APPROXIMATE METHOD FOR THE DETERMINATION OF NATURAL FREQUENCIES OF A MULTI-DEGREE OF FREEDOM BEAM SYSTEM

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Abstract

In this paper, a simplified dynamic model is developed to predict the natural frequencies of a multi-Degree of freedom structural system using the concept of an elastic shear wave in a solid prismatic bar. The vibrating solid prismatic bar and the multi-degree of freedom beam system with distributed mass under self-excited vibration are assumed to be dynamically equivalent. Lumped mass idealization is used to discretize the original beam structure with distributed mass to a weightless beam with distributed masses substituted with lumped masses at different nodal points. A numerical example obtained from literature is given to illustrate the application of the present model. All the natural frequencies predicted using the present formulation compared favorably with those of other authors most especially in the first and second vibration mode. The correlation coefficient value was high showing high (0.991) showing high predictive ability of the present model.

Keywords: MDOF, Natural frequencies, distributed mass, prismatic bar, lumped mass, shear wave

1. Introduction

All real structures have distributed masses and have infinite number of degrees of freedom. The dynamic analysis of a structural system with such a characteristic is tedious as it involves a lot of mathematical manipulations [1-14]. The dynamic analysis is made simpler using lumped mass idealization. The transformed distributed mass system now has finite number of degrees of freedom [15]. The lumped mass configuration is the same structural system but it has its distributed mass substituted with lumped masses located at some chosen nodal points of the structure (Figure 1). The degree of freedom is numerically equal to the number of independent geometric parameters that describe the positions of all masses for all possible displacements of the structural system at any moment in time. The present model is said to be fully defined if the magnitudes of the lumped masses and their respective coordinates are known [15]. For instance, for a cantilevered beam with evenly distributed mass, an infinite number of coordinates is required to define the displacement configuration. If we imagine that the beam mass is lumped into two bodies with the force-displacement properties of the cantilever unchanged and if we assume that the external forces causing the motion are applied at this two masses, then the deflected configuration of the beam at anytime can be completely defined by 12 coordinates, 6 at each mass. The

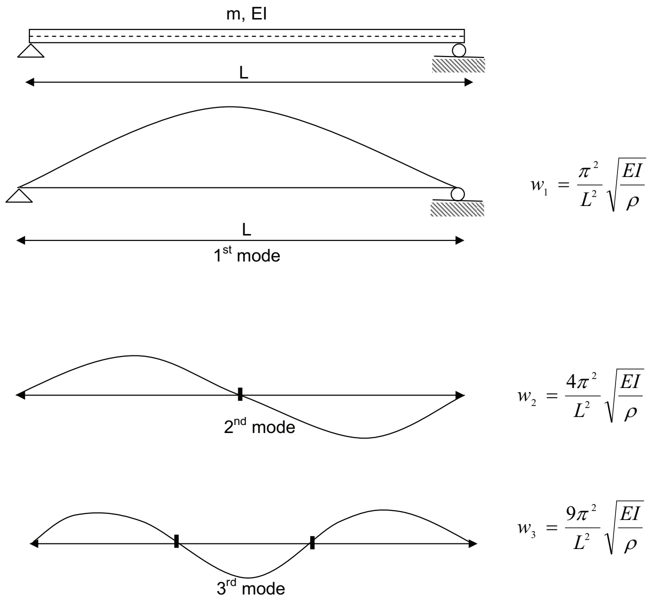

Figure 1: Natural vibration modes and frequencies of a uniform simply supported beam [13].

idealized model having 12 degrees of freedom may therefore conveniently be considered in dynamic analysis in place of the actual system with distributed mass [15]. A more accurate representation can be obtained by using a larger number of masses at a closer interval and consequently requires a greater number of coordinates. The objective of this paper is to develop a simplified method for predicting the natural frequencies of vibration of a structural system with distributed mass under self-excited vibration. The algorithm involved is simple "not involving much computational effort and the results can be achieved using manual computation".

2. Dynamic Model Formulation

The vibration modes of a uniform shear beam representing a weightless beam under self-excited vibration is as shown in Figure 1.

$P_1 = P_2 = P_3 = P =$ Induced unit load at nodal points of a weightless beam. From the natural mode shapes shown in Figure 1, the natural frequency in the ith mode for the

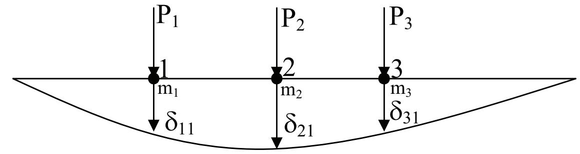

Figure 2: Static beam deflection due to load $P\,(P1 = P2 = P3 = P)$.

actual system [12] is given by

$$w_n = \frac{i^2\pi^2}{L^2}\sqrt{\frac{EI}{P}} \quad (i = 1,\, 2,\, 3) \tag{1}$$

The shear wave speed in a prismatic bar expressed as the shear modulus - to - density ratio is given by;

$$c = \sqrt{\frac{G}{\rho_m}} \tag{2}$$

where $\rho_m = \frac{\gamma}{g}$ = mass density of the material of the beam. γ = unit weight of the material of the beam; g = acceleration due to gravity; c = speed of shear wave in the bar.

Substituting $\frac{\gamma}{g}$ for ρ_m in equation (2) gives;

$$c = \sqrt{\frac{Gg}{\gamma}} \tag{3}$$

The vibration of a uniform simply supported beam (Figure 1) generates a stationary wave.

The shear wave wavelength in the prismatic bar is given by;

$$\lambda_i = \frac{c}{f_i} \quad (i = 1,\, 2,\, 3) \tag{4}$$

where f_i = natural frequency of vibration of the bar in the ith mode; λ_i = wavelength of the shear wave in the ith mode; G = shear modulus of the material of the prismatic bar. But

$$f_i = \frac{1}{T_i} \tag{5}$$

Therefore,

$$\lambda_i = cT_i \quad (i = 1,\, 2,\, 3) \tag{6}$$

Therefore, in the first three modes, the natural vibration periods of a vibrating prismatic bar at the same shear wave speed are given by;

$$T_1 = \frac{\lambda_1}{c} \tag{7}$$

$$T_2 = \frac{\lambda_2}{c} \tag{8}$$

$$T_3 = \frac{\lambda_3}{c} \tag{9}$$

The generalized expression for the wavelength in the i^{th} mode for a vibrating prismatic bar, (Figure 3) is given by;

$$\lambda_i \frac{4L}{(2i-1)}; \quad (i = 1,\, 2,\, 3) \tag{10}$$

Therefore, for an i^{th} mode of vibration, the ith natural period is given by;

$$T_i \frac{4L}{(2i-1)c}; \quad (i = 1,\, 2,\, 3) \tag{11}$$

The first three vibration modes natural periods are thus given by;

$$T_1 \frac{4L}{c} \tag{12}$$

$$T_2 \frac{4L}{3c} \tag{13}$$

$$T_3 \frac{4L}{5c} \tag{14}$$

where $T_1,\ T_2,\ T_3$ = Natural periods in first, second and third modes of vibration respectively. $\frac{L}{c}$ = time of travel of shear wave from one end of the beam to the other end of the beam. $\lambda_1,\ \lambda_2,\ \lambda_3$ = wavelengths of stationary waves in the first three modes.

From the flexural deflection of the dynamically equivalent weightless beam shown in Figure 2,

$$G_b = \frac{P/A}{\delta_{ij}/L} = \frac{PL}{\delta_{ij}A} \tag{15}$$

$$\gamma_b = \frac{mg}{AL} \tag{16}$$

The vibrating solid uniform bar and the weightless beam are dynamically equivalent.

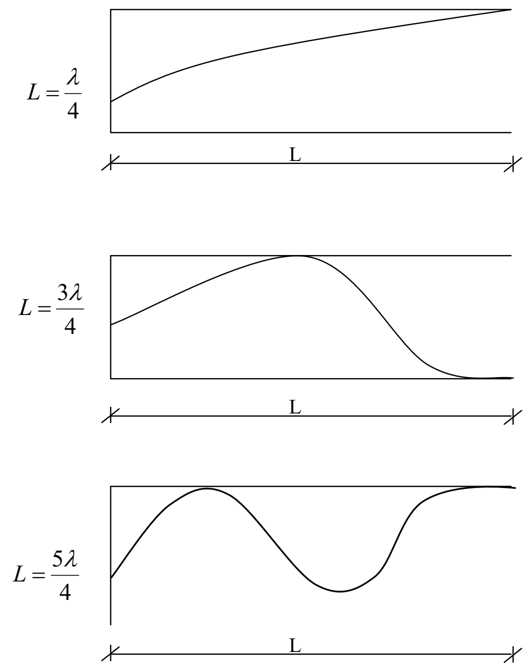

Figure 3: First three normal modes of vibration.

Thus, the speed of shear wave along the weightless beam is given by;

$$c_b = \sqrt{\frac{G_b g}{\gamma_b}} \tag{17}$$

Substitution of equation (15) and (16) into equation (17) gives;

$$c_b = l\sqrt{\frac{P}{m\delta_{ij}}} \tag{18}$$

But $\frac{P}{\delta_{ij}} = k$ = stiffness of a weightless beam

$$\Longrightarrow c_b = l\sqrt{\frac{k}{m}} \tag{19}$$

Substituting equation (19) for c in equation (12), (13), and (14) yields;

$$T_1 = \frac{4\sum_{i=1}^{n} l_i}{l}\sqrt{\frac{m}{k}} \tag{20}$$

$$T_2 = \frac{4\sum_{i=1}^{n} l_i}{3l}\sqrt{\frac{m}{k}} \tag{21}$$

$$T_3 = \frac{4\sum_{i=1}^{n} l_i}{5l}\sqrt{\frac{m}{k}} \tag{22}$$

c_b = shear wave speed in the weightless beam; L = overall length of the weightless beam; l = length of segment of the discretized weightless beam; A = cross-sectional area of a prismatic bar; δ_{ij} = static deflection of a weightless beam at point I due to unit load applied at point j.

For a weightless beam of different magnitudes of lumped masses, different sub-lengths and stiffnesses,

$$c_i = l_i\sqrt{\frac{k_i}{m_i}} \tag{23}$$

where n = total number of a weightless beam-segments where the natural periods of the oscillating lumped masses are considered ($n = 3$ for the present numerical study); l_i = length of an i^{th} beam segment; k_i = stiffness of an ith bema segment; m_i = mass of an ith beam segment; c_i = shear wave velocity in the i_{th} beam segment. The time taken for the propagating shear wave to travel an ith weightless beam segment is given by:

$$\frac{l_i}{c_i} = \sqrt{\frac{m_i}{k_i}} \tag{24}$$

Therefore, the total time taken by the shear wave to travel the entire beam segments is the algebraic sum of the individual times taken to travel the individual beam segments given by;

$$\sum_{i=1}^{n}\frac{l_i}{c_i} = \sum_{ii=1}^{n}\sqrt{\frac{m_i}{k_i}} \tag{25}$$

Using equations (12), (13) and (14), the natural periods in the first three modes of a vibrating weightless beam having different masses and stiffnesses are given by;

$$T_1 = 4\sum_{i=1}^{n} l_i\sqrt{\frac{m_i}{k_i}} \tag{26}$$

$$T_2 = \frac{4}{3}\sum_{i=1}^{n} l_i\sqrt{\frac{m_i}{k_i}} \tag{27}$$

$$T_3 = \frac{4}{5}\sum_{i=1}^{n} l_i\sqrt{\frac{m_i}{k_i}} \tag{28}$$

For a weightless beam system having different masses at different nodal points and different stiffnesses,

$$k_i = \frac{3EIL}{a_i^2 b_i^2} \tag{29}$$

Where E, I = modulus of elasticity and moment of inertia of the prismatic beam respectively. And

$$m_i = \frac{l_i\rho + l_i\rho}{2} \tag{30}$$

where a_i = an i^{th} distance from weightless beam's left support to an i^{th} nodal point carrying an i^{th} lumped mass; b_i = an i^{th} distance from i^{th} nodal point carrying an i^{th} lumped mass to the right support of a weightless beam.

Substitution of equation (29) and (30) into equations (26), (27) and (28) gives a generalized expression for an i^{th} natural period of vibration in the ith mode as;

$$T_i = \sum_{i=1}^{n}\sqrt{\frac{l_i\rho + l_i\rho[a_i^2 \cdot b_i^2]}{6EIL}} \quad (i, j = 1,\ 2,\ 3,\ \ldots,\ n) \tag{31}$$

3. Numerical Study

A numerical example given by Osadebe [1] is used to illustrate the application of the proposed model. A simply supported uniform beam having a distributed mass intensity of 4.75kg/m shown in Figure 4 is used for this numerical study. The MDOF model has three lumped masses m_1, m_2 and m_3 implying three degrees of freedom.

From equation (30),

$$m_1 = m_3 = \frac{\rho + 1,5\rho}{2} = 1.25\rho$$

The natural periods in the first three modes of vibration now transforms to:

$$T_1 = 4\sum_{i=1}^{n}\sqrt{\frac{l_i\rho + l_i\rho(a_i^2 \cdot b_i^2)}{6EIL}} \tag{32}$$

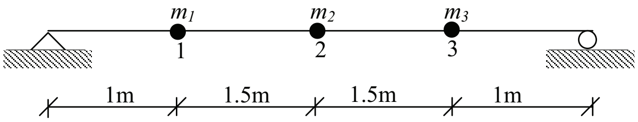

Figure 4: MDOF model of 3 degrees of freedom for numerical study [1].

$$T_2 = \frac{4}{3}\sum_{i=1}^{n}\sqrt{\frac{l_i\rho + l_i\rho(a_i^2 \cdot b_i^2)}{6EIL}} \quad (33)$$

$$T_3 = \frac{4}{5}\sum_{i=1}^{n}\sqrt{\frac{l_i\rho + l_i\rho(a_i^2 \cdot b_i^2)}{6EI}} \quad (34)$$

But circular frequency, $w_i = \frac{2\pi}{T_i}$; $\quad i = 1,\, 2,\, 3.$ Therefore, the circular frequency in the first three modes of vibration are:

$$w_1 = \frac{2\pi}{4\sum_{i=1}^{n}\sqrt{\frac{l_i\rho + l_i\rho(a_i^2 \cdot b_i^2)}{6EI}}} \quad (35)$$

$$w_2 = \frac{2\pi}{\frac{4}{3}\sum_{i=1}^{n}\sqrt{\frac{l_i\rho + l_i\rho(a_i^2 \cdot b_i^2)}{6EI}}} \quad (36)$$

$$w_3 = \frac{2\pi}{\frac{4}{5}\sum_{i=1}^{n}\sqrt{\frac{l_i\rho + l_i\rho(a_i^2 \cdot b_i^2)}{6EI}}} \quad (37)$$

Equations (35), (36) and (37) are the required frequency-predicting equations into equations (32), (33) and (34),

$$\begin{aligned} T_1 &= 4\left[\sqrt{\tfrac{1.25\rho(1^2\times 4^2)}{30EI}} + \sqrt{\tfrac{1.5\rho(2.5^2\times 2.5^2)}{30EI}} + \sqrt{\tfrac{1.25\rho(4^2\times 1^2)}{6EI}}\right] \\ &= \left[\sqrt{\frac{\rho}{EI}}(0.817 + 1.397 + 0.817)\right] \\ &= 12.125\sqrt{\frac{\rho}{EI}} \end{aligned}$$

Therefore, the circular frequency in the first

Table 1: Comparison of results.

	Present model	Osadebe [1]	Distributed mass
w_1	$0.518\sqrt{\frac{EI}{\rho}}$	$0.4039\sqrt{\frac{EI}{\rho}}$	$0.3948\sqrt{\frac{EI}{\rho}}$
w_2	$1.555\sqrt{\frac{EI}{\rho}}$	$1.5929\sqrt{\frac{EI}{\rho}}$	$1.5791\sqrt{\frac{EI}{\rho}}$
w_3	$2.59\sqrt{\frac{EI}{\rho}}$	$3.5142\sqrt{\frac{EI}{\rho}}$	$3.5531\sqrt{\frac{EI}{\rho}}$

Table 2: Comparison of results.

	Present model	Osadebe [1]	Distributed mass
w_1	$0.518\sqrt{\frac{EI}{\rho}}$	$0.4039\sqrt{\frac{EI}{\rho}}$	$0.114\sqrt{\frac{EI}{\rho}}$
w_2	$1.555\sqrt{\frac{EI}{\rho}}$	$1.5929\sqrt{\frac{EI}{\rho}}$	$-0.038\sqrt{\frac{EI}{\rho}}$
w_3	$2.59\sqrt{\frac{EI}{\rho}}$	$3.5142\sqrt{\frac{EI}{\rho}}$	$-0.923\sqrt{\frac{EI}{\rho}}$

three vibrating modes are;

$$\begin{aligned} w_1 &= \frac{2\pi}{12.125}\sqrt{\frac{EI}{\rho}} = 0.518\sqrt{\frac{EI}{\rho}} \\ w_2 &= \frac{2\pi}{\frac{12.125}{3}}\sqrt{\frac{EI}{\rho}} = 1.555\sqrt{\frac{EI}{\rho}} \\ w_3 &= \frac{2\pi}{\frac{12.125}{5}}\sqrt{\frac{EI}{\rho}} = 2.591\sqrt{\frac{EI}{\rho}} \end{aligned}$$

Table 2: Test of adequacy of the model The present model is tested for adequancy using Osadebes[1] results as control points

Average difference $(\bar{d}_i) = \frac{\sum d_i}{n} = \frac{0.847}{3} = -0.282$ Variance $S_d^2 = \sum \frac{d_i - \bar{d}^2}{N-1}$

$$S_d^2 = \frac{0.157 + 0.060 + 0.411}{N-1} = \frac{0.628}{2} = 0.314$$

Standard deviation, $S_d = 0.560$

From t statistics, $t = \frac{\bar{d}}{S_d/\sqrt{n}} = \frac{0.282\sqrt{3}}{0.560} = |-0.881| = 0.881$

Null and Alternative hypothesis:

H_0: all di = 0; There is no difference between the two methods of determining the natural frequencies of a MDOF beam system with distributed mass.

H_1: all $d_i = 0$; There is a significant difference between the two methods.

The hypothesis is tested at 5% level of significance and 2 degrees of freedom.

$T0.025$, $2 = 4.30 > 0.881$. Since the tabulated value is greater than the calculated value, it means acceptance of null hypothesis. There is no difference between the two methods and the model is very much fitted to the control points [1].

4. Discussion of Results

Table 1 shows that the results obtained from the present model which is based on the assumption of dynamic equivalence of a vibrating prismatic bar with a weightless beam under self-excited vibration are almost similar with those of Osadebe[1] and those of the actual system[2] showing the effectiveness of the present model in the prediction of dynamic response of a vibrating MDOF structural system. The difference in responses of the present model and those of Osadebe [1] and the other author[2] may be due to difference in the model assumptions used in the model derivation. Student t-test showed that the present model is very much fitted to the control points [1].

5. Conclusion

In conclusion, the present model provides predictions that are almost similar to those of Osadebe[1] and the actual system[2] most especially in the first and second modes of vibration showing high predictive ability of the present model. The present model can be applied to a multi-storey building undergoing free vibration.

References

1. Osadebe N.N. (1999). An improved MDOF model simulating some system with distributed mass. *Journal of university of Science and Technology Kumasi*, volume 19 Nos. 1, 2 and 3.
2. Biggs, I.M. (1964). *Introduction to structural dynamics* McGraw-Hill Book Company, New York, pp.85-122.
3. Den Hartog, J.P. (1956). *Mechanical Vibrations* McGraw-Hill Book Company, New York, pp.122-169.
4. Taranath, B.S. (1988). *Structural analysis and design of tall buildings*, McGraw-Hill, New York.
5. Smith, B.S. and Coull, A. (1991). *Tall building Structures: Analysis and Design*, Wiley, New York.
6. Rahgozar, r., Safari, H. and Kaviani, P. (2004). Free vibration of tall buildings using timoshenko beams with variable cross-section. *Structures under shock and impact VIII*, N. Jones and C.A. Brebbia eds., WIT Press, U.K.
7. Balendra, T. (1984). Free vibration of a shear wall frame building on an elastic foundation. *Journal sound vibration*, 96(4), 437-446.
8. Clough, R.W. and Penzien, J. (1982). *Dynamics of structures*, McGraw-Hill Int. Students Edition, Tomyo.
9. Kelly, S.G. (1993). *Fundamentals of mechanical vibrations*, McGraw-Hill, New York.
10. Thomson, W.T., (1988). *Theory of vibration with applications* 3rd edition, CBS, Publishers New Delhi.
11. Humar, J.L., (1990). *Dynamics of structures*, Prentice-Hall, Inc.
12. De Silva, C.W. (1999). *Vibration: Fundamentals and Practice*, CRC Press.
13. Rao, S.S. (2003). *Mechanical Vibrations*, 4th edition prentice Hall, Inc.
14. Onundi, L.O. and Adeniji, F.A. (2000). Application of the methods of Initial parameter and substitute cantilever to dynamic analysis of a multi-storey steel framed building. *Journal of Construction Technology and management*, vol.3, No.12000, pp.80-90.
15. Chopra, A.K. (2001) *Dynamics of Structures, theory and applications to earthquake engineering*, 2nd edition, prentice hall, Inc.