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Full Length Research Article

A New Iterative Scheme for the Solution of Tenth Order Boundary Value Problems Using First-Kind Chebychev Polynomials

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ABSTRACT

A new iterative scheme for the solution of tenth order boundary value problems has been implemented using first-kind Chebychev polynomials as trial functions. The method involves transforming tenth order boundary value problems into a system of ordinary differential equations (ODEs). The trial solution is introduced into the ODEs, and is evaluated at the boundaries to obtain the approximate solution. The method avoids quasi-linearization, linearization, discretization or perturbation. Also, the method is computationally simple with round-off and truncation errors avoided. Numerical results obtained with the method show that the method is highly reliable and accurate in obtaining the approximate solution of tenth order boundary value problems as compared with the results generated from the Galerkin method available in literature. All computations were performed with maple 18 software.

Keywords: Boundary Value Problem, First-Kind Chebychev Polynomials, Trial solution, Approximate Solution

INTRODUCTION

Tenth order boundary value problems arise in the mathematical modelling of viscoelastic flows, hydro-magnetic and hydrodynamics stability and other areas of applied mathematics, sciences and engineering. When unstable ordinary convection supersedes stable convection, the eight or tenth boundary value problem is applied to model the instability (Viswanadham and Ballem, 2015). Analytical methods for these problems prove inadequate. Thus, several numerical schemes have been developed by many authors for the solution of boundary value problems due to their mathematical significance in diversified applications in science and engineering. For instance, Njoseh and Mamadu (2016), used the power series approximation method for the numerical solution of generalized nth order boundary value problems. Similarly, Mamadu and Njoseh (2016) applied the Tau-collocation approximation approach for solving first and second order ordinary differential equations. Olagunju and Joseph (2013) solved the boundary value problems in a collocation method using third-kind Chebychev method. Grover and Tomer (2011) applied the homotopy perturbation method for evaluating twelfth order boundary value problems. Islam *et al*. (2009) employed the differential transform method for a special twelfth order boundary value problem.

Furthermore, Viswanadham and Ballem (2015) applied the Galerkin method with septic B-spline for the numerical solution of tenth order boundary value problems. The weighted residual method was adopted by Oderinu (2014) on the numerical solution of tenth and twelfth order boundary value problems. Also, Ali *et al.* (2010) used the optimal homotopy asymptotic method for the solution of special twelfth order boundary value problems. Rashidinia *et al.* (2011) employed the non polynomial spline solutions for special linear tenth-order boundary value problems. The existence and uniqueness solution of the boundary value problems has been explored by Agarwa (1986).

This study develops a new method for solving the tenth order boundary value problems. The method adopts the first-kind Chebychev polynomials as trial functions. The method involves transforming the tenth order boundary value problems into a system of ordinary differential equations (ODEs). Thereafter, the trial solution is introduced into the ODEs, and is evaluated at the boundaries to obtain the approximate solution. The method avoids quasi-linearization, linearization, discretization or perturbation. Also, the method is computationally simple with round-off and truncation errors avoided. Numerical results obtained with the proposed method show that the method is highly reliable and accurate in obtaining the approximate solution of tenth order boundary value problems.

PROPOSED METHODOLOGY

In this section we consider the 10th order boundary value problem of the form (Njoseh and Mamadu, 2016; Viswanadham and Ballem, 2015)

$$f_0(x)u^{(10)}(x)+f_1(x)u^{(9)}(x)+f_2(x)u^{(8)}(x)+f_3(x)u^{(7)}(x)+f_4(x)u^{(6)}(x)+ f_5(x)u^{(5)}(x)+f_6(x)u^{(4)}(x)+f_7(x)u^{(3)}(x)+f_8(x)u^{(2)}(x)+f_9(x)u^{(1)}(x)+ f_{10}(x)u(x)=g(x), \quad x_0<x<x_1, \tag{1}$$

Subject to the boundary conditions

$$u^{(2k)}(x_0)=\alpha_{2k}, \quad k=0(1)(n-1), \tag{2}$$
$$u^{(2k)}(x_1)=\beta_{2k}, \ k=0(1)(n-1), \tag{3}$$

where $f_i(x),\ u^i(x),\ i=0(1)10,$ and $g(x)$ are assumed real and continuous on $x\in[x_0,x_1]$, $\alpha_i,\beta_i, i=0,1,2,3,\cdots,(n-1)$ are finite real constants.
Transforming (1), (2) and (3) to a system of ODEs, we have

$$\frac{du}{dx}=u_1,\ \frac{du_1}{dx}=u_2,\ \frac{du_2}{dx}=u_3,\ \frac{du_3}{dx}=u_4 \ldots \frac{du_n}{dx}=g(x)-\frac{1}{f_0(x)}\sum_{i=1}^{n} f_i(x)u^{(n)-1}\ ,\ n=10, \tag{4}$$

with the boundary conditions

$$u_1(x_0)=\alpha_0,\ u_2(x_0)=\alpha_1,\ u_3(x_0)=\alpha_2,\ldots,\ u_{2n}(x_0)=\alpha_{2n-1}, \tag{5}$$
$$u_1(x_1)=\beta_0,\ u_2(x_1)=\beta_1,\ u_3(x_1)=\beta_2,\ldots,\ u_{2n}(x_1)=\beta_{2n-1}. \tag{6}$$

Let the approximate solution of (1), (2) and (3) be given as

$$y(x)=\sum_{i=0}^{9} a_i T_i\ (x),\ x\in[0,1], \tag{7}$$

where $a_i,\ i=0,1,2,\cdots,(n-1)$ are unknown parameter to be determined, $T_i(x),\ i\geq 0,$ are the first-kind shifted Chebychev polynomials in the interval evaluated using:

$$T_{n+1}(x)-2xT_n(x)+T_{n-1}(x)=0, \tag{8}$$

with the initial conditions, $T_0(x)=1$ and $T_1(x)=\frac{2x-a-b}{b-a}$.

Using Equation (7) in (4), (5) and (6) in the interval $[0,1]$, we obtain the matrix equation

$$Ax=b \tag{9}$$

Where the coefficient matrix A is given as:

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 2 & -8 & 18 & -32 & 50 & -72 & 98 & -128 & 162 \\ 0 & 0 & 16 & -96 & 320 & -800 & 1680 & -3136 & 5376 & -8640 \\ 0 & 0 & 0 & 192 & -1536 & 6720 & -21504 & 56448 & -129024 & 266112 \\ 0 & 0 & 0 & 0 & 3072 & -30720 & 165888 & -645120 & 2027520 & -5474304 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 8 & 18 & 32 & 50 & 72 & 98 & 128 & 162 \\ 0 & 0 & 16 & 96 & 320 & 800 & 1680 & 3136 & 5376 & 8640 \\ 0 & 0 & 0 & 192 & 1536 & 6720 & 21504 & 56448 & 129024 & 266112 \\ 0 & 0 & 0 & 0 & 3072 & 30720 & 165888 & 645120 & 2027520 & 5474304 \end{bmatrix}$$

$$x = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \end{bmatrix}^T$$
$$b = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix}^T$$

The constants a_i, $i = 0,1,2,...,(n-1)$, are determined in (9) with a matrix solver, and substituting these parameters into (7) yields the approximate solution of (1) in the interval $[0,1]$. Similarly, using Equation (7) in (4), (5) and (6) in the interval $[-1,1]$, the coefficient matrix equation A is obtained using the standard first-kind shifted Chebychev polynomials $T_i(x)$, $i \geq 0$, which can be evaluated using the relation in equation (8) with the initial conditions, $T_0(x) = 1$ and $T_1(x) = x$ as:

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & -4 & 9 & -16 & 25 & -36 & 49 & -64 & 81 \\ 0 & 0 & 4 & -24 & 80 & -200 & 420 & -784 & 1344 & -2160 \\ 0 & 0 & 0 & 24 & -192 & 840 & -2688 & 7056 & -16128 & 33264 \\ 0 & 0 & 0 & 0 & 192 & -1920 & 10368 & -40320 & 126720 & -342144 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 \\ 0 & 0 & 4 & 24 & 80 & 200 & 420 & 784 & 1344 & 2160 \\ 0 & 0 & 0 & 24 & 192 & 840 & 2688 & 7056 & 16128 & 33264 \\ 0 & 0 & 0 & 0 & 192 & 1920 & 10368 & 40320 & 126720 & 342144 \end{bmatrix}$$

$$x = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \end{bmatrix}^T$$
$$b = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix}^T$$

The constants a_i, $i = 0,1,2,\cdots,(n-1)$, are determined in (9) with a matrix solver, and substituting these parameters into (7) yields the approximate solution of (1) in the interval $[-1,1]$

The error formulation for this problem is given as $|u(x) - u_n(x)|$, where $u(x)$ is the exact and $u_n(x)$ is the approximate solution.

Numerical Examples

Example 3.1 (Viswanadham and Ballem, 2015)**:**

Consider the given problem below

$$y^{(10)}(x) - y''(x) + xy = (-8 + x - x^2), 0 < x < 1 \tag{10}$$

Subject to the boundary conditions $y(0) = 1, y'(0) = 0, y''(0) = -1,\ y'''(0) = -2,$

$$y^{(iv)}(0) = -3, y(1) = 0, y'(1) = -e, y''(1) = -2e, y'''(1) = -3e, y^{(iv)}(1) = -4e.$$

The exact solution is

$$y(x) = (1 - x)\exp(x).$$

Using the proposed methodology, we have:

$$y(x) = 1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 - \frac{823}{5}x^5 + \frac{227}{6}ex^5 + \frac{4235}{12}x^6 - \frac{779}{6}ex^6 + \frac{339}{2}ex^7 - \frac{1843}{4}x^7 + \frac{2171}{8}x^8 - \frac{599}{6}ex^8 + \frac{67}{3}ex^9 - \frac{1457}{24}x^9$$

Numerical results obtained with the proposed method are given in **Table 1**.

Table 1: Shows the numerical results of the proposed method and error obtained for example 3.1 compared with that obtained from the Galerkin method with Septic B-spline in Viswanadham and Ballem (2015).

x	Exact Solution	Approximate Solution by proposed method	Proposed Method Error	Error in Galerkin method
0.1	0.9946538262	0.9946538263	1.0000E-10	1.537800E-05
0.2	0.9771222064	0.9771222060	4.0000E-10	4.452467E-05
0.3	0.9449011656	0.9449011635	2.1000E-09	3.331900E-05
0.4	0.8950948188	0.8950948151	3.7000E-09	3.552437E-06
0.5	0.8243606355	0.8243606311	4.4000E-09	9.477139E-06
0.6	0.7288475200	0.7288475166	3.4000E-09	2.586842E-05
0.7	0.6041258121	0.6041258103	1.8000E-09	3.975630E-05
0.8	0.4451081856	0.4451081852	4.0000E-10	3.531575E-05
0.9	0.2459603111	0.2459603111	0.0000E+00	2.214313E-05

Example 3.2 (Viswanadham and Ballem, 2015)

Given

$$y^{(10)}(x) - (x^2 - 2x)y = 10cosx - (x-1)^3 sinx, -1 < x < 1 \quad (11)$$

Subject to the boundary conditions

$$y(-1) = 2\sin(1), y'(-1) = -2\cos(1) - \sin(1),$$
$$y''(-1) = 2\cos(1) - \sin(1), y'''(-1) = 2\cos(1) + 3\sin(1),$$
$$y^{(iv)}(-1) = -4\cos(1) + 2\sin(1), y(1) = 0, y'(1) = \sin(1),$$
$$y''(1) = 2\text{cos}(1),\ y'''(1) = -3\text{sin}(1),\ y^{(iv)}(1) = -4\text{cos}(1).$$

The exact solution is

$$y(x) = (1 - x)\text{sin(x)}.$$

Using the proposed methodology, we have

$$y(x) = 0.0000026329 + 0.9999868117x^2 + 0.166666545x^3 - 0.1666402167x^4 - 0.008333090x^5 + 0.0083067614x^6 + 0.0001981685x^7 - 0.00018500425x^8 - 0.0000026329x^9.$$

Numerical results obtained with the proposed method are given in **Table 2**.

Table 2: Shows the numerical results of the proposed method and error obtained for example 3.2 compared with that obtained from the Galerkin method with septic B-spline in Viswanadham and Ballem (2015)

x	Exact Solution	Approximate Solution by proposed method	Proposed Method Error	Error in Galerkin method
-0.8	1.291240964000	1.291240979000	1.5000E-08	4.649162E-06
-0.6	0.903427957400	0.903428238400	2.8100E-07	1.329184E-05
-0.4	0.545185679200	0.545186774600	1.0954E-06	2.050400E-05
-0.2	0.238403197000	0.238405339000	2.1420E-06	9.477139E-06
0.0	0.000000000000	0.000002632900	2.6329E-06	2.731677E-06
0.2	-0.158935464600	-0.158933314700	2.1499E-06	1.458824E-05
0.4	-0.233651005400	-0.233649901800	1.1036E-06	2.110004E-05
0.6	-0.225856989400	-0.225856706000	2.8340E-07	1.908839E-05
0.8	-0.143471218200	-0.143471202200	1.6000E-08	1.342595E-05

Example 3.3 (Njoseh and Mamadu, 2016; Viswanadham and Ballem 2015)

$$y^{(10)}(x) + e^{-x}y^2(x) = e^{-x} + e^{-3x},\ 0 < x < 1 \qquad (13)$$

Subject to the boundary conditions $y(0) = 1, y'(0) = -1, y''(0) = 1,\ y'''(0) = -1,$

$$y^{(iv)}(0) = 1, y(1) = e^{-1}, y'(1) = -e^{-1}, y''(1) = e^{-1}, y'''(1) = -e^{-1}, y^{(iv)}(1) = e^{-1}.$$

The exact solution is

$$y(x) = e^{-x}.$$

Using the proposed methodology, we have

$$y(x) = 1 + \frac{1}{2}x^2 - 0.1666666667x^3 + 0.0416666667x^4 - 0.00833314x^5 + 0.0013880x^6 - 0.0001966x^7 + 0.0000228x^8 - 0.00000168x^9.$$

Numerical results obtained with the proposed method are given in **Table 3**.

Table 3: Shows the numerical results of the proposed method and error obtained for example 3.3 compared with Galerkin method with Septic B-spline in Viswanadham and Ballem (2015).

X	Exact Solution	Approximate Solution by proposed method	Proposed Method Error	Error in Galerkin method
0.1	0.9048374180	0.9048374181	1.0000E-10	6.735325E-06
0.2	0.8187307531	0.8187307531	0.0000E+00	4.410744E-06
0.3	0.7408182207	0.7408182207	0.0000E+00	3.629923E-05
0.4	0.6703200460	0.6703200462	2.0000E-10	4.839897E-05
0.5	0.6065306597	0.6065306599	2.0000E-10	4.929304E-05
0.6	0.5488116361	0.5488116363	2.0000E-10	3.945827E-05
0.7	0.4965853038	0.4965853040	2.0000E-10	9.834766E-06
0.8	0.4493289641	0.4493289643	2.0000E-10	1.996756E-06
0.9	0.4065696597	0.4065696597	0.0000E+00	5.066395E-06

DISCUSSION

In this study, the approximate solution of tenth order boundary value problems have been obtained using first-kind Chebychev polynomials as trial functions. The maximum errors obtained by the proposed method are 4.0000E-10, 1.6000E-08 and 2.0000E-10 in **examples 1, 2** and **3** respectively. These values are by far superior to those obtained from the Galerkin method with septic B-spline in Viswanadham and Ballem (2015) as shown in **Tables 1, 2 and 3.**

CONCLUSION

The method avoids quasi-linearization, linearization, discretization or perturbation. Also, the method is computationally simple with round-off and truncation errors avoided. Numerical results obtained with the proposed method show that the method is highly reliable and accurate in obtaining the approximate solution of tenth order boundary value problems.

The method can be applied to higher order boundary value problems.

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