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Univalent Harmonic Maps on the Unit Disk with Singular Inner and Infinite Blaschke Product Dilatations

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ABSTRACT

We show that a univalent harmonic mapping exhibits a jump when its dilatation is an infinite Blaschke product and it maps the open unit disk univalently into the interior of a convex curve. The results are obtained using boundary behavior analysis of harmonic mappings together with properties of inner functions and their dilatations**.** We identify conditions on the boundary values of the harmonic function under which the dilatation is an infinite Blaschke product and a singular inner function. These characterizations clarify the relationship between boundary behavior and the analytic structure of the dilatation in univalent harmonic mappings. Examples are provided and visualized using Mathematica, and the examples demonstrate that the converses of the main results do not hold in general.

Keywords: Harmonic function, Univalent harmonic function, *Dilatation*, Blaschke product, Jumps, Singular inner function.

1. INTRODUCTION

Throughout this article, we use the notations C, Ω, ∂Ω, Δ, T, ω, Φ and GRM to denote the complex plane, region in the complex plane, boundary of the region, the open unit disk, the unit circle, *dilatation*, boundary function and general Riemann mapping theorem respectively.

Let Φ: ∂Ω→ C be continuous. Then there is a continuous function$f$: $\underline{\Omega}$ → C such that Φ(z) = f(z) for all z ∈$\partial\Omega$ and f is harmonic in Ω (Conway, 2012). The Poisson integral

$$f(z) = \int_{-\pi}^{\pi} p(r, \varphi - \theta)\Phi(e^{i\varphi})d\varphi, z = re^{i\theta} \in \Delta$$

Where, $p(r, \theta)$ is the Poisson kernel of Δ, which is the solution for the Dirichlet problem. Furthermore, it is harmonic mapping of Δ (Bshouty and Lyzzaik, 2010).

The study of harmonic functions can be viewed as a natural extension of the theory of analytic functions. Harmonic functions arise in various areas of physics and engineering, particularly in contexts where Laplace's equation plays a central role. Consequently, research on harmonic mappings is motivated not only by their theoretical significance in mathematics but also by their practical relevance to physical applications (Laugesen, 1997).

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Harmonic univalent mappings also arise in the study of minimal surfaces, where suitable harmonic mappings can be lifted to minimal graphs (Boyd and Dorff, 2014). Univalent harmonic mappings began to attract significant attention from complex analysts following the foundational papers by Clunie and Sheil-Small (1984) and Sheil-Small (1989). In 1986, Hengartner and Schober (1986) proposed a harmonic analogue of the classical Riemann Mapping Theorem. Their approach was inspired by the theory of quasi conformal mappings. The contributions of these researchers, along with many others, have led to the development of a rich theory filled with challenging problems, deep conjectures, and numerous intriguing questions. While several of these problems and conjectures have been resolved, many remain open and continue to stimulate further research.

Complex-valued harmonic mappings whose dilatation are finite **Blaschke** products have been fruitful area of research. For instance, a theorem of Hengartner and Schober (1986) states that if f maps Δ univalently onto bounded, convex region Ω with $\Phi(e^{i\theta})\epsilon\partial\Omega$ a.e. and if ω is a finite **Blaschke** product, then Φ must be piecewise constant on T. Conversely, if Φ is piecewise constant and f maps Δ univalently and harmonically onto a convex (k+2) gon, then dilatation ω is a finite **Blaschke** product of degree k by results of Sheil-Small (Sheil-Small, 1989).

Furthermore, such results show that piecewise constant boundary behavior corresponds to polygonal images and finite Blaschke dilatations (Hengartner and Schober, 1986). This motivates the study of more general situations where the dilatation is not finite but belongs to the class of infinite **Blaschke** products or general inner functions, which exhibit richer boundary behavior.

As Duren (1992) noted, it would be valuable to extend the existing results to the case where the *dilatation* ω is an infinite **Blaschke** product, and to explore the scenario for a broader class of inner functions. In particular, If $\frac{\partial f}{\partial \theta} = 0$ a.e. on T, then *dilatation* ω is inner (Laugesen, 1997). The converse holds provided that f maps onto a convex region (Laugesen, 1997). If Φ is an orientation preserving *homomorphism* of the circle that has $\Phi'(\theta) = 0$ a.e., then the harmonic extension f of Φ maps the closed disk *homeomorphically* onto itself by the Rado-Kneser-Choquet theorem while *dilatation* ω is inner. If f is the Poisson extension of some function $\Phi \in L^1(0, 2\pi)$, then the non-tangential limit exists for almost every θ and in fact, $f(e^{i\theta}) = \Phi(\theta)$ a.e. As needed we may use θ and $e^{i\theta}$ interchangeably.

$$f(e^{i\theta}) = f(z)$$

Numerous problems and conjectures have been proposed by Bshouty and Lyzzaik (2010); and Laugesen (1997). Motivated by these works, we study the relationship between boundary behavior and dilatation structure in univalent harmonic mappings. Therefore, the aim of this paper is to address several of the problems posed by Bshouty and Lyzzaik (2010); and Laugesen (1997).

Problem 1: Find conditions on the boundary values of a univalent harmonic function f defined on Δ that ensures its *dilatatio*n ω is an infinite **Blaschke** product.

Problem 2: Determine conditions on the boundary values of univalent harmonic function f defined on Δ such that its *dilatation* ω is a singular inner function.

Problem 3: Must the boundary function Φ have jump discontinuities if the *dilatation* ω of f is an infinite **Blaschke** product, and f maps Δ univalently into the interior of a convex curve γ with $\Phi(\theta) \in \gamma\ a.e.$?

For Problem 1, Laugesen (1997) attempted to solve partially on his own that the accumulation points of the set of zeros of ω(z) should be T. He also provided an example of an inner function for which the problem holds (Bshouty and Lyzzaik, 2010). Specifically, if A is closed countable subset of the open unit disk Δ, and f jumps at each point of A and is constant on each component of Δ\A, then ω is a **Blaschke** product. Regarding problem 3, if the *dilatation* ω of f is a finite **Blaschke** product, then Φ must be a piece wise constant and therefore exhibits jumps (Bshouty and Lyzzaik, 2010).

Regarding the structure of this paper, we briefly outline it as follows. Section 2 presents methodology and preliminary results used in the proofs. Section 3 contains the main results, proofs and discussions. Moreover, examples are provided to illustrate that the converses of the main results do not hold in general.

2. PRELIMINARIES AND METHODOLOGY

In this section, we review some results in the literature that we will later use to prove the main results of our work.

Definition 2.1. (Geleta, 2026) Let $\Omega \subset C$ be a simply connected domain. A complex-valued function f is called harmonic in Ω if it admits a decomposition: $f(z) = h(z) + g(z)$, where, h(z) and g(z) are analytic in Ω.

Definition 2.2. (Weldegebrial and Geleta, 2026) A complex valued harmonic functions f$(z) = h(z) + \underline{g(z)}$ is said to be sense preserving at z_0 if $J_f(z_0)>0$ and is sense reversing at z_0 if $J_f(z_0)<0$, where $J_f(z_0)$ is a Jacobian of f which given by $J_f(z_0) = |f_z|^2| - \left|f_{\underline{z}}\right|^2 = |h'|^2 - |g'|^2$.

The dilatation ω of f is given by $\omega(z) = \frac{g'(z)}{h'(z)}$. The dilatation may be interpreted geometrically to represent the "stretching" or "distortion" of f. *Dilatation* ω measures how far f is from being conformal. The mapping f is conformal if and only if $= 0$.

Theorem 2.1. (Nguyen, 2020**)** Every function that is convex in one direction is univalent.

Definition 2.3. (Romney, 2013) A harmonic function: $\Delta \rightarrow \Delta$ is inner if $f\ (re^{i\theta})| = 1$ a.e. on T. An example of an inner function is **Blaschke** product. Singular inner function is an inner function having no zeros and can be written as

$$e^{-\int \frac{e^{i\theta}+z}{e^{i\theta}-z} d_\mu(e^{i\theta})},$$

Where, $\theta \in R$ and μ is is positive singular measure. For example: $(z) = e^{\frac{z+1}{z-1}}$.

Definition 2.4. (Ahlfors, 1979) A *Stolz* angle is an angular sector located in the upper half-plane, with its vertex on the real axis, and it intersects the real axis only at the vertex. For $e^{i\theta_0} \in T$,

$$S_\alpha\left(e^{i\theta_0}\right) = \left\{z \in \Delta: \left|z - e^{i\theta_0}\right| < \alpha(1 - |z|)\right\}, 0 < \alpha < \infty.$$

Definition 2.5 (Laugesen, 1997) A **Blaschke** product is an analytic function of the form

$$B(z) = e^{i\theta} \prod_{j=1}^{k} \frac{-\underline{z_j}}{|z_j|} \frac{z - z_j}{1 - z\underline{z_j}} \text{ for } |z|<1,$$

Where, $\theta \in R$, $1 \leq k \leq \infty$, and $|z_j|<1$ for all j, and $\sum_{j=1}^{k} 1 - \left|z_j\right| < \infty$ with convention that $\frac{-\underline{z_j}}{|z_j|}=1$, if $z_j=0$.

When k<∞, we call B(z) a finite **Blaschke** product of degree k, and when k = ∞, we call an infinite **Blaschke** product.

The following theorem shows the local connection between the boundary values of f and its dilatation.

Theorem 2.2. (Laugesen, 1997) Let f be an orientation-preserving harmonic function on the unit disk and the Poisson extension of a function $\Phi \in L^1(0, 2\pi)$. Suppose that Φ has bounded variation on a subinterval I of $(0, 2\pi)$. Then

(a) For almost every $\theta \in I$, we have $\Phi'(\theta)=0 \Longrightarrow|\omega(e^{i\theta})|=1$.

(b) Suppose also that for almost every $\theta \in I$, $\Phi'(\theta) \neq 0$ then $f(\Delta)$ lies strictly to the left of the directed tangent line $\Phi(\theta + t\Phi'(\theta))$, for t $\in \mathbb{R}$. Then for almost every $\theta \in I$, $\Phi'(\theta) = 0$ provided $|\omega(e^{i\theta})|=1$.

(c) part (a) holds under other assumptions also, namely the assumptions that f is an orientation-preserving harmonic function on the unit disk extending continuously to $\Delta \cup I$ for a subinterval I of $[0,2\pi]$ and that $\Phi(\theta) = f(e^{i\theta})$ has bounded variation on I.

The following corollary provides a characterization of the inner *dilatation*.

Colorally 2.1. (Laugesen, 1997**)** Assume that an orientation-preserving harmonic function f on the unit disk is the Poisson integral of a function Φ, and that almost every θ possesses a neighborhood on which Φ has bounded variation. If $\Phi'(\theta) = 0$ a.e., then *dilatation* ω is inner. If in addition to the above hypothesis f has convex image, then $\Phi'(\theta)=0$ provided *dilatation* ω is inner.

The corollary is an immediate consequence of the theorem. The corollary holds even if f is unbounded. It characterizes the inner *dilatation* in terms of the boundary values of the harmonic function (Laugesen, 1997).

Theorem 2.3. (Laugesen, 1997) Let A$\subset T$ be a closed and countable set. Suppose that a function Φ has bounded variation on the unit circle, is constant on every component of the open set $T \setminus$ A and has jump at each point of A. Assume also that the harmonic extension f of Φ is an orientation-preserving harmonic function. Thus, the dilatation ω equals a **Blaschke** product having the property that for each θ, the radial limit $\omega(re^{i\theta})$ exists and has modulus 1.

Furthermore, A is infinite, f is univalent and $f(\Delta)$ is convex. Thus, ω must be an infinite **Blaschke** product.

Theorem 2.4. (Hengartner and Schober, 1986) Suppose Ω be bounded connected open set having a boundary that is locally connected. Assume that $\omega(\Delta) \subset \Delta$ and ω_0 is an invariant point of Ω. Then, for $f_{\underline{z}} = \omega(z) f_z$ there exists a solution satisfying:

a) $f(0) = \omega_0$, $f_z(0)>0$, and $f(\Delta) \subset \Omega$

b) There exist a countable set A$\subset T$ for which the unrestricted limits $\Phi(e^{i\theta}) = f(z)$ exists on $T \setminus$ A and belongs to $\partial\Omega$.

c) $\Phi(e^{it^-}) = \operatorname{ess}\Phi(e^{is})$ and $\Phi(e^{is^+}) = \operatorname{ess}\Phi(e^{is})$ exist on T, is on the boundary of Ω and are the same on $T \setminus$ A

d) The accumulation points of f at $e^{it} \in$A is the line segment connecting $\Phi(e^{it^-})$ to $\Phi(e^{is^+})$.

The mapping f is termed as GRM from Δ onto Ω. It is clear that the boundary function Φ is continuous at each point in T \A and has a discontinuity at every point in A. The term jump characterizes the behavior of Φ at every point of A (Bshouty and Lyzzaik, 2010).

Theorem 2.5. (Lewy, 1936) A complex-valued harmonic function f is locally univalent and orientation- preserving if and only if $|\omega(z)|<1$ for all $z \in \Delta$.

Theorem 2.6. (Laugesen, 1997) Let Φ (θ) be the boundary function of a GRM that maps Δ onto a bounded convex set, and whose *dilatation* is ω . Then Φ' (θ) = 0 a.e. onto T if and only if the *dilatation* ω is an inner function.

2.7. Proof Strategy

The methodology used in this paper is based on analyzing the boundary behavior of the harmonic mapping f through its boundary function Φ. Properties such as bounded variation, constancy and jump discontinuities are examined and then related to the dilatation ω using established results on inner functions, finite and infinite **Blaschke** products and singular inner functions. Classical theorems due to Laugesen (1997); Hengartner and Schober (1986); and Bshouty and Lyzzaik (2010) are applied to classify the dilatation and study its connection with the geometric properties of the image domain. Convexity arguments, boundary limit techniques, and contradiction methods are then used to establish the main results concerning jump behavior and the characterization of the dilatation.

Computational visualizations and symbolic computations supporting the analysis were carried out using Wolfram Mathematica, developed by Wolfram Research (Wolfram Research, 2024; Champaign, IL, USA). These computational illustrations are included only to support and visualize the theoretical results and do not replace the formal mathematical proofs.

The results presented in Theorems 2.1–2.6 are standard results from the literature and are included to provide the theoretical foundation for the paper. The main original contributions of this work are contained in Theorems 3.1–3.3.

3. RESULTS AND DISCUSSION

We apply the following lemma which is stated and proved by Bshouty and Lyzzaik (2010), to demonstrate the next theorem.

Lemma 3.1. Let f be a GRM from Δ onto a bounded convex domain Ω whose dilatation admits an analytic extension across an open interval $J = \{e^{it}: \gamma < t < \delta\}, \gamma < \delta < \gamma + 2\pi,$

Such that $|\omega| \equiv 1$ on J, then the following holds.

a) Φ (θ) has a jump at $e^{i\theta} \in J$ if and only if $\arg\{\sqrt{\omega(e^{i\theta})}\} = 0 \bmod \pi$.

b) $\Delta_J \arg \sqrt{\omega(z)} = \infty$ if and only if $\Phi(\theta)$ has infinitely many jumps in J.

c) If Φ has no jumps in J, then Φ is identically constant on J.

Theorem 3.1. If f maps Δ univalently and harmonically into the interior of a convex curve γ with $\Phi(\theta) \in \gamma$ a.e., and if the *dilatation* of f is an infinite **Blaschke** product, then Φ has jump discontinuity on T.

Proof: It is recognized from Hengartner and Schober (1986) that f maps Δ univalently and harmonically into the interior of a convex curve γ with $\Phi(\theta) \in \gamma$ a.e., and if the *dilatation* ω of f is a finite **Blaschke** product, then the boundary function Φ be a piecewise constant and hence has jumps on T.

However, in this hypothesis, the dilatation ω is given as an infinite **Blaschke** product. Since every **Blaschke** product is an inner function, its dilatation ω is an inner function as well. Furthermore, because the image of f is convex and the dilatation ω of f is inner function , by the corollary 2.1 $\Phi'(\theta) = 0$ a.e. (there exist θ_0 such that $\Phi'(\theta_0) \neq 0$). Consequently, $\Phi(\theta)$ is constant. a.e., but not identically constant.

From Lemma 3.1(c) it is known that; if f has no jumps in J, then Φ is identically constant on J, where J is as in Lemma 3.1. The contrapositive of this lemma states that if Φ is not identically constant on J, then f exhibits jumps on J.

From the hypothesis f maps Δ univalently and harmonically into the interior of a convex curve γ with $\Phi(\theta) \in \gamma$ a.e., (not identically constant) then f must have jumps on T.

Example 1. If we take $f(z) = \frac{-1}{z-1} + cos(\frac{1}{z+1})$, then $g(z) = \frac{1}{(z+1)^2} \sin(\frac{1}{z+1})$ and $h'(z) = \frac{1}{(z+1)^2}$. The *dilatation* $\omega(z) = \frac{g'(z)}{h'(z)} = \sin(\frac{1}{z+1})$ is an infinite **Blaschke** product whose zeros are $z_n = \frac{1}{n\pi}$ -1, $n \in N$ which are infinitely many and lie in the open unit disk. The zero z_n are accumulating near the boundary z = -1. Moreover, $|\sin(\frac{1}{z+1})| < 1$ for $z \in \Delta$. Thus, by theorem 2.5 f is a complex valued harmonic univalent function.

Observe that the *dilatation* ω of f is an infinite **Blaschke** product and has discontinuity at the boundary.

This Theorem illustrates a fundamental connection between the dilatation of a harmonic mapping and the boundary behavior of its extension. Specifically, when the dilatation is an infinite **Blaschke** product, the boundary function Φ cannot be completely smooth; it must exhibit jump discontinuities. This reflects the inherent complexity of infinite **Blaschke** products. Figure 1 (A) and (B) illustrate the geometric behavior of $f(\Delta)$ and ω respectively.

The clustering of zeros near the boundary reflects the infinite **Blaschke** structure and supports the existence of jump discontinuities in Φ.

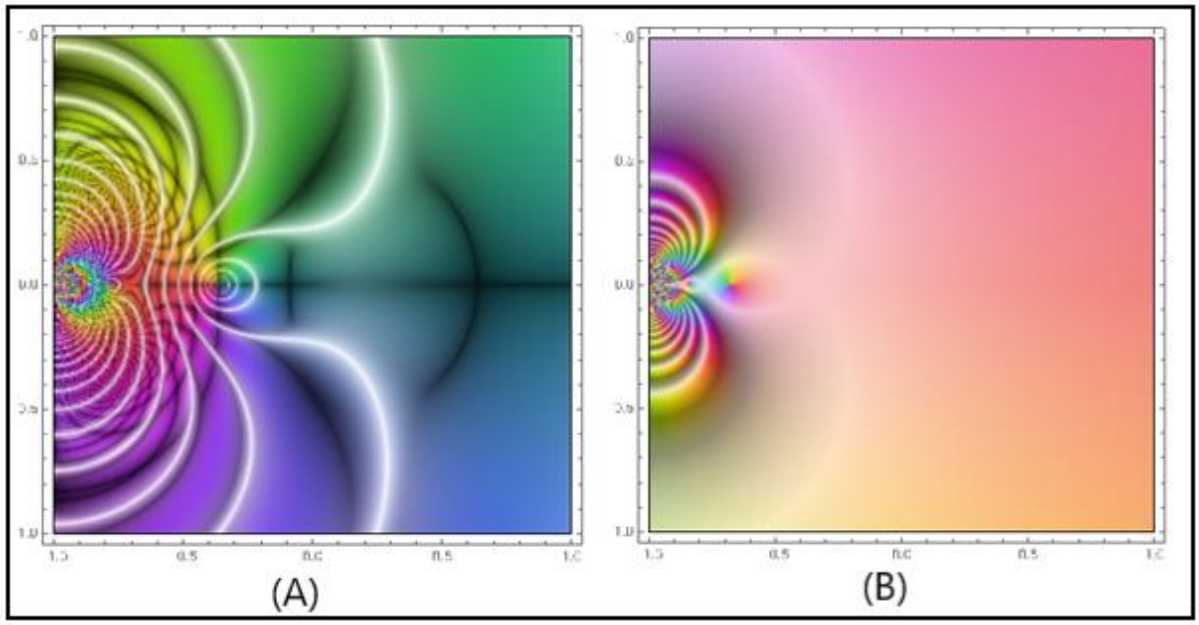


Figure 1. (A) Image of f(Δ); and (B) Image of ω(z) = sin(1/(z+1)).

Theorem 3.2. Suppose $f: \Delta \to \Delta$ is a univalent, sense preserving harmonic mapping whose complex dilatation is ω and assume that Φ is constant on T a.e. Then ω is singular inner function, but the converse is not true.

Proof: Since, Φ is constant on T a.e., we have Φ' $(\theta) = 0$ on T a.e. Furthermore, it is already proved by R. Laugsen (Laugesen, 1997) that $f_\theta(z) = \Phi'(z)$ for almost every $e^{i\theta} \in T$.

Since the analytic function $h(z)$ and $g(z)$ in Δ belongs to hardy space H^p for some p >0 (provided f is univalent). The non-tangential boundary limit h $(e^{i\theta})$, g $(e^{i\theta})$, f $(e^{i\theta})$ exists a.e. (Laugesen, 1997). Then the Poisson formula gives,

$$f(z) = h(z) + \underline{g(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\Phi(\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}}{e^{i\theta}-z}\Phi(\theta)d\theta + \frac{1}{2\pi}\underline{\int_0^{2\pi} \frac{z}{e^{i\theta}-z}\Phi(\theta)d\theta}$$

Differentiating with respect to θ under the integral sign (which is justifiable), we obtain $f_\theta(z) = h_\theta(z) + \underline{g_\theta}(z) = h'(z)\frac{dz}{d\theta} + \underline{g'(z)\frac{dz}{d\theta}} = izh'(z) + \underline{izg'(z)}$, where, $z = re^{i\theta}$. Since, Φ constant on T, we have $\Phi'(z) = f_\theta(z) = 0$. Thus, $|h'(e^{i\theta})| = |g'(e^{i\theta})|$ Implies $|\omega'(e^{i\theta})| = 1$.

This means the distortion (deformation) of f is extreme near the boundary, because the stretching factor ω approach 1. However, inside Δ, |g'(z)|<|h'(z)|. To this effect, $\omega = SB$, where, S singular inner function and B is **Blaschke** product, as ω is an inner function.

It is shown by Sheil-Small (1989) that if Φ is piecewise constant, with f mapping ω univalently and harmonically on to a convex (k+2)-gon, then ω is a finite **Blaschke** product of degree k. If **Ω** has infinitely piecewise constant boundary values (and maps onto a convex infinite-sided polygon) then ω is an infinite **Blaschke** product (Sheil-Smal, 1989).

Therefore, if Φ has neither a jump nor piecewise constant behavior on T then R. Laugsen (1997) showed that for every θ the radial limit $\omega(e^{i\theta})$ exists and has modulus 1. Then it follows that B has no radial limit equal to zero on T, and hence B≡ 1.

Theorem 3.2 strengthens this by showing that when Φ is constant almost everywhere, the dilatation is not only inner but must be a singular inner function, ruling out any **Blaschke** product component. This provides a sharper structural classification of harmonic mappings with boundary collapse

Example 2: Suppose $f(z) = \frac{z}{1-z} - \frac{e^{\frac{z+1}{z-1}}}{2}$. Then $\omega(z) = e^{\frac{z+1}{z-1}}$ which is singular inner function. The function $g'(z)$ has singularities where $h'(z)$ does not vanish but the singularity arises from the exponential terms involving the singular measure. Since the dilatation ω is singular inner function, it has no zeros $in\ \Delta$ meaning that $h'(z)$ and $g'(z)$ do not vanish simultaneously inside Δ. This suggests that f can have non- trivial boundary values. Although the dilatation ω of f is singular inner function, Φ is not constant on T $a.e.$. For instance, $\frac{-1+cos(1)}{2} + \frac{i(1+sin(1))}{2} = f(i) \neq f(-i) = \frac{-1+coscos(1)}{2} - \frac{i(1+sin(1))}{2}$.

Generally, while f may exist almost constant behavior at the boundary in some sense (e.g., being constant on a set of measure zero), it is not necessarily constant on all of T a.e.

Theorem 3.2 highlights the strong connection between the constancy of the boundary function Φ and the type of dilatation associated with a univalent harmonic mapping. However, the converse is not true: a singular inner function does not guarantee that Φ is constant almost everywhere, as shown in Example 2. This explains that singular inner dilatations can correspond to nontrivial or varying boundary behavior. Understanding this relationship helps characterize the boundary regularity and internal geometry of univalent harmonic mappings, especially in cases where the mapping exhibits extreme compression or collapse along the boundary.

Figure 2(A) illustrates the image of $f(\Delta)$, where the internal geometry reveals regions of extreme compression and collapse near the boundary. These distortions reflect the influence of the singular inner dilatation and its impact on the mapping's extension. Figure 2(B) displays the image of the dilatation which is a singular inner function. The figure highlights the non-

constant nature of Φ along the boundary, reinforcing that such dilatations do not ensure smooth or uniform boundary behavior.

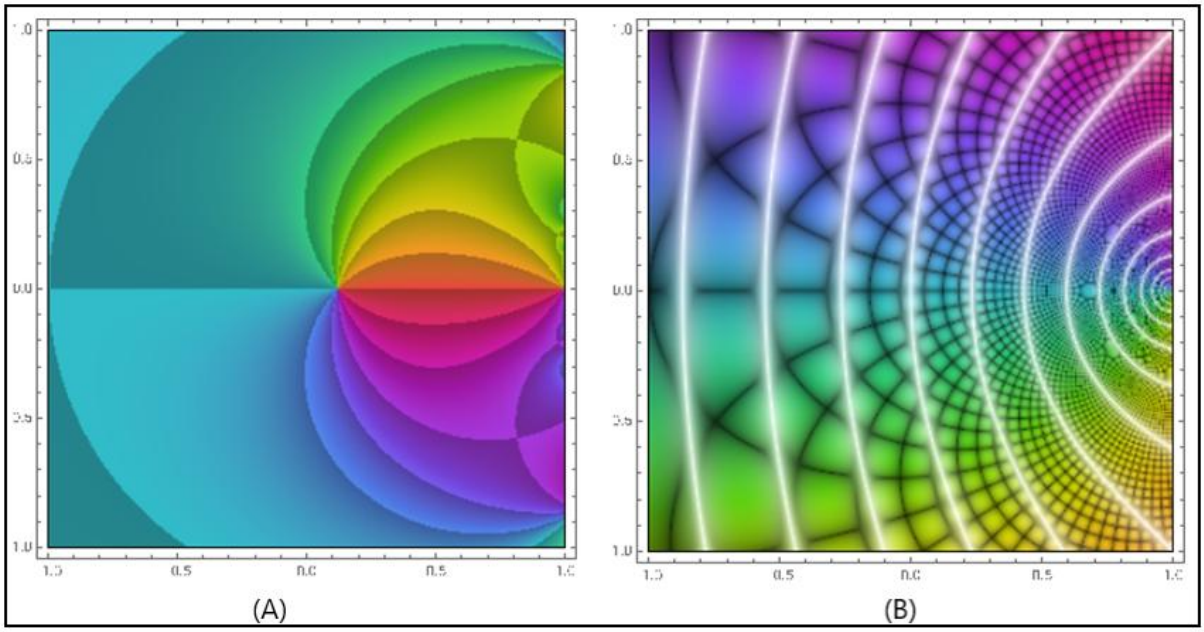


Figure 2. (A) Image of $f(\Delta)$; (B) Image of $\omega(z) = e^{\frac{z+1}{z-1}}$.

Understanding this relationship helps characterize both the boundary behavior and the internal geometry of univalent harmonic mappings, especially in cases where the mapping exhibits extreme boundary distortion. The next theorem demonstrates a necessary condition on Φ for ω to be an infinite **Blaschke** product.

Theorem 3.3. Suppose that the harmonic extension f of Φ is an orientation preserving harmonic mapping and $f(\Delta)$ is convex. Assume also that Φ has bounded variation; a jump on arc γ of length size c occurs on T and is constant on each component of the open set $T \setminus \gamma$. Then ω must be an infinite **Blaschke** product. However, the converse does not hold.

Proof: It is given that f is convex. It follows that f is injective (see theorem 2.1 above). Since Φ has bounded variation (finite angular variation) at the boundary, then the image of the boundary doesn't spiral or fold infinitely. The image of the boundary curve Φ (T) is rectifiable and it has finite length. If a jump of length size c > 0 occurs on T, then it follows that $(1-r)h'(re^{i\theta})\,(1-r)g'(re^{i\theta}) = c$ (Bshouty and Lyzzaik, 2010).

Since, f is an orientation preserving harmonic function, $\omega(e^{i\theta}) = \frac{g\prime(e^{i\theta})}{h\prime(e^{i\theta})}$ has at most removable singularities. On top of that, since $|\omega(z)| < 1$ in Δ and ω is bounded analytic function, It possesses a finite non-tangential limit $\omega(e^{i\theta}) = \omega(z)$ for almost every θ

(Laugesen, 1997). Moreover, if f is an orientation preserving harmonic function, then $h'(e^{i\theta})\neq 0$ on T a.e. (Laugesen, 1997). Thus, $1 = \frac{(1-r)g\prime(e^{i\theta})}{(1-r)h\prime(e^{i\theta})} = \frac{|g\prime(e^{i\theta})|}{|h\prime(e^{i\theta})|} = |\omega(e^{i\theta})|$.

To this effect, the boundary values of $h'(z)$ and $g'(z)$ must satisfy: $|g'(e^{i\theta})| = |h'(e^{i\theta})|$, for almost every $\theta \in [0, 2\pi]$. However, inside Δ, $|g'(z)| < |h'(z)|$ for all $z \in \Delta$. This implies that dilatation ω has a radial limit of modulus 1 at $e^{i\theta}$ For every $e^{i\theta} \in T$. Since $|\omega(e^{i\theta})| = 1$ on the boundary, the Jacobian of f: $J_f(z) = |h'(z)|^2 - |g'(z)|^2 = |h'(z)|^2(1-|\omega(z)|^2$, tends to zeros a.e. on the boundary. Hence, $\omega = SB$.

The contrapositive of Bshouty et al. (2012) confirms that, if it has a jump on T, then its dilatation ω is a **Blaschke** product. It follows that S = 1. But $if\ \omega$ were finite **Blaschke** product, then Φ would be finitely piecewise constant by Hengartner and Schober's result (Hengartner and Schober, 1986).

To simplify the argument, we apply another mechanism using the Identity Theorem for analytic functions (Ahlfors, 1979): if a non-zero analytic function in a domain Ω has boundary values equal to 0 on a set of positive measure on the boundary T, then the function must be identically zero or have infinitely many zeros accumulating at the boundary set. In our case, since f is a univalent harmonic mapping and is constant on $T\backslash\gamma$, it follows that $f'(z)$ has infinitely many zeros in Δ that accumulates towards $T\backslash\gamma$. At each zeros $z_n \in \Delta$, we have $f'(z_n) = h'(z_n) + \underline{g'(z_n)} = 0$. Which leads to $\omega(z_n) \in T$. So for each zeros of f', the modulus of ω is 1: $|\omega(z_n)| = 1$. But the dilatation ω is analytic and bounded by 1 in Δ: This contradicts the strict inequality $|\omega(z_n)| < 1$ unless ω approaches the boundary. But since infinitely many zeros of f'(z) exist in Δ, the modulus of $\omega(z)$ must approach one. This clearly indicates that the *dilatatio*n ω can't be finite **Blaschke** product. Hence, we infer that ω is an infinite **Blaschke** product.

Theorem 3.3 extends Laugesen's framework (1997) by linking boundary jump discontinuities directly to the infinite **Blaschke** structure of the dilatation. While Laugesen characterizes inner dilatation mainly via boundary derivative conditions, this result shows that even a single jump of finite size on an arc forces the dilatation to be an infinite **Blaschke** product, revealing a stronger boundary–zero distribution rigidity.

The following example demonstrates the converse is not true.

Example 3: let $f(z) = \frac{1}{z+1} + \underline{sin(\frac{1}{z+1})}$, where $g'(z) = \frac{-1}{(z+1)^2} cos\ cos\left(\frac{1}{z+1}\right)$, $h'(z) = \frac{-1}{(z+1)^2}$. $h'(z)$ and $g'(z)$ are analytic in the open unit disk. The *dilatation* $\omega(z) = \frac{g\prime(z)}{h\prime(z)} =$

$cos(\frac{1}{z+1})$ is an infinite **Blaschke** product whose zeros $z_n = \frac{1}{\pi(n+\frac{1}{2})} - 1,\ n \in N$ are infinite in the open unit disk. The zeros z_n are accumulating near the boundary z = -1. Moreover, $|\cos(\frac{1}{z+1})| < 1$ for z ∈ Δ. Thus, we infer by theorem 2.5 that f is a complex- valued harmonic univalent function.

The dilatation ω of f is an infinite **Blaschke** product, it doesn't exhibit a jump discontinuity on an arc γ of length c. Furthermore, it is not constant on every component of the open set $T \setminus \{-1\}$. Thus, this clearly demonstrates that the converse of the theorem doesn't hold.

Theorem 3.3 establishes a clear link between jump discontinuities of the boundary function Φ and the nature of the dilatation ω for harmonic univalent mappings into convex domains. Specifically, a jump on an arc γ of T forces ω to be an infinite **Blaschke** product, reflecting the necessity of infinitely many zeros in the derivative f to accommodate the boundary jump. The above example demonstrates that the converse does not hold: an infinite **Blaschke** product dilatation does not necessarily produce jump discontinuities in Φ. This highlights that while jumps in Φ imply a certain complexity in ω, the analytic structure of ω alone does not dictate the exact boundary behavior of f.

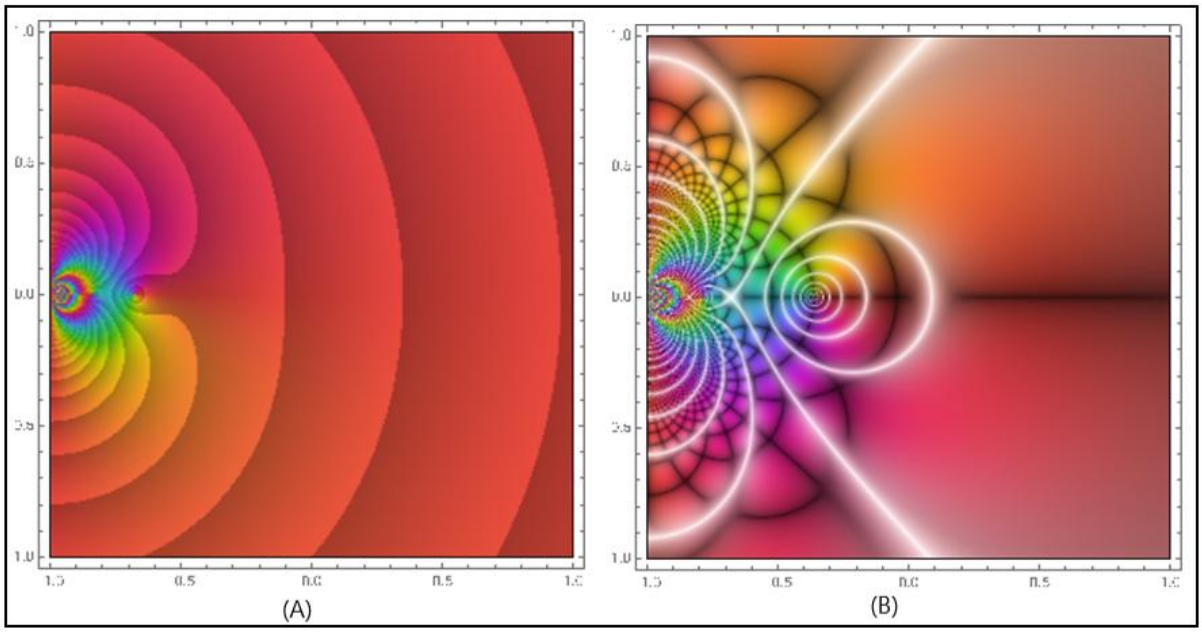


Figure 3. (A) Image of f(Δ); (B)Image of ω(z) = cos ($\frac{1}{z+1}$).

Figure 3(A) illustrates the image of $f(\Delta)$, where the geometry reveals localized distortions near arcs of discontinuity. These visual features correspond to the analytic complexity introduced by the infinite **Blaschke** product structure of the dilatation. Figure 3(B)

presents the image of the dilatation which serves as an example of an infinite **Blaschke** product. Despite its analytic richness, the figure demonstrates that such a dilatation does not necessarily produce jump discontinuities in Φ, reinforcing the asymmetry in the relationship.

The corollary that follows is a direct consequence of the theorem 3.3.

Corollary 3.1. Assume that the harmonic extension f of Φ is an orientation-preserving harmonic function and f (Δ) is convex. If Φ is jump discontinuous on a dense subset of T, then the dilatation ω of f is an infinite **Blaschke** product.

4. CONCLUSION

In this paper, we have proved that an univalent harmonic function f has a jump (collapse) if ω is an infinite **Blaschke** product and f maps Δ univalently into the interior of convex curve γ with $\Phi \in \gamma$ a.e. We have found conditions on the boundary values of f such that its *dilatation* ω is an infinite **Blaschke** product and singular inner function where the domain is an open unit disk. In addition, we have demonstrated that the converse is not true by displaying examples and creating visual representations using *mathematica* software.

For future research we have the following to be considered:

1) How does the boundary function influence the precise distribution of the dilatation zeros, and under what conditions do these zeros concentrate near specific boundary arcs or within angular regions?
2) How does the argument of the dilatation control the geometric features of the image domain, especially the asymptotic direction, curvature, and distortion of boundary curves?
3) Under what conditions does extreme boundary behavior lead to continuity of the harmonic extension versus the formation of infinitely many discontinuities?

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6. CONFLICT OF INTEREST

There is no conflict of interest.

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