CLASSICAL ANALYSIS OF THE SHEAR VIBRATION CHARACTERISTICS OF AN EMBANKMENT DAM

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ABSTRACT

In this paper, the governing differential equation for the vibration of a homogeneous, isotropic embankment dam was established by the shear beam theory. The governing equation was developed for a typical wedge shaped embankment dam with rectilinear side slopes and having the same gradient on both the upstream and downstream slope. For harmonic displacement response, it was found that the governing partial differential equation reduces to an ordinary differential equation of the Bessel type. This was then solved, subject to the boundary conditions, to obtain the modal shape functions and natural frequencies of vibration. The shear stress distribution along the embankment dam was also obtained.

INTRODUCTION

Theoretical analysis using shear beam theory, numerical analysis as a two or three dimensional elastic structure, and the use of model tests are some of the methods for determining the dynamic characteristics of embankment dams.[1]

Shear beam theory idealizes the dam as a plate obtained by introducing imaginary sections at two arbitrary cross-sections perpendicular to the axis of the dam, and analyzes the resulting substructure as a wedge-shaped shear beam. The effect of both banks of the valley is ignored in the analysis. The displacements at various points at the same elevation are all assumed to be equal.

Another more rigorous method considers the resulting plate as a two-dimensional structure with the thickness of the dam taken into account in the determination of the vibration characteristics. The effect of the banks is also disregarded here. [2]

However, a closed form solution, especially for complicated boundary conditions, is usually difficult, if not impossible to obtain in some cases. Hence, numerical procedures like the finite difference method and the finite element techniques have been developed to yield approximate solutions to boundary value problems of engineering and mathematical physics [3].

Recent developments in the finite element analysis of dynamic structures have made it possible to determine the dynamic characteristics by considering the three-dimensional geometry of the dam, the anisotropy and heterogeneity of the dam materials and the influence of the banks. [4]

However, the procedure is extremely demanding of computer storage space due to the large number of equations obtained.

In the similitude tests, a physical model of the dam is built, and submitted to an elastic vibration test in order to determine the periodic time, mode of natural vibrations, and the displacement and surface strain of the dam during vibrations. Then, using the principles of similarity, the vibration characteristics of the actual dam can he obtained.

Similitude tests employ the principles of similarity between the model and the prototype in order to determine the vibration characteristics of the real dam from the vibration characteristics of the physical model. However, dams are such complex structures that it is usually impossible to obtain similarities for all the characteristics. This is a major limitation of the use of similitude tests.

In this work, we seek to:

- obtain an equation governing the vibration behaviour of a wedge-shaped embankment dam assumed to be made of homogenous, isotropic and linear elastic material
- solve the resulting governing equation to obtain the vibration period, natural frequencies and mode shapes of the embarkment dam.
- calculate the shear stress variation with depth on the axis of the dam for each mode of vibration.

Formulation of the Governing Differential Equation by the Shear Beam Theory

The derivation is based on the following assumptions:

(i) the dam is idealized as a beam with a cross-section shaped in the form of a wedge since the trapezoidal shape of some embankment dams can be built up from a consequence of the superposition principle applied to triangular wedges. Again if the crest of the dam is very small relative to the base width, the dam can be approximated, for ease of analysis, as a triangular wedge.

(ii) the displacement field is a function of the depth coordinate alone.

(iii) according to the shear beam theory, the deformation of the dam is due to shear stress effects only [1] and [4], hence flexural deformations are ignored.

(iv) the material of the dam is homogeneous, linear elastic and isotropic

(v) the dam is very long compared with its cross-sectional base width that the dam is idealized as infinitely long. Consequent upon this assumption, end effects are ignored.

The problem is required to satisfy simultaneously

- the differential equations of dynamic equilibrium
- the stress-strain law for the material of the dam, and
- the strain displacement requirement.

We consider an elemental segment of the dam, as shown in Fig.1. For equilibrium of forces acting on the elemental segment, we apply D'Alembert's principle of dynamic equilibrium to yield:

$$\sum F = ma$$

$$-S_{yz} + S_{zy} + \frac{\partial S_{zy}}{\partial z} dz = \rho b dz \frac{\partial^2 v}{dt^2}$$

Using the principle of complimentary stress, $S_{zy} = S_{yz}$, and the equation of motion for the elemental shear beam becomes

$$pbdz\frac{\partial^2 v}{\partial t^2} = \frac{\partial S_{zy}}{\partial z}dz = \frac{\partial}{\partial z}(b\tau)dz \qquad (1)$$

where ρ = mass density of the material of the dam

S = shearing force

H = dam height

G = shear modulus of elasticity of the material of the dam

τ = shear stress

v = displacement in the direction of the *y*-coordinate

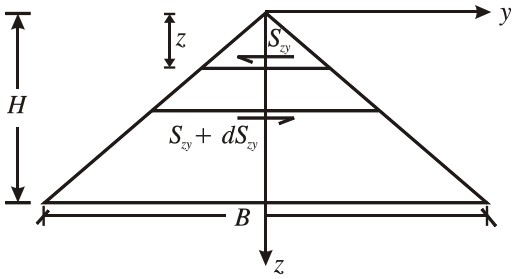

Fig. 1: The Embankment Dam

The stress-strain law for the material of the dam assuming linear isotropic elasticity is

$$\tau = G\gamma \qquad (2)$$

where γ = shear strain

The strain displacement equation is given, assuming finite strain behaviour, by

$$\gamma = \frac{\partial v}{\partial z} \qquad (3)$$

Using equations (2) and (3) in equation (1) we obtain the differential equation of equilibrium as:

$$pb\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial z}\left(bG\frac{\partial v}{\partial z}\right) \qquad (4)$$

For a wedge-shaped cross-section, which we assumed, the dam width can be expressed as $b(z) = b_0 z$ where b_0 is a measure of the gradient of the side slopes, and $0 \le z \le H$, then

$$pb_o z\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial z}\left(b_o zG\frac{\partial v}{\partial z}\right) \qquad (5)$$

Simplifying we obtain

$$\frac{p}{G}\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial z^2} + \frac{1}{2}\frac{\partial v}{\partial z} \qquad (6)$$

Equation (6) is the governing differential equation for the natural vibrations of the dam.

Closed Form Solution

Assuming harmonic vibrations, and harmonic response, we seek a closed form solution for $v(z, t)$ in the variable separable form

$$v(z,t) = v(z)\cos(\omega t + \alpha) \qquad (7)$$

where $v(z)$ is the mode of natural vibrations, ω is the circular frequency of natural vibrations, and α is the phase.

Then the governing equation becomes,

$$\left(\frac{d^2 v}{dz^2} + \frac{1}{z}\frac{dv}{dz} + \frac{\omega^2 p}{G}v\right)\cos(\omega t + \alpha) = 0 \qquad (8)$$

For nontrivial solutions, $\cos(\omega t + \alpha) \neq 0$, and the characteristic equation becomes

$$\frac{d^2 v}{dz^2} + \frac{1}{z}\frac{dv}{dz} + \frac{\omega^2 p}{G}v = o \qquad (9)$$

Let $C_s = \sqrt{\frac{G}{p}}$, where C_s =velocity of shear waves, then we obtain

$$\frac{d^2 v}{dz^2} + \frac{1}{z}\frac{dv}{dz} + \frac{\omega^2}{c_s^2}v = 0 \qquad (9a)$$

Equation (9a) is a Bessel differential equation,

the general solution of which is,

$$v(z) = c_1 J_o\left(\frac{\omega z}{C_s}\right) + C_2 Y_o\left(\frac{\omega z}{C_s}\right) \qquad (10)$$

where J_0 is the Bessel function of the first kind of order zero, and Y_0 is the Bessel function of the second kind of order zero, and c_1 and c_2 are constants of integration which are determined from the boundary conditions.

The boundary conditions are

$$\tau(z = o) = 0 \qquad (11a)$$

$$v(z = H) = 0 \qquad (11b)$$

Using equation 11a, we find: $c_2 = 0$

Thus the modal shape function becomes

$$v(z) = c_1 J_o\left(\frac{\omega z}{c_s}\right) \qquad (10a)$$

Using equation (11b), we obtain the frequency equation

$$J_o = \left(\frac{\omega H}{C_s}\right) = 0 \qquad (12)$$

Solving, $\frac{\omega H}{C_s} = z_n$

where z_n are the n roots of $J_0(z) = 0$

So, $\omega_n = \frac{z_n C_s}{H}$

where z_n = 2.4048, 5.5201, 8.6537, 11.7915, 14.931, etc.

So, $\omega_1 = \frac{2.4048}{H}\sqrt{\frac{G}{p}}$

$\omega_2 = \frac{5.5201}{H}\sqrt{\frac{G}{p}}$

$\omega_3 = \frac{8.6537}{H}\sqrt{\frac{G}{P}}$

$\omega_4 = \frac{11.7915}{H}\sqrt{\frac{G}{P}}$

$\omega_5 = \frac{14.931}{H}\sqrt{\frac{G}{P}}$

This compares remarkably well with the first two natural frequencies obtained by Ashok K.C. in [2] as follows

$\omega_1 = \frac{2\pi}{2.62H}\sqrt{\frac{G}{P}} = \frac{2.401}{H}\sqrt{\frac{G}{P}}$

$\omega_2 = \frac{2\pi}{2.12H}\sqrt{\frac{G}{P}} = \frac{5.616}{H}\sqrt{\frac{G}{P}}$

The natural period is

$T_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{z_n}\frac{H}{C_s} = \frac{2\pi H}{z_n}\sqrt{\frac{P}{G}}$

The vibration mode is

$v_n(z) = c_1 J_o\left(\frac{\omega z}{C_s}\right) = C_1 J_o\left(\frac{z_n z}{H}\right)$

Shear stress

The shearing stress is

$\tau = G\frac{\partial v}{\partial z} = -\frac{GH}{z_n} J_1\left(\frac{z_n z}{H}\right)\cos(\omega t + \alpha)$

The modes and shearing stress amplitudes of the first, second and third order natural vibrations are as given in Tables 1, 2, and 3 below:

First order mode

Table 1 Mode function value and shear stress amplitude variation with depth factor ($^z/_H$): First Order Mode

z/H	v_n	$\tau / \left(-GH / z_n\right)$
0	1.00	0
0.1	0.9850	0.1202
0.2	0.94	0.2405
0.3	0.88	0.3295
0.4	0.77	0.44
0.5	0.67	0.498
0.6	0.53	0.55
0.7	0.398	0.577
0.8	0.28	0.58
0.9	0.11	0.548
1.0	0	0.519

Table 2 Mode function value and shear stress amplitude variation with depth factor: *Second Order Mode*

z/H	v_n	$\tau / \left(-GH / z_n\right)$
0	1.00	0
0.1	0.92	0.259
0.2	0.72	0.469
0.3	0.42	0.569
0.4	0.11	0.556
0.5	-0.16	0.4195
0.6	-0.34	0.2208
0.7	-0.40	-0.027
0.8	-0.34	-0.204
0.9	-0.18	-0.325
1.00	0	-0.341

1st order natural vibration

- - - - normalized shear stress $\tau / \left(-GH / z_n\right)$

Deflection mode amplitude

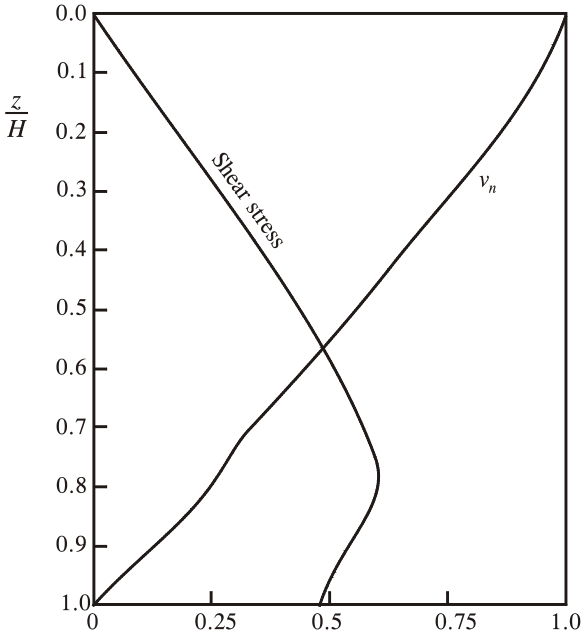

Fig. 2 Mode shapes and shearing stress due to shear vibration for 1st mode of vibration

2nd order natural vibration

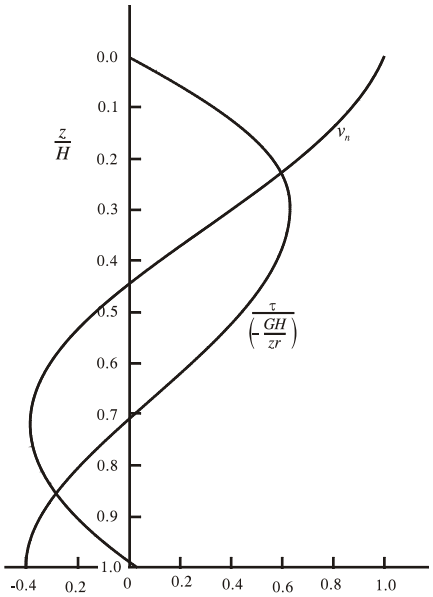

Fig. 3 Deflection mode shape and shearing stress due to shear vibration for 2nd mode of vibration

Table 3 Mode function value and shear stress amplitude variation with depth factor: 3rd order mode

z/H	v_n	$\tau / \left(-GH / z_n\right)$
0	1	0
0.1	0.82	0.381
0.2	0.38	0.58
0.3	–0.0968	0.467
0.4	–0.370	0.147
0.5	–0.361	–0.1730
0.6	–0.1103	–0.3460
0.7	0.16	–0.1644
0.8	0.2981	–0.0350
0.9	0.2154	0.199
1	0	0.271

The rate of distribution to each normal coordinate is measured by the modal participation factor which is defined, for the *n*th mode, by

$$\mu_n = \frac{\int_o^H z J_o\left(\frac{z_n z}{H}\right) dz}{\int z\left(J_o\left(\frac{z_n z}{H}\right)\right)^2 dz} = \frac{2}{z_n J_1(z_n)}$$

The modal participation factors can be calculated as: $\mu_1 = 1.60$, $\mu_2 = -1.06$, $\mu_3 = 0.86$

Conclusion

The governing partial differential equation for a wedge-shaped embankment dam made of homogeneous, isotropic, linear elastic material has been obtained using the shear beam theory.

Closed form expressions for the natural frequencies, period, and modal shape functions of an embankment dam have been obtained from a solution of the governing partial differential equation. As in all classical solutions, an infinite set of values

was obtained for the natural frequencies and the mode shapes of the dam.

According to shear beam theory, the stress due to the natural vibration of a dam is distributed as shown in Figs 2, and 3 and, as tabulated in Tables 1, 2, and 3.

From the charts, it is observed that shear stresses are maximum at $z = 0.75H$ with first order vibration and at $z = 0.33H$ with second order vibration. This agrees remarkably well with the observed response of physical models of dams from the technical literature, such as Ashok K.C. in [2] and Gazetas G. in [1].

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