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Operator Algebras Associated with the Multiplicative Semigroup of Natural Numbers and Integer Groups ℤ

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ABSTRACT

This paper investigates operator algebras arising from the left regular representations of the multiplicative semigroup of natural numbers and the integer group. It is shown that the Banach algebra generated by it contains neither non-trivial projections nor compact operators. Additionally, it is proven that the -*algebra*, generated by a family of bounded operators in, does not contain any non-zero compact operators. The study further examines the von Neumann algebras generated by in. Moreover, it is shown that the, where is the C∗-algebra generated by in and represents the space of continuous functions on the unit circle. The results connect naturally with number theory through arithmetic semigroup actions and with quantum mechanics as models for noncommutative observables and symmetries. These links suggest that the structural rigidity established in this work may have implications for both arithmetic and quantum spectral analysis.

Keywords: Banach algebra, C∗-algebra, Projection, Compact operator, Group algebra.

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1. INTRODUCTION

Motivated by some problems in the study of quantum mechanics, von Neumann (1929) introduced operator algebras around 1930 (Liu, 2011). More specifically, he introduced algebraic structures consisting of a class of bounded operators on a Hilbert space, which he called ’rings of operators’, later renamed ’von Neumann algebras’ (Dixmier, 1957). Such algebras are self-adjoint, strong-operator closed, and contain the identity operator. Soon after, von Neumann and his collaborator Murray laid the foundation of operator algebras their notions and methods were successfully applied in many mathematical and physical problems.

In recent years, operator algebras have been further related to dynamical systems, number theory, and quantum information (Grigoletto et al., 2024; Bhattacharya (2025); Leutheusser and Liu, 2025). Most classical examples of operator algebras arise from group algebras and actions of groups on manifolds or measure spaces. In many applications, however, semigroups—and their associated algebras—serve as natural generalizations of groups and group algebras (Murray and von Neumann, 1937; 1939; Murphy, 1990; Choi et al., 2023). This

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motivates the study of operator algebras associated with the multiplicative semigroup of natural numbers $\mathbb{N}$ and its actions.

A central object in this setting is the multiplicative convolution operator on $\ell^2(\mathbb{N})$ induced by an arithmetic function f. It is defined by

$$(\mathcal{M}_f g)(n) = \sum_{d|n} f(d) g(\frac{n}{d}).$$

Such operators naturally connect operator theory with problems in number theory and noncommutative analysis, thereby building a bridge between these fields (Chib and Komal, 2014; Kadison and Ringrose, 1983, 1997).

The study of operator algebras related to number theory is currently an active area of research. Much of the existing work has focused on boundedness properties of these operators (Codecà and Nair, 2008; Dong et al., 2018; Hilberdink, 2013; Kibrom and LinZhe, 2019; Thorn, 2018). Codecà and Nair were among the first to address boundedness explicitly. They showed that for any completely multiplicative function $f \in \ell^1$, the operator $\mathcal{M}_f : \ell^P(\mathbb{N}) \to \ell^P(\mathbb{N})$, $p \geq 1$, is bounded with $\|\mathcal{M}_f\|_2 = \|f\|_1$, where the norm in this regard is independent of. Hilberdink (2013) later established the corresponding result for $p = 2$, proving that $\mathcal{M}_f : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is bounded for all nonnegative $f \in \ell^1$ with $\|\mathcal{M}_f\|_2 = \|f\|_1$ (Hilberdink, 2013).

More recently, Dong et al. (2018) initiated a systematic study of the Banach, C*, and von Neumann algebras generated by the left regular representation of the multiplicative semigroup of natural numbers. They showed that the maximal ideal space of the Banach algebra $\mathfrak{B}$ generated by $\Omega_{\mathbb{N}} \coloneqq \{\mathcal{M}_n : n \in \mathbb{N}\}$ is homeomorphic to $\mathbb{D}^{\mathrm{P}}$, where $\mathbb{D}$ is the closed unit disk in the complex plane. Despite these advances, the internal structural properties of the algebra $\mathfrak{B}$, the existence of non-trivial projections and compact operators, remain largely unexplored.

In this paper, we address this gap by investigating whether $\mathfrak{B}$ contains any non-trivial projections or non-zero compact operators, and by examining the weak operator topology closure $\overline{\mathfrak{B}}^{\mathrm{WOT}}$. We prove that $\mathfrak{B}$ admits neither non-trivial projections nor non-zero compact operators, while the characterization of $\overline{\mathfrak{B}}^{\mathrm{WOT}}$ remains open.

From a broader perspective, these questions are motivated by applications to arithmetic dynamics in number theory and to quantum mechanics, where operator algebras model algebras of observables and discrete symmetries. Understanding rigidity phenomena such as the absence of compact operators provides insight into the spectral and structural behavior of systems arising in both arithmetic and quantum settings.

2. RESULTS

The following is our first main result:

Theorem 1. The Banach algebra $\mathfrak{B}$ generated by $\Omega_{\mathbb{N}} \coloneqq \{\mathcal{M}_n : n \in \mathbb{N}\}$ in $\mathcal{B}(\ell^2(\mathbb{N}))$ contains no non-trivial projections.

Next, we consider the adjoint operators $\mathcal{M}_k$, where $k \in \mathbb{N}$. For all $f, g \in \ell^2(\mathbb{N})$,

$$\langle \mathcal{M}_k f, g\rangle = \sum_{n=1}^{\infty} (\mathcal{M}_k f)(n)\overline{g(n)} = \sum_{k|n} f(\tfrac{n}{k})\,\overline{g(n)} = \sum_{m=0}^{\infty} f(m)\overline{g(km)},$$

which shows that $(\mathcal{M}_k^* g)(n) = g(kn), n \in \mathbb{N}$. Moreover, $\|\mathcal{M}_k^*\| = 1$. Replacing g by δ_m, for any $, m \in \mathbb{N}$,

$$\mathcal{M}_k^* \delta_m = \begin{cases} \delta_{m/k}, & if\ k|m \\ 0, & otherwise \end{cases}$$

The C*-algebra $\mathfrak{A}$ generated by a family of bounded operators by $\Omega_{\mathbb{N}} \coloneqq \{\mathcal{M}_n : n \in \mathbb{N}\}$ in $\mathcal{B}(\ell^2(\mathbb{N}))$ contains no projection of finite rank other than 0, as shown by Dong et al. (2018). From the importance and success of the theory of compact operators, it is natural to extend and investigate the existence of the compact operators to C*-algebra $\mathfrak{A}$. So, the next natural question is: “Does the C*-algebra $\mathfrak{A}$ contain any non-zero compact operators?” We have the following theorem.

Theorem 2. The C$*$-algebra $\mathfrak{A}$ generated by Ω does not contain any non-zero compact operators.

Let $\mathcal{S}$ be a set of bounded linear operators on a Hilbert space $\mathcal{H}$, i.e., $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$.

The **commutant** of $\mathcal{S}$ is

$$\mathcal{S}' = \{T \in \mathcal{B}(\mathcal{H}) : TS = ST \ for\ all\ S \in \mathcal{S}\}.$$

The **double commutant** of $\mathcal{S}$ is the commutant of the commutant:

$$\mathcal{S}'' = (\mathcal{S}')' = \{T \in \mathcal{B}(\mathcal{H}) : TR = RT \ for\ all\ R \in \mathcal{S}'\}.$$

Theorem 3. (Double Commutant Theorem): If $\mathcal{R}$ is a *-subalgebra of $\mathcal{B}(\mathcal{H})$ that contains the identity operator, then

$$\mathcal{R}'' \coloneqq (\mathcal{R}')' = \bar{\mathcal{R}}^{\mathrm{WOT}} = \bar{\mathcal{R}}^{\mathrm{SOT}}.$$

That is, the double commutant equals the weak (or strong) operator topology closure of $\mathcal{R}$. Hence, $\mathcal{R}''$is the **von Neumann algebra generated by** $\mathcal{R}$. That is, a von Neumann algebra is a subalgebra $\mathcal{R}$ of $\mathcal{B}(\mathcal{H})$ that contains the identity operator, is self-adjoint, and is closed under the weak operator topology(WOT).

The double commutant of $\mathcal{R}$ is $\mathcal{R}'' := (\mathcal{R}')'$. It is a trivial consequence that $\mathcal{R} \subseteq \mathcal{R}''$ for all $\emptyset \neq \mathcal{R} \subseteq \mathcal{B}(\mathcal{H})$. Remarkably, this relates a topological property (WOT-closure) to an algebraic property (double commutant). In particular, if $\mathcal{R}$ is a WOT-closed, unital C∗-subalgebra of $\mathcal{B}(\mathcal{H})$, then $\mathcal{R} = \mathcal{R}''$ (Kadison and Ringrose, 1983; 1997) The authors Kibrom and LinZhe (2019) applied the double commutant theorem and determined the von Neumann algebra generated by Ω on weighted Hilbert spaces $\ell^2(\mathbb{N}, 1/n^\beta)$, $\beta \geq 0$ (Kibrom and LinZhe, 2019; Konca et al., 2015). We next consider the integer group $\mathbb{Z}$ and determine the von Neumann algebra generated by $\{L_n : n \in \mathbb{N}\}$ in $\mathcal{B}(\ell^2(\mathbb{Z}))$. The following theorem is proved in section 3.

Theorem 4. $\{L_n : n \in \mathbb{N}\}$ in $\mathcal{B}(\ell^2(\mathbb{Z}))$ generates a von Neumann algebra

$$\mathcal{L}_{\mathbb{Z}} = \{L_\alpha \in \mathcal{B}(\ell^2(\mathbb{Z})) : \alpha \in \ell^2(\mathbb{Z})\}.$$

Let $\mathcal{S}^1$ denote the unit circle $\{ z \in \mathbb{C} : |z| = 1\}$ and P($\mathcal{S}^1$) the polynomials over the circle, i.e., finite sums of the form

$$\mathcal{A} = \sum_{n\in\mathbb{Z}} A_n z^n,\ A_n \in \mathbb{Z}.$$

We denote $L^2(\mathcal{S}^1)$ as the set of all square-integrable functions. Recall that ($\mathcal{S}^1$,×) is a multiplicative abelian group isomorphic to the additive group $(\mathbb{R}/\mathbb{Z}, +)$. Furthermore, $\mathbb{R}/\mathbb{Z}$ is in one-to-one correspondence with [0, 1). Thus, we have an isomorphism between [0, 1) and $\mathcal{S}^1$ via t $\mapsto e^{2\pi i t}$. Topologically, $\mathbb{R}/\mathbb{Z}$, [0, 1), and $\mathcal{S}^1$ are homeomorphic (Katznelson, 2004). We will show that the Fourier transform is an algebra isomorphism $\mathrm{C}\mathbb{Z} \cong \mathrm{P}(\mathcal{S}^1)$, extending to an isometric isomorphism of C*-algebras: $\mathrm{C}^*\mathbb{Z} \cong \mathrm{C}(\mathcal{S}^1)$, where $\mathrm{C}^*\mathbb{Z}$ is the $\mathrm{C}^* -$ algebra generated by $\mathrm{C}\mathbb{Z}$ in $\mathcal{B}(\ell^2(\mathbb{Z}))$, and $\mathrm{C}(\mathcal{S}^1)$, is the space of continuous functions on the unit circle $\mathcal{S}^1$.

Theorem 5. $\mathrm{C}^*\mathbb{Z} \cong \mathrm{C}(\mathcal{S}^1)$.

3. DISCUSSION AND PROOFS OF THE RESULTS

In the study of commutative Banach algebras, the Gelfand representation plays a crucial role. Let A denote a commutative Banach algebra. The spectrum of A, denoted $\sigma(\mathrm{A})$, is the set of all non-zero homomorphisms from A to C. The space $\mathrm{C}(\sigma(\mathrm{A}))$ consists of continuous complex-valued functions on $\sigma(\mathrm{A})$. For $x \in \mathrm{A}$, define $\hat{x} : \sigma(\mathrm{A}) \to \mathrm{C}$ by $\hat{x}(\varphi) = \varphi(x)$. Then $\hat{x}$ is a continuous function, called the Gelfand transform of x. The mapping

$$\Gamma_{\mathrm{A}}: A \to \mathrm{C}(\sigma(\mathrm{A})), \qquad \mathrm{x} \longmapsto \hat{x}$$

is a homomorphism, called the Gelfand homomorphism. Note that

$$\sigma(\mathrm{x}) = \{ \lambda \in \mathrm{C} : x - \lambda I \notin \mathrm{A}^\times\}$$

$$= \{ \lambda \in \mathbb{C}: \hat{x} - \lambda \text{ vanishes at least at one point } \notin \mathrm{A}\}$$
$$= \hat{x}\big(\sigma(\mathrm{A})\big),$$

where $\mathrm{A}^{\times}$ denotes the group of invertible elements of A.

Example 1. Let $\mathrm{A} = \mathrm{C}([0,1])$, the Banach algebra of continuous functions on [0,1] with sup norm. For each $x \in [0,1]$, define $\phi_x(f) = f(x)$. The Gelfand transform: for $f \in \mathrm{A}$ is given as follows:

$$\hat{f}(\phi_x) = \phi_x(f) = f(x), x \in [0,1].$$

3.1. Projections in Banach Spaces

The study of projections, or idempotents, in Banach spaces is a central topic in functional analysis. A projection in a Banach space is a bounded linear operator P satisfying $\mathrm{P}^2 = \mathrm{P}$ (Randrianantoanina, 2001).

The following lemma is useful for our main results.

Lemma 1. If A is in the Banach algebra $\mathfrak{B}$ and $\sigma(\mathrm{A})$ is not connected, then $\mathfrak{B}$ contains an idempotent different from 0 and I.

We now prove Theorem **1.**

Proof. From Theorem 1.1 of Dong et al. (2018), we have $\sigma(\mathfrak{B}) \cong \Pi_{\mathcal{P}\in \mathrm{P}}\mathbb{D}^{\mathcal{P}}$, where $\mathbb{D}$ is the closed unit disk in the complex plane. By the Gelfand transform, there is a homomorphism $\pi: \mathfrak{B} \longrightarrow \mathrm{C}\big(\sigma(\mathfrak{B})\big) = \mathrm{C}(\Pi_{\mathcal{P}\in \mathrm{P}}\mathbb{D}^{\mathcal{P}})$.

Consider an idempotent P in $\mathfrak{B}$. There is a corresponding idempotent $\widehat{\mathrm{P}}$ in $\mathrm{C}\big(\sigma(\mathfrak{B})\big)$, i.e.,

$$\mathrm{P} \longmapsto \widehat{\mathrm{P}}.$$

Since $\Pi_{\mathcal{P}\in \mathrm{P}}\mathbb{D}^{\mathcal{P}}$ is connected, the only idempotents in $\mathrm{C}\big(\sigma(\mathfrak{B})\big)$ are 0 and 1; i.e.,

$$\mathrm{P} \longmapsto \widehat{\mathrm{P}} \in \{0,1\}.$$

Since the Gelfand transform maps $\mathrm{I} \longmapsto 1$, let $\widehat{\mathrm{P}} = 0$. Then $\sigma(\mathrm{P}) = 0$, which implies $\mathrm{r}(\mathrm{P}) = 0$, where r denotes the spectral radius. Thus,

$$\lim_{n\to\infty} \|\mathrm{P}^n\|^{1/n} = 0,$$

so $\mathrm{P} = 0$. Hence, $\mathfrak{B}$ contains no non-trivial projections.

Definition 1. Let $\mathcal{A}$ be a commutative Banach algebra over C. The radical of $\mathcal{A}$, $\mathrm{rad}(\mathcal{A})$, is defined by $\mathrm{rad}(\mathcal{A}) = \{\mathrm{T} \in \mathcal{A}: \mathrm{r}(\mathrm{T}) = 0\}$. A is semi-simple if $\mathrm{rad}(\mathcal{A}) = \{0\}$, i.e.,

$\mathrm{r}(\mathrm{T}) = \lim_{n\to\infty} \|\mathrm{T}^n\|^{1/n} = 0$, which implies $\mathrm{T} = 0$.

A character on $\mathcal{A}$ is a non-zero homomorphism $\tau: \mathcal{A} \to \mathbb{C}$ satisfying $\tau(ab) = \tau(a)\tau(b)$ for $a, b \in \mathcal{A}$. We denote $\Omega(\mathcal{A})$ as the set of characters on $\mathcal{A}$.

Example 2. Let $\mathcal{A} = \mathrm{C}(\mathrm{X})$, where X is a compact topological space. For each $x \in \mathrm{X}$, the map $\tau: \mathcal{A} \to \mathbb{C}$, $f \longmapsto f(x)$, is a character on $\mathcal{A}$.

Lemma 2. The Banach algebra $\mathfrak{B}$ generated by Ω is semi-simple, i.e., if $T \in \mathfrak{B}$ and $\sigma(T) = 0$, then $T = 0$.

Proof. Let $T \in \mathfrak{B}$, so $T = M_f \in \mathcal{B}(\ell^2(\mathbb{N}))$, where $f \in \ell^2(\mathbb{N})$. We show semi-simplicity by proving that $\lim_{n\to\infty} \|T^n\|^{1/n} = 0$ implies $T = 0$. Assume $T \neq 0$, so $f \neq 0$. Take a minimal $n \in \mathbb{N}$ such that $f(n) \neq 0$.

$$\begin{aligned}\|T^m\| \geq \|\langle T^m\delta_e, \delta_{n^m}\rangle\| &= \left\| \langle M_{\underbrace{f*\cdots*f}_{m-times}}\delta_e, \delta_{n^m}\rangle \right\| \\ &= \left\| \underbrace{(f*\cdots*f)}_{m-times}(n^m) \right\| \\ &= |f(n)|^m,\end{aligned}$$

So, $\|T^m\|^{1/m} \geq |f(n)| > 0$. Thus,

$\mathrm{r}(T) = \lim_{m\to\infty} \|T^m\|^{1/m} \geq |f(n)| > 0$, contradicting $\mathrm{r}(T) = 0$. Therefore, $T = 0$.

Corollary 1. For $\mathfrak{B}$, a unital commutative Banach algebra,

$$\sigma(a) = \{\tau(a): \tau \in \Omega(\mathfrak{B})\}, \quad a \in \mathfrak{B}.$$

Proof. This follows from Theorem 1.3.4 of Dong et al. (2018).

3.2. Compact Operators

The theory of compact operators is a significant area in operator algebras. A bounded linear transformation $T: \mathcal{H} \to \mathcal{H}$ is compact if the closure of $T(\mathcal{S})$ is compact in $\mathcal{H}$, where $\mathcal{S}$ is the closed unit ball. The set of compact operators is the closure of the set of finite-rank operators in the operator norm.

Definition 2. A $*$-algebra is an algebra $\mathcal{A}$ with an involution $*: \mathcal{A} \to \mathcal{A}$ such that for all $a, b \in \mathcal{A}$, $\lambda \in \mathbb{C}$, $(a^*)^* = a$, $(\alpha a + b)^* = \bar{\alpha}a^* + b^*$, $(\lambda a)^* = \bar{\lambda}a^*$ and $(ab)^* = b^*a^*$. A normed algebra $\mathcal{A}$ satisfies $\| ab \| \leq \| a \| \| b \|$ for all $a, b \in \mathcal{A}$. A C$*$-algebra $\mathcal{A}$ is a Banach algebra with an involution $a \longmapsto a^*$ satisfying $\|a^* \mathrm{a}\| = \|a\|^2$ for all $a \in \mathcal{A}$. We now prove Theorem 2.

Proof. Suppose $T \in \mathcal{A}$ is a compact operator with $\|T\| = 1$. For any $\varepsilon > 0$, there exists $\sum_{finite} a_{mn}M_nM_m^* \in \mathfrak{A}$ such that $\left\|\sum_{finite} a_{mn}M_nM_m^* - T\right\| < \varepsilon$. Since T is compact, there exists a finite-rank operator $F = \sum_{i=1}^{N} \xi_i \otimes \eta_i \in \mathcal{B}(\ell^2(\mathbb{N}))$ such that

$$\|F - T\| < \in. \tag{2.1}$$

For $x \in \ell^2(\mathbb{N})$, $Fx = \sum_{i=1}^{N}\langle x, \eta_i\rangle\, \xi_i$. For a large prime p, $E_{\mathrm{P}} = M_{\mathrm{P}}M_{\mathrm{P}}^*$ is a projection onto the closed subspace generated by $\{\delta_{\mathrm{P}},\ \delta_{2\mathrm{P}}, \delta_{3\mathrm{P}}, \cdots\}$. Then,

$$F = \operatorname*{Sup}_{\|x\|=1} \|Fx\| \le \operatorname*{Sup}_{\|x\|=1} \sum_{i=1}^{N} |\langle x, \eta_i\rangle|\, \|\xi_i\|$$

$$\le \operatorname*{Sup}_{\|x\|=1} \sum_{i=1}^{N} \|x\|\|\eta_i\|\, \|\xi_i\| = \operatorname*{Sup}_{\|x\|=1} \sum_{i=1}^{N} \|\eta_i\|\, \|\xi_i\|$$

For all $\xi_i \in \ell^2(\mathbb{N})$, $\|E_{\mathrm{P}}\xi_i\| \to 0$ as $\mathrm{p} \to \infty$. Thus, $\|E_{\mathrm{P}}F\| \to 0$. From (2.1),

$$\|E_{\mathrm{P}}(F - T)\| \le \|E_{\mathrm{P}}\|\|F - T\| < \varepsilon.$$

For large p, $\|E_{\mathrm{P}}F\| < \varepsilon$, so

$$\|E_{\mathrm{P}}T\| < \|E_{\mathrm{P}}F\| + \varepsilon < 2\varepsilon \qquad (2.2)$$

Also,

$$\|\textstyle\sum_{n=1}^{N} a_{mn}M_nM_m^* - T\| < \varepsilon \qquad (2.3)$$

Thus,

$$\|E_{\mathrm{P}}(\textstyle\sum_{n=1}^{N} a_{mn}M_nM_m^*) - E_{\mathrm{P}}T\| < \varepsilon,$$

implying

$$\|E_{\mathrm{P}}(\textstyle\sum_{n=1}^{N} a_{mn}M_nM_m^*)\| < 3\varepsilon$$

For $\mathrm{p} > \max\{m, n\}$,

$$\|E_{\mathrm{P}}(\textstyle\sum_{n=1}^{N} a_{mn}M_nM_m^*)\| = \|\textstyle\sum_{n=1}^{N} a_{mn}M_nM_m^*\|.$$

Thus,

$$\|\textstyle\sum_{n=1}^{N} a_{mn}M_nM_m^*\| < 3\varepsilon.$$

From (2.3), $\|T\| < 4\varepsilon$, but $\|T\| = 1$, a contradiction. Hence, $\mathfrak{A}$ contains no non-zero compact operators.

It follows that the Banach algebra $\mathfrak{B}$ generated by Ω in $\mathcal{B}(\ell^2(\mathbb{N})$ contains no non-zero compact operators. We provide an alternative proof.

Lemma 3. The Banach algebra $\mathfrak{B}$ generated by Ω in $\mathcal{B}(\ell^2(\mathbb{N})$ contains no compact operator other than zero.

Proof. The Gelfand transform gives a homomorphism

$$\pi: \mathfrak{B} \longrightarrow \mathrm{C}\big(\sigma(\mathfrak{B})\big) = \mathrm{C}(\Pi_{\mathcal{P}\in\mathrm{P}}\mathbb{D}^{\mathcal{P}}).$$

If $T \in \mathfrak{B}$ is compact, there is a corresponding $\widehat{\mathrm{T}} \in \mathrm{C}\big(\sigma(\mathfrak{B})\big)$. Since $\sigma(\mathrm{T}) = \{0\} \cup \sigma_{pt}(\mathrm{T})$ is disconnected, if $\sigma(\mathrm{T}) = \{0\}$ or $\sigma_{pt}(\mathrm{T}) = \emptyset$, then by Lemma 2, $\mathrm{T} = 0$. Otherwise, by Lemma 1, $\mathfrak{B}$ contains a non-trivial projection, which contradicts Theorem 1. Thus, $\mathfrak{B}$ contains no non-zero compact operators.

In operator theory, compact operators often behave like finite-dimensional matrices—they can be "approximated" by simpler, finite-rank operators. Showing that certain operators do not have compact components indicates that their behavior is inherently infinite-dimensional. This matter because it means these operators cannot be simplified to finite approximations without losing essential structure, which is important for both theoretical understanding and practical computation in quantum information or dynamical systems.

3.3. Integer Group and Operator Algebras

Let G be a group (or semigroup). The complex group algebra $\mathbb{C}G$ is the set of finitely supported complex functions on G:

$$\mathbb{C}G = \left\{\sum_{g\in G} \lambda_g\, g \mid \lambda_g \in \mathbb{C}, g \in G\right\}.$$

Addition is pointwise:

$$\sum_{g\in G} \lambda_g\, g + \sum_{g\in G} \lambda_g{}^*\, g = \sum_{g\in G} (\lambda_g + \lambda_g{}^*)$$

and multiplication is defined by

$$\alpha\beta = \sum_{g,h\in G} \lambda_g \lambda_h\, gh = \sum_{g\in G} \left(\sum_{h\in G} \lambda_{gh^{-1}} \lambda_h\right) h,$$

for $\alpha = \sum_{g\in G} \lambda_g\, g$ and $\beta = \sum_{h\in G} \lambda_h\, g$. For $G = \mathbb{Z}$,

$$\mathbb{C}\mathbb{Z} = \left\{\alpha : \alpha = \sum_{i=1}^{k} \lambda_i\, g(n_i), \lambda_i \in \mathbb{C}, g(n_i) = n_i \in \mathbb{Z}\right\}.$$

Similarly, for $\mathbb{N}$,

$$\mathbb{C}\mathbb{N} = \left\{\alpha : \alpha = \sum_{i=1}^{k} \lambda_i\, g(n_i), \lambda_i \in \mathbb{C}, g(n_i) = n_i \in \mathbb{N}\right\}.$$

Proposition 1. Both $\mathbb{C}\mathbb{Z}$ and $\mathbb{C}\mathbb{N}$ are integral domains with units.

Proof. We prove for $\mathbb{C}\mathbb{Z}$ (since $\mathbb{C}\mathbb{N}$ is a subring). The unit is $\delta_0 = g(0)$. Take $\alpha = \sum_{i=1}^{k} \alpha_i g(n_i)$ and $\beta = \sum_{i=1}^{m} \beta_i g(t_i)$, with $\alpha_i, \beta_i \neq 0$ and $n_1 < n_2 < \cdots < n_k$, $t_1 < t_2 < \cdots < t_m$. Then, $(\alpha\beta)(t_1 + n_1) = \alpha_1\beta_1$. Since $g(n_1 + t_1)$ and other $g(n_i + t_j)$ are linearly independent, $\alpha\beta \neq 0$, $\mathbb{C}\mathbb{Z}$is an integral domain.

3.3.1. Regular Representation of $\mathbb{Z}$

Consider a discrete group G with unit e. Define

$$\ell^2(G) = \left\{\xi : G \longrightarrow \mathbb{C} \mid \sum_{g\in G} |\xi(g)|^2 < \infty\right\}.$$

Let δ_g be the function that is 1 at g and 0 elsewhere; $\{\delta_g\}$ is an orthonormal basis for $\ell^2(G)$. Let G act on $\ell^2(G)$ by left multiplication:

$$\mathcal{L}_g : \xi \longmapsto \xi(g^{-1}.).$$

$\mathcal{L}_g$ is isometric, called the left regular representation. The right regular representation is

$$\mathcal{R}_g : \xi \longmapsto \xi(.g).$$

Represent G on $\ell^2(G)$:

$$\pi: G \longrightarrow \mathcal{B}\big(\ell^2(G)\big), g \longmapsto \mathcal{L}_g.$$

$\mathcal{L}_g$is unitary with $\mathcal{L}_g{}^* = \mathcal{L}_{g^{-1}}$, and $\mathcal{L}_{gh} = \mathcal{L}_g\mathcal{L}_h$. Elements in $\ell^2(G)$ can be written as

$$\left\{\sum_{g\in G}\lambda_g\mathcal{L}_g : \sum_{g\in G}\left|\lambda_g\right|^2 < \infty\right\}.$$

For $\alpha \in \mathbb{C}G, \mathcal{L}_\alpha: \ell^2(G) \to \ell^2(G)$ is bounded, representing $\mathbb{C}G$ by bounded operators:

$$\mathcal{L}_g: \mathbb{C}G \ \longrightarrow \mathcal{B}\big(\ell^2(G)\big).$$

For $G = \mathbb{Z}$,

$$\ell^2(\mathbb{Z}) = \{\sum_{n\in\mathbb{Z}}\lambda_n\mathcal{L}_n : \sum_{n\in\mathbb{Z}}|\lambda_n|^2 < \infty\},$$

and

$$\mathcal{L}_m(\sum_{n\in\mathbb{Z}}\lambda_n\mathcal{L}_n) = \sum_{n\in\mathbb{Z}}\lambda_n\mathcal{L}_{m+n} = \sum_{n\in\mathbb{Z}}\lambda_{n-m}\mathcal{L}_n.$$

Lemma 4. Let $T \in \mathcal{B}\big(\ell^2(G)\big)$, $\xi \in \ell^2(\mathbb{Z})$. If $\langle T\delta_m, \delta_n\rangle = \langle \xi * \delta_m, \delta_n\rangle$ for all $m, n \in \mathbb{Z}$, then $T = \mathcal{L}_\xi$.

Proof. Since $\xi \in \ell^2(\mathbb{Z}), \mathcal{L}_\xi$ is defined. For any $y \in \ell^2(\mathbb{Z})$, prove $\xi * y = T(y)$. For fixed $n, \forall m$

$$(\xi * \delta_m)(n) = \langle \xi * \delta_m, \delta_n\rangle = \langle T\delta_m, \delta_n\rangle\,.$$

Since $y \longmapsto (\xi * y)(n)$ and $y \longmapsto T(y)(n)$ are continuous linear functionals agreeing on the basis $\{\delta_m\}, \xi * y = T(y)$. Thus, $\mathcal{L}_\xi y = T(y)$, so $T = \mathcal{L}_\xi$.

Proposition 2. If $\mathcal{L}_\alpha$ is bounded, then $\alpha \in \ell^2(\mathbb{Z})$. If $\mathcal{L}_\beta$ is bounded, then $\mathcal{L}_{\alpha*\beta} = \mathcal{L}_\alpha\mathcal{L}_\beta$.

Proof. Since $\mathcal{L}_\alpha\delta_n(m+n) = \alpha(m)$,

$$\sum_{|i|<n}|\alpha(i)|^2 \le \|\mathcal{L}_\alpha\delta_n\|_{\ell^2}^2 \le \|\mathcal{L}_\alpha\|^2.$$

Thus, $\alpha \in \ell^2(\mathbb{Z})$. For $\alpha * (\beta * \delta_m) = \mathcal{L}_\alpha\mathcal{L}_\beta\delta_m$, set $m = 0$ to get $\alpha * \beta \in \ell^2(\mathbb{Z})$. Then,

$$\langle \mathcal{L}_{\alpha*\beta}\delta_m, \delta_n\rangle = (\alpha * \beta)(n-m),$$

$$\langle \mathcal{L}_\alpha\mathcal{L}_\beta\delta_m, \delta_n\rangle = \big(\alpha * (\beta * \delta_m)\big)(n) = (\alpha * \beta)(n-m)\,.$$

By Lemma 2.4, $\mathcal{L}_{\alpha*\beta} = \mathcal{L}_\alpha\mathcal{L}_\beta$.

Proposition 3.3. C$\mathbb{Z}$ acting on $\ell^2(\mathbb{Z})$ is a unital $*$-subalgebra of $\mathcal{B}\big(\ell^2(\mathbb{Z})\big)$.

Proof. Define $\mathcal{L}_{\mathrm{C}\mathbb{Z}} \coloneqq \{\mathcal{L}_\alpha : \alpha \in \mathrm{C}\mathbb{Z}\}$. For $\alpha = \sum_{i=1}^k \lambda_i g(n_i)$,

$$\|\mathcal{L}_\alpha(\beta)\|_{\ell^2} = \|\alpha * \beta\|_{\ell^2} = (\sum_n|\sum_l \alpha(l)\beta(n-l)|^2)^{1/2}$$

$$\le \sum_l |\alpha(l)|\|\beta\|_{\ell^2} \qquad \text{(Hölder's inequality)}$$

Thus, $\|\mathcal{L}_\alpha\| \le \sum_l|\alpha(l)|$, so $\mathcal{L}_\alpha \in \mathcal{B}\big(\ell^2(\mathbb{Z})\big)$. For $\beta = \sum_{i=1}^k \widehat{\lambda_\iota} g(-n_i)$, $\mathcal{L}_\alpha{}^* = \mathcal{L}_\beta$. Also, $\mathcal{L}_{\alpha+\beta}$ $= \mathcal{L}_\alpha + \mathcal{L}_B$. Hence, $\mathcal{L}_{\mathbb{C}\mathbb{Z}}$ is a unital $*$-subalgebra.

Corollary 2. $\ell^1(\mathbb{Z})$ is a subalgebra of $\mathcal{B}\big(\ell^2(\mathbb{Z})\big)$.

Proof. For $\alpha, \beta \in \ell^1(\mathbb{Z})$,

$\sum_n |(\alpha * \beta)(n)| \leq \sum_n \sum_l |\alpha(l)\beta(n-l)| = \|\alpha\|_{\ell^1} \|\beta\|_{\ell^1}$. Thus, $\alpha * \beta \in \ell^1(\mathbb{Z})$.

3.4. Group von Neumann algebra

The weak operator topology on $\mathcal{B}(\mathcal{H})$ is induced by $\{\omega_{x,y}(\mathrm{A}) = \langle \mathrm{A}x, y\rangle : x, y \in \mathcal{H}\}$. The closure of a C^* algebra under this topology is a von Neumann algebra.

Lemma 5. $\{L_n : n \in \mathbb{Z}\}$ in $\mathcal{B}(\ell^2(\mathbb{Z}))$ generates a von Neumann algebra $\mathcal{L}_{\mathbb{Z}} = \{L_\alpha \in \mathcal{B}(\ell^2(\mathbb{Z})) : \alpha \in \ell^2(\mathbb{Z})\}$.

Proof. Let $\mathcal{L}$ be the commutant of $\{L_n : n \in \mathbb{Z}\}$. Since $\{L_n\}$ is self-adjoint, prove $\mathcal{L}' = \mathcal{L}_{\mathbb{Z}}$. Since $L_n L_m = L_m L_n$, $\{L_n\} \subseteq \mathcal{L}$, so $\mathcal{L}' = \{L_n\}'' \subseteq \mathcal{L}$. For $T \in \mathcal{L}$, let $\alpha = T\delta_0 \in \ell^2(\mathbb{Z})$. For any $m, n \in \mathbb{Z}$,

$$\langle T\delta_m, \delta_n\rangle = \langle T L_m \delta_0, \delta_n\rangle = \langle L_m T \delta_0, \delta_n\rangle = \langle L_m \alpha, \delta_n\rangle .$$

By Lemma 4, $T = L_\alpha$, so $\mathcal{L} \subseteq \mathcal{L}_{\mathbb{Z}}$. Since $\mathcal{L}_{\mathbb{Z}}$ is commutative, $\mathcal{L}_{\mathbb{Z}} \subseteq \mathcal{L}_{\mathbb{Z}}{}' \subseteq \mathcal{L}'$. Thus,

$$\mathcal{L} = \mathcal{L}_{\mathbb{Z}} = \mathcal{L}_{\mathbb{Z}}{}' = \mathcal{L}' .$$

Denote $\mathrm{P}(\mathcal{S}^1)$ as polynomials over the circle:

$$A = \sum_{n \in \mathbb{Z}} A_n z^n , \qquad A_n \in \mathbb{Z} .$$

$\mathrm{P}(\mathcal{S}^1)$ is a pre-Hilbert space with inner product

$$\langle A, B\rangle = \int_{\mathcal{S}^1} A(z)\overline{B(z)} dz,$$

extending to $\mathcal{L}^2(\mathcal{S}^1) = \{F : \mathcal{S}^1 \longrightarrow \mathbb{C} \mid \int_{\mathcal{S}^1} |F(z)|^2 dz < \infty\}$. The set $\{z^n\}_{n \in \mathbb{Z}}$ is an orthonormal basis for $\mathrm{P}(\mathcal{S}^1)$, since

$$\int_{\mathcal{S}^1} z^n dz = \begin{cases} 1, & if\ n = 0 \\ 0, & otherwise. \end{cases}$$

The Fourier transform is

$$\mathcal{F} : \mathbb{C}\mathbb{Z} \longrightarrow C(\mathcal{S}^1), \qquad f \longmapsto \mathcal{F}(f)(z) = \sum_{n \in \mathbb{Z}} f(n) z^n,$$

an algebra isomorphism since $\mathcal{F}(\delta_n * \delta_m) = z^{n+m} = \mathcal{F}(\delta_n)\mathcal{F}(\delta_m)$.

Lemma 6. $\mathbb{C}\mathbb{Z}$ and $\mathrm{P}(\mathcal{S}^1)$ are isomorphic.

We now prove Theorem 4.

Proof. Consider $\mathcal{S}^1$ as $\mathbb{R}/\mathbb{Z}$ via $t \longmapsto e^{2\pi i t}$. Define

$$F : \ell^2(\mathbb{Z}) \longrightarrow L^2([0,1), dx), F(\delta_n) = e^{2\pi i n t}.$$

Since $\|\delta_n\| = \|e^{2\pi i n t}\| = 1$, F is norm-preserving. It induces an isometric isomorphism from

$\mathcal{B}(\ell^2(\mathbb{Z}))$ to $\mathcal{B}(L^2([0,1),dx))$. For $f \in L^\infty([0,1),dx)$, the multiplication $\mathcal{M}_f: g \longmapsto fg$ satisfies $\|\mathcal{M}_f\| = \|f\|_{L^\infty}$. Since $F(\mathbb{C}\mathbb{Z})$ are polynomials on $\mathcal{S}^1$, the norm closure of

$\mathcal{M}_{F(\mathbb{C}\mathbb{Z})} = \{\mathcal{M}_f \,;\, f \in \mathbb{C}\mathbb{Z}\}$is $M_{\mathrm{C}(\mathcal{S}^1)} = \{\mathcal{M}_f : f \in \mathrm{C}(\mathcal{S}^1)\}$. Thus, $C^*\mathbb{Z} \cong \mathrm{C}(\mathcal{S}^1)$.

Corollary 3. $\sigma(L_1) = \mathcal{S}^1$.

Proof. Since $L_1^n = L_n$ and $L_{-1}^n = L_{-n}$, $\mathrm{C}^*\mathbb{Z}$ is generated by the unitary L_1, $\mathrm{C}^*\mathbb{Z} \cong \mathrm{C}(\sigma(L_1))$. By Theorem 4., $\sigma(L_1) = \mathcal{S}^1$.

Alternatively, since L_1 is unitary, $\sigma(L_1) \subseteq \mathcal{S}^1$. For $\lambda \in \mathcal{S}^1$, take $\alpha_n(i) = \frac{\lambda^{-i}}{\sqrt{n}}$, $i = -1, \cdots, -n$, zero elsewhere. Then $\|\alpha_n\| = 1$, and

$$\|L_1\alpha_n - \lambda\alpha_n\| = \left\|\frac{\lambda}{\sqrt{n}}\delta_0 - \frac{\lambda^{1+n}}{\sqrt{n}}\delta_n\right\| = \frac{\sqrt{2}}{\sqrt{n}} \longrightarrow 0.$$

Thus $\lambda \in \sigma(L_1)$.

4. CONCLUSION

Inspired by von Neumann's foundational work linking operator algebras to quantum mechanics, this study investigates Banach algebras generated by multiplicative convolution operators on $\ell^2(\mathbb{N})$, connecting number theory with operator theory. Building on previous results regarding boundedness and C*-algebraic structures, we examined structural properties of the algebra $\mathfrak{B}$ generated by $\Omega_{\mathbb{N}} \coloneqq \{M_n : n \in \mathbb{N}\}$. We showed that $\mathfrak{B}$ contains no non-trivial projections and no non-zero compact operators, highlighting its rigidity and revealing new insights into the interplay between arithmetic functions and operator algebra. These findings further the understanding of how multiplicative semigroups of natural numbers act in operator-theoretic frameworks, offering a foundation for future exploration of $\mathfrak{B}$ under the weak operator topology and potential applications in non-commutative analysis and quantum information.

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