov, 1999).

Definition 2.2. Let $f_{\mathcal{H}}$ be an SS. If $f_{\mathcal{H}}(w) = \emptyset$ for all $w \in E$, then $f_{\mathcal{H}}$ is called a null SS and indicated by $\emptyset_E$, and if $f_{\mathcal{H}}(w) = U$, for all $w \in E$, then $f_{\mathcal{H}}$ is called an absolute SS and indicated by U_E (Çağman and Enginoğlu, 2010).

Definition 2.3. Let $\mathscr{f}_{\mathcal{H}}$ and $\mathscr{g}_{\aleph}$ be two SSs. If $\mathscr{f}_{\mathcal{H}}(w) \subseteq \mathscr{g}_{\aleph}(w)$, for all $w \in E$, then $\mathscr{f}_{\mathcal{H}}$ is a soft subset of $\mathscr{g}_{\aleph}$ and indicated by $\mathscr{f}_{\mathcal{H}} \widetilde{\subseteq} \mathscr{g}_{\aleph}$. If $\mathscr{f}_{\mathcal{H}}(w) = \mathscr{g}_{\aleph}(w)$, for all $w \in E$, then $\mathscr{f}_{\mathcal{H}}$ is called soft equal to $\mathscr{g}_{\aleph}$, and denoted by $\mathscr{f}_{\mathcal{H}} = \mathscr{g}_{\aleph}$ (Çağman and Enginoğlu, 2010).

Definition 2.4. Let $\mathscr{f}_{\mathcal{H}}$ and $\mathscr{g}_{\aleph}$ be two SSs. Then, the symmetric difference of $\mathscr{f}_{\mathcal{H}}$ and $\mathscr{g}_{\aleph}$ is the soft set $\mathscr{f}_{\mathcal{H}} \tilde{\Delta} \mathscr{g}_{\aleph}$, where $(\mathscr{f}_{\mathcal{H}} \tilde{\Delta} \mathscr{g}_{\aleph})(w) = \mathscr{f}_{\mathcal{H}}(w) \Delta \mathscr{g}_{\aleph}(w)$, for all $w \in E$ (Enginoğlu, 2008).

Definition 2.5. Let $\mathscr{f}_K$ and $\mathscr{g}_{\aleph}$ be two SSs. Then, $\mathscr{f}_K$ is called a soft S-subset of $\mathscr{g}_{\aleph}$, denoted by $\mathscr{f}_K \widetilde{\subseteq}_S \mathscr{g}_{\aleph}$ if for all $w \in E$, $\mathscr{f}_K(w) = \mathcal{M}$ and $\mathscr{g}_{\aleph}(w) = \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are two fixed sets and $\mathcal{M} \subseteq \mathcal{N}$. Moreover, two SSs $\mathscr{f}_K$ and $\mathscr{g}_{\aleph}$ are said to be soft S-equal, denoted by $\mathscr{f}_K =_S \mathscr{g}_{\aleph}$, if $\mathscr{f}_K \widetilde{\subseteq}_S \mathscr{g}_{\aleph}$ and $\mathscr{g}_{\aleph} \widetilde{\subseteq}_S \mathscr{f}_K$ (Sezgin et al., 2025c).

It is obvious that if $\mathscr{f}_K =_S \mathscr{g}_{\aleph}$, then $\mathscr{f}_{\mathrm{K}}$ and $\mathscr{g}_{\aleph}$ are the same constant functions, that is, for all $w \in E$, $\mathscr{f}_K(w) = \mathscr{g}_{\aleph}(w) = \mathcal{M}$, where $\mathcal{M}$ is a fixed set (Sezgin et al., 2025c).

Definition 2.6. Let $\mathscr{f}_K$ and $\mathscr{g}_{\aleph}$ be two SSs. Then, $\mathscr{f}_K$ is called a soft A-subset of $\mathscr{g}_{\aleph}$, denoted by $\mathscr{f}_K \widetilde{\subseteq}_A \mathscr{g}_{\aleph}$, if, for each $\mathscr{a}, \mathscr{b} \in E$, $\mathscr{f}_K(\mathscr{a}) \subseteq \mathscr{g}_{\aleph}(\mathscr{b})$ (Sezgin et al., 2025c).

Definition 2.7. Let $\mathscr{f}_K$ and $\mathscr{g}_{\aleph}$ be two SSs. Then, $\mathscr{f}_K$ is called a soft S-complement of $\mathscr{g}_{\aleph}$, denoted by $\mathscr{f}_K =_S (\mathscr{g}_{\aleph})^c$, if, for all $w \in E$, $\mathscr{f}_K(w) = \mathcal{M}$ and $\mathscr{g}_{\aleph}(w) = \mathcal{D}$, where $\mathcal{M}$ and $\mathcal{N}$ are two fixed sets and $\mathcal{M} = \mathcal{D}'$. Here, $\mathcal{D}' = U \backslash \mathcal{D}$. (Sezgin et al., 2025c).

From now on, Δ represents the symmetric difference operation, which is commutative and associative, in classical set theory. Then, the symmetric difference of the family $\mathfrak{B} = \{C_i : i \in I\}$ such that I is an index set, is denoted by

$$\triangle \mathfrak{B} = \triangle_{i \in I} C_i = C_1 \Delta C_2 \Delta \ldots \Delta C_n$$

Definition 2.8. Let $\mathscr{f}_G$ and $\mathscr{g}_G$ be two SSs, where G is a group. Then, the soft symmetric difference-union product $\mathscr{f}_G \otimes_{s/u} \mathscr{g}_G$ is defined by

$$(\mathscr{f}_G \otimes_{s/u} \mathscr{g}_G)(x) = \triangle_{x=yz} (\mathscr{f}_G(y) \cup \mathscr{g}_G(z)), \ \ y, z \in G$$

for all $x \in G$ (Sezgin et al., 2025b).

The symmetric difference complement of the family $\mathfrak{D} = \{K_i : i \in I\}$ such that I is an index set, is denoted by

$$\coprod \mathfrak{D} = \coprod_{i \in I} K_i = (K_1 \Delta K_2 \Delta \ldots \Delta K_n)'$$

Definition 2.9. Let $\mathscr{f}_G$ and $\mathscr{g}_G$ be two SSs. Then, the soft symmetric difference complement-intersection product $\mathscr{f}_G \otimes_{s'/i} \mathscr{g}_G$ is defined by

$$\left(f_G \otimes_{s'/i} g_G\right)(x) = \coprod_{x=yz} \left(f_G(y) \cap g_G(z)\right), \quad y, z \in G$$

for all $x \in G$ (Ay and Sezgin, 2025).

Definition 2.10. Let f_G and g_G be two SSs, where G is a group. Then, the soft union-intersection product $f_G \otimes_{u/i} g_G$ is defined by

$$\left(f_G \otimes_{u/i} g_G\right)(x) = \bigcup_{x=yz} \left(f_G(y) \cap g_G(z)\right) \ y, z \in G$$

for all $x \in G$ (Kaygisiz, 2012).

Definition 2.11. A semiring $(R, +, \cdot)$ is an algebraic structure composed of a non-empty set R and two binary operations (addition and multiplication, respectively), such that multiplication is distributive over addition from both sides and $(R, +)$ and $(R, \cdot)$ are both semigroups. An element 0 in R is referred to as the zero of R if $0 \cdot \varpi = \varpi \cdot 0 = 0$ and $0 + \varpi = \varpi + 0 = \varpi$, for all $\varpi \in R$. The term "hemiring" refers to a semiring with commutative addition and zero element (Vandiver, 1934).

For additional information on SSs, refer to Khan et al. (2017); Atagün and Sezgin (2015); Sezer et al. (2014); Atagün and Sezgin (2017); Gulistan et al. (2018); Sezer et al. (2013); Jana et al. (2019); Atagün et al. (2019); Sezgin and Orbay (2022); Atagün and Sezgin (2018); Manikantan et al. (2023); Naeem (2017); Riaz and Naeem (2016); Riaz and Naeem (2017); Memiş (2022); Naeem and Memis (2023) and the generalizations of determinant divisibility in structured matrices refer to Kholil et al. (2025).

3. MORE ON SYMMETRIC DIFFERENCE OPERATION OF SOFT SETS

From now on, G denotes a group, $S_G(U)$ is the collection of SSs over U, whose parameter set is G, and all the SSs are the elements of $S_G(U)$. Çağman and Enginoğlu (2010) defined symmetric difference operation of soft sets (see Definition 2.4); however, the basic algebraic properties of the operation are not investigated up to now. In this section, we conduct a detailed investigation on the symmetric difference operation of soft sets in $S_G(U)$ focusing on the basic algebraic properties of the operation, and we show that $S_G(U)$ together with the soft symmetric difference operation is an abelian group.

Proposition 3.1. The set $S_G(U)$ is closed under the symmetric difference operation of SSs.That is, if f_G and g_G are two SSs over U, then so is $f_G \tilde{\Delta} g_G$.

PROOF. It is obvious that the symmetric difference operation of SSs is a binary operation in $S_G(U)$. Thereby, $S_G(U)$ is closed under the soft symmetric difference operation.

Proposition 3.2. The symmetric difference operation of SSs is associative in $S_G(U)$.

PROOF. Let f_G, g_G, and h_G be three SSs and $f_G\tilde{\Delta}g_G = m_G$, where $m_G(x) = f_G(x)\Delta g_G(x)$. Let $m_G\tilde{\Delta}h_G = k_G$, where $k_G(x) = m_G(x)\Delta h_G(x)$, for all $x \in G$. Thus, $k_G(x) = \big(f_G(x)\Delta g_G(x)\big)\Delta h_G(x)$. Let $g_G\tilde{\Delta}h_G = n_G$, where $n_G(x) = g_G(x)\Delta h_G(x)$ and $f_G\tilde{\Delta}n_G = l_G$, where $l_G(x) = f_G(x)\Delta n_G(x)$, for all $x \in G$. Thus, $l_G(x) = f_G(x)\Delta\big(g_G(x)\Delta h_G(x)\big)$, for all $x \in G$, implying that $k_G(x) = l_G(x)$, for all $x \in G$. Thereby, $(f_G\tilde{\Delta}g_G)\tilde{\Delta}h_G = f_G\tilde{\Delta}(g_G\tilde{\Delta}h_G)$.

Proposition 3.3. The symmetric difference operation of SSs is commutative in $S_G(U)$.

PROOF. Let f_G and g_G be two SSs. Then, $(f_G\tilde{\Delta}g_G)(x) = f_G(x)\Delta g_G(x) = g_G(x)\Delta f_G(x) = (g_G\tilde{\Delta}f_G)(x)$, for all $x \in G$. Thus, $f_G\tilde{\Delta}g_G = g_G\tilde{\Delta}f_G$.

Proposition 3.4. $\emptyset_G$ is the identity element of the symmetric difference operation of SSs in $S_G(U)$.

PROOF. Let f_G be an SS. Then, $\big(f_G\tilde{\Delta}\emptyset_G\big)(x) = f_G(x)\Delta\emptyset_G(x) = f_G(x)\Delta\emptyset = f_G(x)$, for all $x \in G$. Similarly, $\big(\emptyset_G\tilde{\Delta}f_G\big)(x) = \emptyset_G(x)\Delta f_G(x) = \emptyset\Delta f_G(x) = f_G(x)$. Thus, $f_G\tilde{\Delta}\emptyset_G = \emptyset_G\tilde{\Delta}f_G = f_G$, implying that $\emptyset_G$ is the identity element for soft symmetric difference operation in $S_G(U)$.

Proposition 3.5. The inverse of each element for the symmetric difference operation of SSs is itself in $S_G(U)$.

PROOF. Let $x \in G$ and f_G be an SS. Thus, $\big(f_G\tilde{\Delta}f_G\big)(x) = f_G(x)\Delta f_G(x) = \emptyset = \emptyset_G(x)$. Thereby, $f_G\tilde{\Delta}f_G = \emptyset_G$, implying that the inverse of each element for the symmetric difference operation of SSs is itself in $S_G(U)$. Moreover, the symmetric difference operation of SSs is not idempotent in $S_G(U)$.

Theorem 3.6. $(S_G(U), \tilde{\Delta})$ is an abelian group with the identity $\emptyset_G$.

PROOF. The proof follows from Proposition 3.1, Proposition 3.2, Proposition 3.3, Proposition 3.4, and Proposition 3.5.

4. SOFT SYMMETRIC DIFFERENCE-INTERSECTION PRODUCT OF GROUPS

In this section, we introduce a novel binary product on SSs, termed the soft symmetric difference–intersection product, defined for SSs whose parameter sets are endowed with a group structure. A comprehensive algebraic analysis of this product is undertaken, with particular emphasis on its structural behavior in the context of diverse soft equality relations and classifications of soft subsets. Theoretical developments are supplemented with illustrative examples to elucidate the underlying concepts and to demonstrate the practical utility of the proposed construct.

Definition 4.1. Let f_G and g_G be two SSs over U. Then, the soft symmetric difference-intersection product $f_G \otimes_{s/i} g_G$ is defined by

$$\left(f_G \otimes_{s/i} g_G\right)(x) = \mathop{\triangle}_{x=yz}\left(f_G\,(y) \cap g_G(z)\right), \quad y, z \in G$$

for all $x \in G$.

Note here that since G is a group, there always exists $y, z \in G$ such that $x = yz$, for all $x \in G$. Let the order of the group G be n, that is $|G| = n$. Then, it is obvious that there exist n distinct representation for each $x \in G$ such that $x = yz$, where $y, z \in G$.

Note 4.2. The soft symmetric difference-intersection product is well-defined in $S_G(U)$. In fact, let $f_G,\ g_G,\ m_G,\ k_G \in\ S_G(U)$ such that $(f_G, g_G) = (m_G, k_G)$. Then, $f_G = m_G$ and $g_G = k_G$, implying that $f_G(x) = m_G(x)$ and $g_G(x) = k_G(x)$, for all $x \in G$. Thereby, for all $x \in G$,

$$\begin{aligned}\left(f_G \otimes_{s/i} g_G\right)(x) &= \mathop{\triangle}_{x=yz}\left(f_G\,(y) \cap g_G(z)\right)\\ &= \mathop{\triangle}_{x=yz}\left(m_G\,(y) \cap k_G(z)\right)\\ &= \left(m_G \otimes_{s/i} k_G\right)(x)\end{aligned}$$

Hence, $f_G \otimes_{s/i} g_G =\ m_G \otimes_{s/i} k_G$.

Example 4.3. Consider the group $G = \{\varpi, \tau\}$ with the following operation**:**

$\cdot$	ϖ	τ
ϖ	ϖ	τ
τ	τ	ϖ

Let f_G and g_G be two SSs over $U = D_2 = \{< x, y >: x^2 = y^2 = e, xy = yx\} = \{e, x, y, yx\}$ as follows:

$f_G = \{(\varpi, \{e, x, yx\}), (\tau, \{x, yx\})\}$ and $g_G = \{(\varpi, \{e, y, yx\}), (\tau, \{e, y\})\}$

Since $\varpi = \varpi\varpi = \tau\tau, \left(f_G \otimes_{s/i} g_G\right)(\varpi) = \left(f_G(\varpi) \cap g_G(\varpi)\right)\Delta\left(f_G(\tau) \cap g_G(\tau)\right) = \{e, yx\}$, and since $\tau = \varpi\tau = \tau\varpi$, $\left(f_G \otimes_{s/i} g_G\right)(\tau) = \left(f_G(\varpi) \cap g_G(\tau)\right)\Delta\left(f_G(\tau) \cap g_G(\varpi)\right) = \{e, yx\}$ is obtained. Hence,

$$f_G \otimes_{s/i} g_G = \{(\varpi, \{e, yx\}), (\tau, \{e, yx\})\}$$

Proposition 4.4. The set $S_G(U)$ is closed under the soft symmetric difference-intersection product. That is, if f_G and g_G are two SSs over U, then so is $f_G \otimes_{s/i} g_G$.

PROOF. It is obvious that the soft symmetric difference-intersection product is a binary operation in $S_G(U)$. Thereby, $S_G(U)$ is closed under the soft symmetric difference-intersection product.

Proposition 4.5. The soft symmetric difference-intersection product is associative in $S_G(U)$.

PROOF. Let f_G , g_G, and h_G be three SSs. Thus, for all $x \in G$,

$$((f_G \otimes_{s/i} g_G) \otimes_{s/i} h_G)(x) = \underset{x=yz}{\triangle} \left((f_G \otimes_{s/i} g_G)(y) \cap h_G(z)\right)$$

$$= \underset{x=yz}{\triangle} \left(\left(\underset{y=mn}{\triangle} f_G(m) \cap g_G(n)\right) \cap h_G(z)\right)$$

$$= \underset{x=(mn)z}{\triangle} \left(f_G(m) \cap g_G(n) \cap h_G(z)\right)$$

$$= \underset{x=m(nz)}{\triangle} \left(f_G(m) \cap \left(g_G(n) \cap h_G(z)\right)\right)$$

$$= \underset{x=m(nz)}{\triangle} \left(f_G(m) \cap \left(\underset{k=nz}{\triangle} \left(g_G(n) \cap h_G(z)\right)\right)\right)$$

$$= \underset{x=mk}{\triangle} \left(f_G(m) \cap \left(g_G \otimes_{s/i} h_G\right)(k)\right)$$

$$= \left(f_G \otimes_{s/i} (g_G \otimes_{s/i} h_G)\right)(x)$$

Thereby, $\left(f_G \otimes_{s/i} g_G\right) \otimes_{s/i} h_G = f_G \otimes_{s/i} \left(g_G \otimes_{s/i} h_G\right)$. □

Example 4.6. Consider the SSs f_G and g_G over $U = \{e, x, y, yx\}$ in Example 4.3. Let $h_G = \{(\varpi, \{e, y\}), (\tau, \{y, yx\})\}$ be an SS over U. Since $f_G \otimes_{s/i} g_G = \{(\varpi, \{e, yx\}), (\tau, \{e, yx\})\}$, then

$$\left(f_G \otimes_{s/i} g_G\right) \otimes_{s/i} h_G = \{(\varpi, \{e, yx\}), (\tau, \{e, yx\})\}$$

Moreover, since $g_G \otimes_{s/i} h_G = \{(\varpi, \{e\}), (\tau, \{y, yx\})\}$, then

$$f_G \otimes_{s/i} \left(g_G \otimes_{s/i} h_G\right) = \{(\varpi, \{e, yx\}), (\tau, \{e, yx\})\}$$

Thereby, $\left(f_G \otimes_{s/i} g_G\right) \otimes_{s/i} h_G = f_G \otimes_{s/i} \left(g_G \otimes_{s/i} h_G\right)$.

Proposition 4.7. The soft symmetric difference-intersection product is not commutative in $S_G(U)$. However, if G is an abelian group, then the soft symmetric difference-intersection product is commutative in $S_G(U)$.

PROOF. Let f_G and g_G be two SSs and G be an abelian group. Then, for all $x \in G$,

$$\left(f_G \otimes_{s/i} g_G\right)(x) = \underset{x=yz}{\triangle} \left(f_G(y) \cap g_G(z)\right)$$

$$= \underset{x=zy}{\triangle} \left(g_G(z) \cap f_G(y)\right)$$

$$= \left(g_G \otimes_{s/i} f_G\right)(x)$$

implying that $\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G = \mathscr{g}_G \otimes_{s/i} \mathscr{f}_G$. □

Proposition 4.8. The soft symmetric difference-intersection product is not idempotent in $S_G(U)$.

PROOF. Consider the SS $\mathscr{f}_G$ in Example 4.3. Then, $\mathscr{f}_G \otimes_{s/i} \mathscr{f}_G = \{(\varpi, \{e\}), (\tau, \emptyset)\}$, implying that $\mathscr{f}_G \otimes_{s/i} \mathscr{f}_G \neq \mathscr{f}_G$. □

Proposition 4.9. Let $\mathscr{f}_G$ be a constant SS. Then,

i. $\mathscr{f}_{\mathrm{G}} \otimes_{\mathrm{s/i}} \mathscr{f}_{\mathrm{G}} = \mathscr{f}_{\mathrm{G}}$, where $|\mathrm{G}| = \mathscr{r}$ and $\mathscr{r}$ is a positive odd integer.

ii. $\mathscr{f}_{\mathrm{G}} \otimes_{\mathrm{s/i}} \mathscr{f}_{\mathrm{G}} = \emptyset_{\mathrm{G}}$, where $|\mathrm{G}| = \mathscr{r}$ and $\mathscr{r}$ is a positive even integer.

PROOF. Let $\mathscr{f}_G$ be a constant SS such that, for all $x \in G$, $\mathscr{f}_G(x) = A$, where A is a fixed set.

i. Let $|G| = \mathscr{r}$, where $\mathscr{r}$ is a positive odd integer. Then, for all $x \in G$,

$$(\mathscr{f}_G \otimes_{s/i} \mathscr{f}_G)(x) = \underset{x=yz}{\triangle}(\mathscr{f}_G(y) \cap \mathscr{f}_G(z)) = \underbrace{A\Delta A\Delta \ldots \Delta A}_{\mathscr{r} \text{ times } A,\ \text{where } \mathscr{r} \text{ is odd}} = A = \mathscr{f}_G(x)$$

Thereby, $\mathscr{f}_G \otimes_{s/i} \mathscr{f}_G = \mathscr{f}_G$.

ii. Let $|G| = \mathscr{r}$, where $\mathscr{r}$ is a positive even integer. Then, for all $x \in G$,

$$(\mathscr{f}_G \otimes_{s/i} \mathscr{f}_G)(x) = \underset{x=yz}{\triangle}(\mathscr{f}_G(y) \cap \mathscr{f}_G(z)) = \underbrace{A\Delta A\Delta \ldots \Delta A}_{\mathscr{r} \text{ times } A,\ \text{where } \mathscr{r} \text{ is even}} = \emptyset_G(x)$$

Thereby, $\mathscr{f}_G \otimes_{s/i} \mathscr{f}_G = \emptyset_G$.

Remark 4.10. Let $S_G{}^*(U)$ be the collection of all constant SSs. Then, the soft symmetric difference-intersection product is idempotent in $S_G{}^*(U)$, where $|G| = \mathscr{r}$ and $\mathscr{r}$ is a positive odd integer.

Proposition 4.11. $\emptyset_G$ is the absorbing element of the soft symmetric difference-intersection product in $S_G(U)$.

PROOF. Let $\mathscr{f}_G$ be an SS. Then, for all $x \in G$,

$$(\emptyset_G \otimes_{s/i} \mathscr{f}_G)(x) = \underset{x=yz}{\triangle}(\emptyset_G(y) \cap \mathscr{f}_G(z)) = \underset{x=yz}{\triangle}(\emptyset \cap \mathscr{f}_G(z)) = \emptyset_G(x)$$

Similarly, for all $x \in G$,

$$(\mathscr{f}_G \otimes_{s/i} \emptyset_G)(x) = \underset{x=yz}{\triangle}(\mathscr{f}_G(y) \cap \emptyset_G(z)) = \underset{x=yz}{\triangle}(\mathscr{f}_G(y) \cap \emptyset) = \emptyset_G(x)$$

Thus, $\emptyset_G \otimes_{s/i} \mathscr{f}_G = \mathscr{f}_G \otimes_{s/i} \emptyset_G = \emptyset_G$. □

Proposition 4.12. Let $\mathscr{f}_G$ be a constant SS. Then,

i. $U_G \otimes_{s/i} \mathscr{f}_G = \mathscr{f}_G \otimes_{s/i} U_G = \mathscr{f}_G$, where $|G| = \mathscr{r}$ and $\mathscr{r}$ is a positive odd integer.

ii. $\mathscr{f}_G \otimes_{s/i} \mathscr{f}_G = \emptyset_G$, where $|G| = \mathscr{r}$ and $\mathscr{r}$ is a positive even integer.

PROOF. Let $\mathscr{f}_G$ be an SS.

i. Suppose that $|G| = \mathscr{r}$, where $\mathscr{r}$ is a positive odd integer. Then, for all $x \in G$,

$$\left(U_G \otimes_{s/i} \mathscr{f}_G\right)(x) = \underset{x=yz}{\triangle}\left(U_G(y) \cap \mathscr{f}_G(z)\right) = \underset{x=yz}{\triangle}\left(U \cap \mathscr{f}_G(z)\right) = \mathscr{f}_G(x)$$

Thereby, $U_G \otimes_{s/i} \mathscr{f}_G = \mathscr{f}_G$. Similarly, for all $x \in G$,

$$\left(\mathscr{f}_G \otimes_{s/i} U_G\right)(x) = \underset{x=yz}{\triangle}\left(\mathscr{f}_G(y) \cap U_G(z)\right) = \underset{x=yz}{\triangle}\left(\mathscr{f}_G(y) \cap U\right) = \mathscr{f}_G(x)$$

Thus, $\mathscr{f}_G \otimes_{s/u} U_G = \mathscr{f}_G$.

ii. Suppose that $|G| = \mathscr{r}$, where $\mathscr{r}$ is a positive even integer. Then, for all $x \in G$,

$$\left(U_G \otimes_{s/i} \mathscr{f}_G\right)(x) = \underset{x=yz}{\triangle}\left(U_G(y) \cap \mathscr{f}_G(z)\right) = \underset{x=yz}{\triangle}\left(U \cap \mathscr{f}_G(z)\right) = \emptyset_G(x)$$

Thereby, $U_G \otimes_{s/i} \mathscr{f}_G = \emptyset_G$. Similarly, for all $x \in G$,

$$\left(\mathscr{f}_G \otimes_{s/i} U_G\right)(x) = \underset{x=yz}{\triangle}\left(\mathscr{f}_G(y) \cap U_G(z)\right) = \underset{x=yz}{\triangle}\left(\mathscr{f}_G(y) \cap U\right) = \emptyset_G(x)$$

Thereby, $\mathscr{f}_G \otimes_{s/i} U_G = \emptyset_G$. □

Remark 4.13. U_G is the identity element of the soft symmetric difference-intersection product in $S_G{}^*(U)$, where $|G| = \mathscr{r}$ and $\mathscr{r}$ is a positive odd integer.

Theorem 4.14. $\left(S_G(U), \otimes_{s/i}\right)$ is a noncommutative semigroup with the absorbing element $\emptyset_G$. If G is abelian, then $\left(S_G(U), \otimes_{s/i}\right)$ is a commutative semigroup with the absorbing element $\emptyset_G$.

PROOF. The proof follows from Proposition 4.4, Proposition 4.5, Proposition 4.7, Proposition 4.8 and Proposition 4.11. □

Theorem 4.15. $\left(S_G{}^*(U), \otimes_{s/i}\right)$ is a noncommutative idempotent semigroup with the identity element U_G and the absorbing element $\emptyset_G$, where $|G| = \mathscr{r}$ and $\mathscr{r}$ is a positive odd integer. If G is abelian, then $\left(S_G{}^*(U), \otimes_{s/i}\right)$ is a commutative idempotent monoid with the identity element U_G and the absorbing element $\emptyset_G$, that is, a bounded semilattice, where $\mathscr{r}$ is a positive odd integer.

PROOF. The proof follows from Proposition 4.4, Proposition 4.5, Proposition 4.7, Remark 4.10, Proposition 4.11, and Remark 4.13. □

Proposition 4.16. Let $\mathscr{f}_G$ be a constant SS. Then, $\mathscr{f}_G \otimes_{s/i} \mathscr{f}_G{}^c = \mathscr{f}_G{}^c \otimes_{s/i} \mathscr{f}_G = \emptyset_G$.

PROOF. Let $\mathscr{f}_G$ be a constant SS. Then, for all $x \in G$,

$$\left(\mathscr{f}_G \otimes_{s/i} \mathscr{f}_G{}^c\right)(x) = \underset{x=yz}{\triangle}\left(\mathscr{f}_G(y) \cap \mathscr{f}_G{}^c(z)\right) = \emptyset = \emptyset_G(x)$$

Thereby, $\mathscr{f}_G \otimes_{s/i} \mathscr{f}_G{}^c = \emptyset_G$. Similarly, for all $x \in G$,

$$\left(\mathscr{f}_G{}^c \otimes_{s/i} \mathscr{f}_G\right)(x) = \underset{x=yz}{\triangle}\left(\mathscr{f}_G{}^c(y) \cap \mathscr{f}_G(z)\right) = \emptyset = \emptyset_G(x)$$

Thus, $\mathscr{f}_G{}^c \otimes_{s/i} \mathscr{f}_G = \emptyset_G$. □

Proposition 4.17. Let $\mathscr{f}_G$ and $\mathscr{g}_G$ be two SSs such that $\mathscr{f}_G \,\tilde{\subseteq}_S\, \mathscr{g}_G$. Then,

i. $\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G = \mathscr{f}_G$, where $|G| = r$ and r is a positive odd integer.

ii. $\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G = \emptyset_G$, where $|G| = r$ and r is a positive even integer.

PROOF. Let $\mathscr{f}_G$ and $\mathscr{g}_G$ be two SSs and $\mathscr{f}_G \,\tilde{\subseteq}_S\, \mathscr{g}_G$. Then, for all $x \in G$, $\mathscr{f}_G(x) = A$, $\mathscr{g}_G(x) = B$, where A and B are two fixed sets and $A \subseteq B$.

i. Let $|G| = r$, where r is a positive odd integer. Then, for all $x \in G$,

$$\left(\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G\right)(x) = \mathop{\triangle}_{x=yz}\left(\mathscr{f}_G(y) \cap \mathscr{g}_G(z)\right) = \mathscr{f}_G(x)$$

Thereby, $\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G = \mathscr{f}_G$.

ii. Let $|G| = r$, where r is a positive even integer. Then, for all $x \in G$,

$$\left(\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G\right)(x) = \mathop{\triangle}_{x=yz}\left(\mathscr{f}_G(y) \cap \mathscr{g}_G(z)\right) = \emptyset_G(x)$$

Thereby, $\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G = \emptyset_G$.

Proposition 4.18. Let $\mathscr{f}_G$ and $\mathscr{g}_G$ be two SSs such that $\mathscr{g}_G \,\tilde{\subseteq}_S\, \mathscr{f}_G$. Then,

i. $\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G = \mathscr{g}_G$, where $|G| = r$ and r is a positive odd integer.

ii. $\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G = \emptyset_G$, where $|G| = r$ and r is a positive even integer.

PROOF. Let $\mathscr{f}_G$ and $\mathscr{g}_G$ be two SSs and $\mathscr{g}_G \,\tilde{\subseteq}_S\, \mathscr{f}_G$. Then, for all $x \in G$, $\mathscr{f}_G(x) = A$, $\mathscr{g}_G(x) = B$, where A and B are two fixed sets and $B \subseteq A$.

i. Let $|G| = r$, where r is a positive odd integer. Then, for all $x \in G$,

$$\left(\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G\right)(x) = \mathop{\triangle}_{x=yz}\left(\mathscr{f}_G(y) \cap \mathscr{g}_G(z)\right) = \mathscr{g}_G(x)$$

Thereby, $\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G = \mathscr{g}_G$.

ii. Let $|G| = r$, where r is a positive even integer. Then, for all $x \in G$,

$$\left(\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G\right)(x) = \mathop{\triangle}_{x=yz}\left(\mathscr{f}_G(y) \cap \mathscr{g}_G(z)\right) = \emptyset_G(x)$$

Thereby, $\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G = \emptyset_G$.

Proposition 4.19. Let $\mathscr{f}_G$ and $\mathscr{g}_G$ be two SSs such that $\mathscr{f}_G \,\tilde{\subseteq}_A\, (\mathscr{g}_G)^c$. Then, $\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G = \emptyset_G$.

PROOF. Let $\mathscr{f}_G$ and $\mathscr{g}_G$ be two SSs and $\mathscr{f}_G \,\tilde{\subseteq}_A\, (\mathscr{g}_G)^c$. Then, $\mathscr{f}_G(x) \subseteq (\mathscr{g}_G)^c(y)$, for each $x, y \in G$. Then, for all $x \in G$,

$$\left(\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G\right)(x) = \mathop{\triangle}_{x=yz}\left(\mathscr{f}_G(y) \cap \mathscr{g}_G(z)\right) = \emptyset_G(x)$$

Thereby, $\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G = \emptyset_G$. Here note that, in classical set theory if $A \subseteq B'$, then $A \cap B = \emptyset$, where A and B are two fixed sets.

Proposition 4.20. Let $\mathscr{f}_G$ and $\mathscr{g}_G$ be two SSs such that $\mathscr{f}_G \,\tilde{\subseteq}_S\, (\mathscr{g}_G)^c$. Then, $\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G = \emptyset_G$.

PROOF. The proof is similar to the proof of Proposition 4.19.

Proposition 4.21. Let f_G and g_G be two SSs. Then, $(f_G \otimes_{s/i} g_G)^c = f_G \otimes_{s'/i} g_G$.

PROOF. Let f_G and g_G be two SSs. Then, for all $x \in G$,

$$\begin{aligned}(f_G \otimes_{s/i} g_G)^c(x) &= \left(\underset{x=yz}{\triangle}\big(f_G(y) \cap g_G(z)\big)\right)' \\ &= \coprod_{x=yz}\big(f_G(y) \cap g_G(z)\big)' \\ &= \coprod_{x=yz}\big(f_G(y) \cap g_G(z)\big) \\ &= (f_G \otimes_{s'/i} g_G)(x)\end{aligned}$$

Thus, $(f_G \otimes_{s/i} g_G)^c = f_G \otimes_{s'/i} g_G$. Here note that, in classical set theory, $(A'\Delta B')' = (A\Delta B)'$, where A and B are fixed sets. □

Proposition 4.22. Let f_G and g_G be two SSs. If $f_G = g_G = U_G$ and $|G| = r$, where r is a positive odd integer, then $f_G \otimes_{s/i} g_G = U_G$.

PROOF. Let $f_G = g_G = U_G$ and $|G| = r$, where r is a positive odd integer. Then, for all $x \in G$,

$$\begin{aligned}(f_G \otimes_{s/i} g_G)(x) &= \underset{x=yz}{\triangle}\big(f_G(y) \cap g_G(z)\big) \\ &= \underset{x=yz}{\triangle}\big(U_G(y) \cap U_G(z)\big) \\ &= U_G(x)\end{aligned}$$

Thus, $f_G \otimes_{s/i} g_G = U_G$.□

Proposition 4.23. Let f_G and g_G be two SSs. Then, $f_G \otimes_{s/i} g_G \,\tilde{\subseteq}\, f_G \otimes_{u/i} g_G$.

PROOF. Let f_G and g_G be two SSs. Then, for all $x \in G$,

$$\begin{aligned}(f_G \otimes_{s/i} g_G)(x) &= \underset{x=yz}{\triangle}\big(f_G(y) \cap g_G(z)\big) \\ &\subseteq \bigcup_{x=yz}\big(f_G(y) \cap g_G(z)\big) \\ &= (f_G \otimes_{u/i} g_G)(x)\end{aligned}$$

Thus, $f_G \otimes_{s/i} g_G \,\tilde{\subseteq}\, f_G \otimes_{u/i} g_G$. □

Proposition 4.24. Let f_G and g_G be two SSs. If one of the assertions following is satisfied, then $f_G \otimes_{s/i} g_G = f_G \otimes_{u/i} g_G$:

i. $f_G \,\tilde{\subseteq}_S\, g_G$ and $|G| = r$, where r is a positive odd integer

ii. $g_G \,\tilde{\subseteq}_S\, f_G$ and $|G| = r$, where r is a positive odd integer

iii. $f_G =_S g_G$ and $|G| = r$, where r is a positive odd integer *iv.* $\mathscr{f}_G \,\tilde{\subseteq}_A\, (\mathscr{g}_G)^c$

v. $\mathscr{f}_G \,\tilde{\subseteq}_S\, (\mathscr{g}_G)^c$

vi. $\mathscr{f}_G = \emptyset_G$ or $\mathscr{g}_G = \emptyset_G$

vii. $\mathscr{g}_G$ be a constant, $\mathscr{f}_G = U_G$, $|G| = r$ and r is a positive odd integer

viii. $\mathscr{f}_G$ be a constant and $\mathscr{g}_G = U_G$, $|G| = r$ and r is a positive odd integer

PROOF. Let $\mathscr{f}_G$ and $\mathscr{g}_G$ be two SSs.

i. Let $\mathscr{f}_G \,\tilde{\subseteq}_S\, \mathscr{g}_G$ and $|G| = r$, where r is a positive odd integer. Hence, for all $\varpi \in G$, $\mathscr{f}_G(\varpi) = \mathcal{M}$ and $\mathscr{g}_G(\varpi) = \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are two fixed sets and $\mathcal{M} \subseteq \mathcal{N}$. Thus, for all $x \in G$,

$$
\begin{aligned}
(\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G)(x) &= \mathop{\triangle}_{x=yz}\left(\mathscr{f}_G(y) \cap \mathscr{g}_G(z)\right)\\
&= \mathscr{f}_G(x)\\
&= \bigcup_{x=yz}\left(\mathscr{f}_G(y) \cap \mathscr{g}_G(z)\right)\\
&= (\mathscr{f}_G \otimes_{u/i} \mathscr{g}_G)(x)
\end{aligned}
$$

Thus, $\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G = \mathscr{f}_G \otimes_{u/i} \mathscr{g}_G$.

ii. Let $\mathscr{g}_G \,\tilde{\subseteq}_S\, \mathscr{f}_G$ and $|G| = r$, where r is a positive odd integer. Hence, for all $\varpi \in G$, $\mathscr{f}_G(\varpi) = \mathcal{M}$ and $\mathscr{g}_G(\varpi) = \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are two fixed sets and $\mathcal{N} \subseteq \mathcal{M}$. Thus,

$$
\begin{aligned}
(\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G)(x) &= \mathop{\triangle}_{x=yz}\left(\mathscr{f}_G(y) \cap \mathscr{g}_G(z)\right)\\
&= \mathscr{g}_G(x)\\
&= \bigcup_{x=yz}\left(\mathscr{f}_G(y) \cap \mathscr{g}_G(z)\right)\\
&= (\mathscr{f}_G \otimes_{u/i} \mathscr{g}_G)(x)
\end{aligned}
$$

Then, $\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G = \mathscr{f}_G \otimes_{u/i} \mathscr{g}_G$.

iii. The proof follows from the proof of Proposition 4.24 (ii).

iv. Let $\mathscr{f}_G \,\tilde{\subseteq}_A\, (\mathscr{g}_G)^c$. Then, $\mathscr{f}_G(x) \subseteq (\mathscr{g}_G)^c(y)$, for each $x, y \in G$. Thus, for all $x \in G$,

$$
\begin{aligned}
(\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G)(x) &= \mathop{\triangle}_{x=yz}\left(\mathscr{f}_G(y) \cap \mathscr{g}_G(z)\right)\\
&= \emptyset\\
&= \bigcup_{x=yz}\left(\mathscr{f}_G(y) \cap \mathscr{g}_G(z)\right)\\
&= (\mathscr{f}_G \otimes_{u/i} \mathscr{g}_G)(x)
\end{aligned}
$$

Thereby, $\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G = \mathscr{f}_G \otimes_{u/i} \mathscr{g}_G$.

v. The proof follows from the proof of Proposition 4.24 (iv).

vi. Without loss of generality, let $f_G = \emptyset_G$. Then, for all $x \in G$, $f_G(x) = \emptyset$. Thus,

$$
\begin{aligned}
(f_G \otimes_{s/i} g_G)(x) &= \mathop{\triangle}_{x=yz} \big(f_G(y) \cap g_G(z)\big) \\
&= \emptyset \\
&= \bigcup_{x=yz} \big(f_G(y) \cap g_G(z)\big) \\
&= (f_G \otimes_{u/i} g_G)(x)
\end{aligned}
$$

Thereby, $f_G \otimes_{s/i} g_G = f_G \otimes_{u/i} g_G$.

vi. Let g_G be a constant SS such that, for all $x \in G$, $g_G(x) = B$, where B is a fixed set, $f_G = U_G$, where $|G| = r$ and r is a positive odd integer. Then, for all $x \in G$, $f_G(x) = U_G(x) = U$. Thus, for all $x \in G$,

$$
\begin{aligned}
\big(f_G \otimes_{s/i} g_G\big)(x) &= \mathop{\triangle}_{x=yz} \big(f_G(y) \cap g_G(z)\big) \\
&= \bigcup_{x=yz} \big(f_G(y) \cap g_G(z)\big) \\
&= \big(f_G \otimes_{u/i} g_G\big)(x)
\end{aligned}
$$

Thereby, $f_G \otimes_{s/i} g_G = f_G \otimes_{u/i} g_G$.

vii. The proof follows from the proof of Proposition 4.24 (vi). □

Proposition 4.25. The soft symmetric difference-intersection product distributes over the soft symmetric difference from the right side.

PROOF. Let f_G, g_G, and h_G be three SSs. Then, for all $x \in G$,

$$
\begin{aligned}
\big((f_G \tilde{\Delta} g_G) \otimes_{s/i} h_G)\big)(x) &= \mathop{\triangle}_{x=yz} \Big(\big((f_G \tilde{\Delta} g_G)(y)\big) \cap h_G(z)\Big) \\
&= \mathop{\triangle}_{x=yz} \Big(\big(f_G(y) \Delta g_G(y)\big) \cap h_G(z)\Big) \\
&= \mathop{\triangle}_{x=yz} \big[\big(f_G(y) \cap h_G(z)\big) \Delta \big(g_G(y) \cap h_G(z)\big)\big] \\
&= \Big[\mathop{\triangle}_{x=yz} \big(f_G(y) \cap h_G(z)\big)\Big] \Delta \Big[\mathop{\triangle}_{x=yz} \big(g_G(y) \cap h_G(z)\big)\Big] \\
&= (f_G \otimes_{s/i} h_G)(x) \Delta (g_G \otimes_{s/i} h_G)(x) \\
&= \big((f_G \otimes_{s/i} h_G) \tilde{\Delta} (g_G \otimes_{s/i} h_G)\big)(x)
\end{aligned}
$$

Thus, $(f_G \tilde{\Delta} g_G) \otimes_{s/i} h_G = (f_G \otimes_{s/i} h_G) \tilde{\Delta} (g_G \otimes_{s/i} h_G)$. □

Example 4.26. Let f_G, g_G, and h_G be three SSs over $U = \{e, x, y, yx\}$ as follows:

$\mathscr{f}_G = \{(\varpi, \{e, x, yx\}), (\tau, \{x, yx\})\}$, $\mathscr{g}_G = \{(\varpi, \{e, y, yx\}), (\tau, \{e, y\})\}$, and $\mathscr{h}_G = \{(\varpi, \{e, y\}), (\tau, \{x, y\})\}$

Since, $\mathscr{f}_G \otimes_{s/i} \mathscr{h}_G = \{\varpi, \{e, x\}), (\tau, \{x\})\}$ and $\mathscr{g}_G \otimes_{s/i} \mathscr{h}_G = \{(\varpi, \{e\}), (\tau, \{e\})\}$, then

$$(\mathscr{f}_G \otimes_{s/i} \mathscr{h}_G) \tilde{\Delta} (\mathscr{g}_G \otimes_{s/i} \mathscr{h}_G) = \{(\varpi, \{x\}), (\tau, \{e, x\})\}$$

Moreover, since, $\mathscr{f}_G \tilde{\Delta} \mathscr{g}_G = \{(\varpi, \{x, y\}), (\tau, \{e, x, y, yx\})\}$,

$$(\mathscr{f}_G \tilde{\Delta} \mathscr{g}_G) \otimes_{s/i} \mathscr{h}_G = \{(\varpi, \{x\}), (\tau, \{e, x\})\}$$

Thus, $(\mathscr{f}_G \tilde{\Delta} \mathscr{g}_G) \otimes_{s/i} \mathscr{h}_G = (\mathscr{f}_G \otimes_{s/i} \mathscr{h}_G) \tilde{\Delta} (\mathscr{g}_G \otimes_{s/i} \mathscr{h}_G)$.□

Proposition 4.27. The soft symmetric difference-intersection product distributes over the soft symmetric difference from the left side.

PROOF. Let $\mathscr{f}_G$, $\mathscr{g}_G$, and $\mathscr{h}_G$ be three SSs. Then, for all $x \in G$,

$$\begin{aligned}
\left(\mathscr{f}_G \otimes_{s/i} (\mathscr{g}_G \tilde{\Delta} \mathscr{h}_G)\right)(x) &= \bigtriangleup_{x=yz} \left(\mathscr{f}_G(y) \cap (\mathscr{g}_G \tilde{\Delta} \mathscr{h}_G)(z)\right) \\
&= \bigtriangleup_{x=yz} \left(\mathscr{f}_G(y) \cap \left(\mathscr{g}_G(z) \Delta\, \mathscr{h}_G(z)\right)\right) \\
&= \bigtriangleup_{x=yz} \left[\left(\mathscr{f}_G(y) \cap \mathscr{g}_G(z)\right) \Delta \left(\mathscr{f}_G(y) \cap \mathscr{h}_G(z)\right)\right] \\
&= \left[\bigtriangleup_{x=yz} \left(\mathscr{f}_G(y) \cap \mathscr{g}_G(z)\right)\right] \Delta \left[\bigtriangleup_{x=yz} \left(\mathscr{f}_G(y) \cap \mathscr{h}_G(z)\right)\right] \\
&= (\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G)(x) \Delta (\mathscr{f}_G \otimes_{s/i} \mathscr{h}_G)(x) \\
&= \left((\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G) \tilde{\Delta} (\mathscr{f}_G \otimes_{s/i} \mathscr{h}_G)\right)(x)
\end{aligned}$$

Thus, $\mathscr{f}_G \otimes_{s/i} (\mathscr{g}_G \tilde{\Delta} \mathscr{h}_G) = (\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G) \tilde{\Delta} (\mathscr{f}_G \otimes_{s/i} \mathscr{h}_G)$.□

Example 4.28. Let $\mathscr{f}_G$, $\mathscr{g}_G$, and $\mathscr{h}_G$ be three SSs over $U = \{e, x, y, yx\}$ as follows:

$\mathscr{f}_G = \{(\varpi, \{e, x, yx\}), (\tau, \{x, yx\})\}$, $\mathscr{g}_G = \{(\varpi, \{e, y, yx\}), (\tau, \{e, y\})\}$, and $\mathscr{h}_G = \{(\varpi, \{e, y\}), (\tau, \{x, y\})\}$

Since $\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G = \{\varpi, \{e, yx\}), (\tau, \{e, yx\})\}$ and $\mathscr{f}_G \otimes_{s/i} \mathscr{h}_G = \{(\varpi, \{e, x\}), (\tau, \{x\})\}$, then

$$(\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G) \tilde{\Delta} (\mathscr{f}_G \otimes_{s/i} \mathscr{h}_G) = \{(\varpi, \{x, yx\}), (\tau, \{e, x, yx\})\}$$

Moreover, since $\mathscr{g}_G \tilde{\Delta} \mathscr{h}_G = \{(\varpi, \{yx\}), (\tau, \{e, x\})\}$,

$$\mathscr{f}_G \otimes_{s/i} (\mathscr{g}_G \tilde{\Delta} \mathscr{h}_G) = \{(\varpi, \{x, yx\}), (\tau, \{e, x, yx\})\}$$

Thus, $\mathscr{f}_G \otimes_{s/i} (\mathscr{g}_G \tilde{\Delta} \mathscr{h}_G) = (\mathscr{f}_G \otimes_{s/i} \mathscr{g}_G) \tilde{\Delta} (\mathscr{f}_G \otimes_{s/i} \mathscr{h}_G)$.

Remark 4.29. The soft symmetric difference-intersection product distributes over the soft symmetric difference from both sides.

Theorem 4.30. $\left(S_G(U), \tilde{\Delta}, \otimes_{s/i}\right)$ is a noncommutative ring without identity. However, if G is an abelian group, then $\left(S_G(U), \tilde{\Delta}, \otimes_{s/i}\right)$ is a commutative ring without identity.

PROOF. By Theorem 3.6, $\left(S_G(U), \tilde{\Delta}\right)$ is an abelian group. Moreover, by Theorem 4.14, $\left(S_G(U), \otimes_{s/i}\right)$ is a semigroup. Furthermore, by Remark 4.29, the soft symmetric difference-intersection product distributes over the soft symmetric difference from both sides. Hence, $\left(S_G(U), \tilde{\Delta}, \otimes_{s/i}\right)$ is a ring. Since the soft symmetric difference-intersection product is not commutative and does not possess an identity element in $S_G(U)$, $\left(S_G(U), \tilde{\Delta}, \otimes_{s/i}\right)$ is a noncommutative ring without identity. However, if G is abelian, then $\left(S_G(U), \tilde{\Delta}, \otimes_{s/i}\right)$ is a commutative ring without identity.

Theorem 4.31. $\left(S_G(U), \tilde{\Delta}, \otimes_{s/i}\right)$ is a noncommutative hemiring without identity. However, if G is an abelian group, then $\left(S_G(U), \tilde{\Delta}, \otimes_{s/i}\right)$ is a commutative hemiring without identity.

PROOF. Let $\not{f}_G \in S_G(U)$. By Theorem 4.30, $\left(S_G(U), \tilde{\Delta}, \otimes_{s/i}\right)$ is a ring, hence a semiring. Further, by Proposition 3.4, $\not{f}_G \tilde{\Delta} \emptyset_G = \emptyset_G \tilde{\Delta} \not{f}_G = \not{f}_G$, and by Proposition 4.11, $\not{f}_G \otimes_{s/i} \emptyset_G = \emptyset_G \otimes_{s/i} \not{f}_G = \emptyset_G$. That is to say, $\emptyset_G$ is the zero element of $\left(S_G(U), \tilde{\Delta}, \otimes_{s/i}\right)$. Moreover, since the symmetric difference of SSs is commutative in $S_G(U)$, $\left(S_G(U), \tilde{\Delta}, \otimes_{s/i}\right)$ is a hemiring. Since the soft symmetric difference-intersection product is not commutative and does not possess an identity element in $S_G(U)$, $\left(S_G(U), \tilde{\Delta}, \otimes_{s/i}\right)$ is a noncommutative hemiring without identity. However, if the group G is abelian, then $\left(S_G(U), \tilde{\Delta}, \otimes_{s/i}\right)$ is a commutative hemiring without identity.

Theorem 4.32. $\left(S_G{}^*(U), \tilde{\Delta}, \otimes_{s/i}\right)$ is a Boolean ring, where $|G| = r$ and r is a positive odd integer.

PROOF. Let $\not{f}_G \in S_G{}^*(U)$ and $|G| = r$, where r is a positive odd integer. By Theorem 4.30, $\left(S_G{}^*(U), \tilde{\Delta}, \otimes_{s/i}\right)$ is a ring and by Theorem 4.14, $\left(S_G{}^*(U), \otimes_{s/i}\right)$ is an idempotent monoid with the identity element U_G and the zero element $\emptyset_G$, where r is a positive odd integer That is, $(\not{f}_G)^2 = \not{f}_G \otimes_{s/i} \not{f}_G = \not{f}_G$, $U_G \otimes_{s/i} \not{f}_G = \not{f}_G \otimes_{s/i} U_G = \not{f}_G$, $\not{f}_G \tilde{\Delta} \emptyset_G = \emptyset_G \tilde{\Delta} \not{f}_G = \not{f}_G$, and $\not{f}_G \otimes_{s/i} \emptyset_G = \emptyset_G \otimes_{s/i} \not{f}_G = \emptyset_G$. Hence, $\left(S_G{}^*(U), \tilde{\Delta}, \otimes_{s/i}\right)$ is a Boolean ring, where $|G| = r$ and r is a positive odd integer. The fact that $\not{f}_G \tilde{\Delta} \not{f}_G = \emptyset_G$, that is, the characteristic of $\left(S_G{}^*(U), \tilde{\Delta}, \otimes_{s/i}\right)$ is 2, where $|G| = r$ and r is a positive odd integer, is a natural consequence of the algebraic structure $\left(S_G{}^*(U), \tilde{\Delta}, \otimes_{s/i}\right)$ being a Boolean ring.

5. CONCLUSION

This study commences with an in-depth investigation of the basic algebraic properties of the symmetric difference operation of soft sets. It is rigorously demonstrated that the set of all soft sets defined with a fixed parameter set, under this operation, forms an abelian group. Building upon this structural foundation, we introduce a novel soft product, referred to as the soft symmetric difference–intersection product, specifically formulated for soft sets whose parameter set is a group. A thorough algebraic investigation of this product is carried out, with a particular focus on its interaction with diverse classes of soft subsets and soft equality relations. Moreover, it is established that the collection of soft sets, endowed with the symmetric difference operation of soft sets and the soft symmetric difference–intersection product, satisfies the axiomatic requirements of both a ring and a hemiring. The introduction and systematic study of such operations and products within a well-defined universe of discourse is of fundamental significance in algebra. In particular, identifying whether a given binary operation satisfies closure, associativity, commutativity, identity, and inverse properties over a set enables the classification of that structure within the hierarchy of algebraic systems. Furthermore, investigating how multiple operations distribute over one another provides essential insights into the internal coherence and expressive capacity of the underlying algebraic framework. Such characterizations not only reveal intrinsic structural features but also determine the extent to which these constructions can be employed to generalize existing theories or resolve open problems in the field. In this context, the theoretical framework proposed herein not only addresses existing gaps but also offers a potential pathway for the development of a new branch of soft group theory grounded in the proposed product. Potential avenues for future inquiry include the design of novel soft algebraic operations and a deeper investigation of soft equality frameworks, both of which are expected to substantially advance the theoretical underpinnings and expand the scope of applications within soft set theory and its algebraic generalizations.

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7. CONFLICT OF INTEREST

There is no conflict of interest.

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