MATHEMATICAL MODELLING OF ALUMINUM SURFACE WHEN
DIPPED IN MOLTEN METAL

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Abstract. A mathematical model is presented to describe the undulating surface of aluminium casting
during an industrial process involving the dipping of the mould, at a particular velocity, into the molten
metal. The problem of air gap formation between the mould and the casting was also considered. Below
certain value of the mould velocity the shape of the casting as well as its thickness remain practically
unchanged with changes in mould velocities. The undulating surface disappears when the mould
temperature is in excess of 120°C.

1. INTRODUCTION.
The process of metal solidification needs to be
carefully controlled in order to avoid casting defects
that could reduce the quality of such castings [1,2].
One of the ways of obtaining good castings is to
reduce the air gap between the mould and the
solidified metal [3]. Also, by reducing the absorbed
gases in the molten metal, a high quality casting can
be produced [4]. Aluminium, for example, is known to
absorb hydrogen from the atmosphere thereby
leading to the formation of micro pores in its
castings.

The present work focuses on the problem of
surface undulation in aluminium casting during a
particular industrial process described in the
following. When a flat metal mould was dipped
inside a molten aluminium at a velocity, \( v \), and
immediately withdrawn, the surface of the
solidified layer was found to be undulated. Fig.1a
shows a sketch of such process and the resulting
undulating surface in Fig.1b. The cause of the
surface undulation is still not well known but it may
be related to the occurrence of air gaps at the
boundary between the mould and the molten metal.
The surface undulation may also be due to thermal
stresses arising from temperature difference
between the mould and the molten metal. The air
gap formation at mould wall could serve as barrier
to heat transfer across the mould and it will
therefore be taken into consideration in the
modelling. The aim of the present work is to
develop a model predicting the surface undulation
during aluminium solidification in relation to the
mould velocity and the initial mould temperature.

1. MATHEMATICAL FORMULATION

Fig.1a shows the mould entering the molten
metal at velocity, \( v \), in the \( x \)-direction. The moving
boundary of the solidifying layer is represented by
\( s(x,t) \) in Fig.1b. Heat is removed through the wall
of the mould. At the boundary between the
solidifying metal and the molten metal, the
temperature is taken as \( T_m \). Considering a two-
dimensional case and taking an element with
dimension \( dx \) and \( dz \) in the already solidified layer,
the rate at which heat is removed across the surface
of the element must be equal to the rate of decrease
of heat in it. Thus,

\[
\frac{c_p \rho}{\frac{dT}{dt}} + c_p v \frac{dT}{dz} = \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right), \quad 0 < z < s(x,t)
\]

(1)

The second term, on the left hand side of the
equation, represents the rate of decrease of heat due
to the introduction of the mould into the molten
metal at a velocity, \( v \). The boundary of the
solidifying layer is moving and latent heat is
generated due to the transformation of liquid to
solid state and this can be represented as follows:

\[
\frac{c_p \rho}{\frac{dT}{dt}} + c_p v \frac{\partial s}{\partial x} = t_h \frac{\partial T}{\partial n}; \quad z = s(x,t).
\]

where

\[
\frac{\partial T}{\partial n} = \frac{-\frac{\partial T}{\partial x} \frac{\partial x}{\partial n} - \frac{\partial T}{\partial z} \frac{\partial z}{\partial n}}{\sqrt{1 + \left( \frac{\partial x}{\partial n} \right)^2}}
\]

(2)
Lₘ is the latent heat of melting and n is the normal to the surface of the solidifying metal as shown in Fig.1b. The meaning of the symbols and their values as relevant to aluminum alloy are presented in Table1. The temperature at the moving boundary is assumed to be equal to that of the melting point of aluminum alloy i.e.

\[ T = T_m; \quad z = s(x,t) \]  \hfill (3)

and the heat flux at the mould wall is given by:

\[ K \frac{\partial T}{\partial z} = h(T - T_r); \quad z = 0 \]  \hfill (4)

Considering also that at the initial stage

\[ s(x,t) = 0 \text{ at } x = 0 \]  \hfill (5)

The governing eqns (1) and (2) with the boundary conditions (3) – (5) can be non-dimensionalised to obtain the following eqns (6) to (10):

\[ \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} = a \frac{\partial^2 T}{\partial x^2} + g \frac{\partial^2 T}{\partial z^2}; \quad 0 < z < s(x,t) \]  \hfill (6)

\[ \frac{\partial s}{\partial t} + v \frac{\partial s}{\partial x} = \left( -b \frac{\partial T}{\partial x} \frac{\partial s}{\partial x} \frac{\partial T}{\partial z} \right) \left( \frac{-b \frac{\partial T}{\partial x} \frac{\partial s}{\partial x} \frac{\partial T}{\partial z}}{1 + c \left( \frac{\partial s}{\partial x} \right)^2} \right); \quad z = s(x,t) \]  \hfill (7)

\[ T = 1; \quad z = s(x,t) \]  \hfill (8)

\[ \frac{\partial T}{\partial z} = h(T - T_r); \quad z = 0 \]  \hfill (9)

\[ s = 0; \quad z = 0 \]  \hfill (10)

However when the parameters relevant to the alloy are substituted, the values of \( a \approx 10^3 \), \( b \approx 10^3 \) and \( c \approx 10^3 \), which are small, compared to the value of \( g \). The expression for \( g \) is given as:

\[ g = \frac{k_κ(T_m - T_c)}{L_κ \rho ν} \]  \hfill (11)

For this reason the problem can be further simplified by neglecting the terms containing \( a, b \) and \( c \).
\[ \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} = \kappa \frac{\partial^2 T}{\partial z^2}; 0 < z < s(x,t) \]  

(12)

The heat transfer coefficient, \( h \), at the mould wall may be taken as constant but if then we consider the presence of an air gap we can model it in the form of a sine function:

\[ h(x) = i + \sin (2\pi x) \]  

(17)

\[ s(x,t) + \frac{\partial h}{\partial t} = h(T - T_r); z = 0 \]  

(15)

\[ s = 0; x = 0 \]  

(16)

\[ T_r = 1 - e^{2\pi} \]  

(18)

Table 1: Definition of symbols and values of parameters used in the model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thermal conductivity</td>
<td>( \lambda = 230 J/m^2/K )</td>
</tr>
<tr>
<td>Mould density</td>
<td>( \rho = 2650 Kg/m^3 )</td>
</tr>
<tr>
<td>Metal thermal diffusivity</td>
<td>( K = 8.2 \times 10^{-7} m^2/s )</td>
</tr>
<tr>
<td>Specific heat capacity</td>
<td>( c = 3 \times 10^3 J/Kg/K )</td>
</tr>
<tr>
<td>Latent heat of melting</td>
<td>( L = 3.9 \times 10^5 J/Kg )</td>
</tr>
<tr>
<td>Velocity of mould</td>
<td>( v = 0-0.25 m/s )</td>
</tr>
<tr>
<td>Room temperature</td>
<td>( T_r = 25^\circ C )</td>
</tr>
<tr>
<td>Melting temperature</td>
<td>660°C</td>
</tr>
<tr>
<td>Heat transfer coefficient</td>
<td>( h = 1100 J/m^2 )</td>
</tr>
</tbody>
</table>

The parameter \( \lambda \) is taken as a positive constant. Eqs (12) - (18) can be solved so as to obtain the expression for \( s(x,t) \) in terms of \( x \). Attempts were made to solve this problem using integral balance method and the resulting differential equations were solved using Mathematica [5].

1. NUMERICAL RESULTS AND DISCUSSION

In formulating the governing eqns (12 - 18) of the model certain parameters were introduced viz \( g \), \( \lambda \) and \( h \) but they are subject to some variations. The physical significance of these parameters needs to be discussed. The parameter, \( g \), is inversely proportional to the velocity of the mould as given by the expression in eqn 11. Table 2 gives the values of \( g \) utilized in the modeling and the corresponding values of the mould velocity. It is important to examine how the mould velocity affects the shape and the thickness of the casting. The shape of the casting and its thickness is represented by the curve of \( s(x) \) against \( x \). These curves are shown in Fig. 2 for eight different values of the mould velocity presented in Table 2. The curves of \( s(x) \) against \( x \) for velocities below 0.001625 m/s, practically coincide and the undulating surface is observed. This implies that neither the thickness of the casting nor the undulating surface is sensitive to changes when the mould velocity is below certain value.

Table 2: Parameter \( g \) and the corresponding mould velocities

<table>
<thead>
<tr>
<th>( g )</th>
<th>Velocity, ( v ) (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>4.06x10^-3</td>
</tr>
<tr>
<td>4</td>
<td>8.12x10^-3</td>
</tr>
<tr>
<td>2</td>
<td>1.625x10^-2</td>
</tr>
<tr>
<td>0.5</td>
<td>6.5x10^-3</td>
</tr>
<tr>
<td>0.25</td>
<td>1.3x10^-2</td>
</tr>
<tr>
<td>0.1</td>
<td>3.25x10^-2</td>
</tr>
<tr>
<td>0.01</td>
<td>3.25x10^-3</td>
</tr>
</tbody>
</table>

The second parameter, \( \lambda \), is related to the temperature of the mould before inserting it into the molten metal, given in the expression, \( T_r = 1 - e^{-\lambda x} \). The values of \( \lambda \), utilized in the present modeling are presented in Table 3 with the corresponding temperatures of the mould. The effects of the mould temperature on the thickness and shape of the casting are predicted in Fig.3. It can be seen
that the surface undulation is remarkable only when
the mould temperature is lower than about 120°C.
The thickness of the casting is reduced as the
mould temperature increases. The results are
plausible since lower mould temperature will cause
a large thermal difference between the mould and
the molten metal so that thermal stresses may
provoke surface undulation.

| Table 3: Parameter, λ and the corresponding initial mould temperature |
|----------------|------------------|
| λ     | T₀ (°C)       |
| 0.1   | 63.7           |
| 0.2   | 119            |
| 0.5   | 259            |
| 0.8   | 364            |
| 1     | 417            |
| 4     | 570            |

Fig.2: The curve of s(x) against x showing the
thickness and shape of the casting at various mould
temperatures. No undulation at temperatures above
120°C.

Fig.3: The curve of s(x) against x showing the
thickness and shape of the casting at various
mould velocities. Thickness and surface
undulation remains the same for mould
velocities less than 1.63x10⁻² m/s.

4. CONCLUSION
The surface undulation of the solidified layer
was predicted from the model. The prediction
shows that neither the thickness of the casting nor
its undulating shape is subject to changes when the
mould velocity is below certain value. The surface
undulation is prominent only when the mould
temperature is below about 120°C. It was not
possible however to model how the air gap got into
the surface of the mould. A main factor affecting
the surface undulation is the variation of the heat-
transfer coefficient of the mould surface.

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