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ANALYTICAL BENDING SOLUTION OF ALL CLAMPED ISOTROPIC RECTANGULAR PLATE ON WINKLER'S FOUNDATION USING CHARACTERISTIC ORTHOGONAL POLYNOMIAL

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ABSTRACT
The analytical bending solution of all clamped rectangular plate on Winkler foundation using characteristic orthogonal polynomials (COPs) was studied. This was achieved by partially integrating the governing differential equation of rectangular plate on elastic foundation four times with respect to its independents x and y axis. The foundation was assumed to be homogeneous, elastic and isotropic. The governing differential equation was non-dimensionalised to make it consistent. The deflection polynomials functions were formulated. Thereafter, the Galerkin's works method was applied to the governing differential equation of the plate on Winkler foundation to obtain the deflection coefficient, W_{uv}. Numerical example was presented at the end to compare the results obtained by this method and those from earlier studies. The percentage difference obtained for central deflection of all clamped rectangular plate loaded with UDL using the method and earlier research works for K = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 are: 0.000042, 0.000052, 0.000043, -0.000011, -0.000068, 0.000001, 0.000001, 0.000001, -0.000001, 0.000000, 0.000001. 0.000033, 0.000035, 0.000033, -0.000018, -0.000072, -0.000003, -0.000003, -0.000003, 0.000002, -0.000002, -0.000001. The result showed that an easy to use and understandable model was developed for determination of deflections of all clamped rectangular plates on Winkler's elastic foundation using principle of COPs.

Keywords: *Analytical solution, Winkler foundation, Isotropic rectangular plate, Characteristic Orthogonal Polynomials, works method.*

1. INTRODUCTION

The study of theoretical analysis of rectangular plates on elastic foundation is very important since they have wide application in structural engineering such as foundations, storage tanks, swimming pools, floor system of buildings, highways, air field pavement, etc. Therefore, the deflection of these plates have been studied by different analytical and numerical methods.

Cauchy and Poisson, were first to formulate the problem of plate bending based on general equations of elasticity. They obtained the governing differential equation for deflections that coincides completely with the well-known Lagrange equation [1]. However, the first satisfactory theory of bending of plates is associated with Navier, who considered the plate thickness in the general plate equation as a function of rigidity. Navier equally introduced a method known as the "exact" method. This method was used to transform the differential equation into algebraic expressions using Fourier trigonometric series [1].

Timoshenko and Woinowsky-Krieger [2] studied the analytical bending of rectangular plates on elastic foundation using Fourier trigonometric series. They assumed that the bottom plate is loaded by the elastic reactions of the foundations which at any part of the bottom of the plate is proportional to the deflection at that point. Using Navier's solution, they were able to formulate the deflection equation for a rectangular plates on elastic foundation in the form of a trigonometric series. One of the limitations of the trigonometric series is that it is extremely difficult to formulate the shape functions of some rectangular plates with unsymmetrical edges [3].

The fundamental problem in the analysis of structure-foundation problem lies in the determination of the constant pressure which depends on the subgrade. This is as a result of the complexity of the real behaviour of foundations. This led to the formulation of foundation models of which the simplest of them is the Winkler's model which assumes that the vertical displacement of the soil media at any point on the surface is directly

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proportional to the load applied at that point and independent of the loads applied at other locations [4]. This model can be considered as an idealization of the soil medium by a number of mutually independent spring elements [5].

Al-Khaiat and West [6] developed an approximate solutions for rectangular plates on elastic foundation. Their work was based on Winkler model. They used initial value method and presented the dimensionless deflection of the plate. Ozgan and Daloglu [7] used a computer coded program based on finite element method to analyse thin and thick plates on elastic foundations. In their study, a four-noded plate bending quadrilateral (PBQ4) and an eight-noded plate bending quadrilateral (PBQ8) elements based on Mindlin plate theory were adopted for the analysis of the plates resting on Winkler elastic foundation.

Mishra and Chakrabarti [8] worked on shear and attachment effects on the behaviour of rectangular plates resting on tensionless elastic foundation using finite element techniques. In their work a nine-noded Mindlin element was used to account for transverse shear effects. Liu [9] developed differential equation element method (DQEM) for the static analysis of homogeneous isotropic rectangular plates on Winkler foundation, his formulation was based on the basis of first order shear deformation theory. Daloglu and Vallabhan [10] using non-dimensional parameters attempted to evaluate the value of subgrade reaction modulus for use in the Winkler model for the analysis of slab. They provided graphs from which values of an equivalent subgrade reaction modulus can be computed from the complete geometry and properties of the overall system.

These methods have their different limitations, either as a result of their inability to satisfy a particular plate boundary conditions or the domain equation.

This study tends to come up with an easy to use and understandable model for determination of deflections of all clamped rectangular plate on elastic foundation by providing the design parameters necessary for determination of deflection of the plate.

In this paper, characteristics orthogonal polynomial (COP) shape functions are formulated for the all clamped isotropic rectangular plate on Winkler elastic foundation. The formulation is based on the assumption that the function for a rectangular plate is a product of two functions; one of which is a pure function of x and the other is of y [11]. A detailed analysis is carried out using the dimensionless parameters of both x and y and therefore the deflection co-efficient is generated taking the effect of Winkler foundation into account. The elements are tested with various values of subgrade reactions for an all clamped rectangular plate loaded with uniformly distributed loads.

2. GOVERNING DIFFERENTIAL EQUATION OF RECTANGULAR PLATE ON WINKLER'S FOUNDATION

Figure 1 shows an all clamped rectangular thin plate on Winkler foundation subjected to uniformly distributed load. The length of the plate is a and width b.

The governing differential equation of the plate when it is supported by Winkler foundation is given as;

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{kw}{D} - \frac{q}{D} = 0 \quad (1)$$

Where,

$$D = \frac{Eh^3}{12(-\mu^2)} \quad (2)$$

D is the plate flexural rigidity, E is the Young's modulus of elasticity of the plate, q is the uniformly distributed load, w is the deflection, k is the modulus of subgrade reaction, h is the thickness of the plate and $\mu$is the Poisson's ratio of the plate.

The independent co-ordinates, x and y, of the clamped plate can be expressed in the form of non-dimensional co-ordinates, R and Q, for the x and y directions as in [12] That is:

$$x = aR;\ 0 \le R \le 1 \quad (3)$$

$$y = bQ; \quad 0 \le Q \le 1 \quad (4)$$

Then the derivatives in Equation (1) in non-dimensional co-ordinates R and Q transform as follows

$$w = W_{uv}H; \frac{\partial^4 w}{\partial x^4} = \frac{W_{uv}}{a^4}\frac{\partial^4 H}{\partial R^4};\ \frac{\partial^4 w}{\partial y^4} = \frac{W_{uv}}{b^4}\frac{\partial^4 H}{\partial Q^4};\ \frac{\partial^4 w}{\partial x^2 \partial y^2} = \frac{W_{uv}}{a^2 b^2}\frac{\partial^4 H}{\partial R^2 \partial Q^2} \quad (5)$$

Where, H is Shape function matrix, u and v are the deflection in x and y directionsand w_{uv} is the deflection function. Substituting Equation (5) into Equation (1) gives:

$$\frac{W_{uv}}{a^4}\frac{\partial^4 H}{\partial R^4} + 2\frac{W_{uv}}{a^2 b^2}\frac{\partial^4 H}{\partial R^2 \partial Q^2} + \frac{W_{uv}}{b^4}\frac{\partial^4 H}{\partial Q^4} + \frac{kW_{uv}H}{D} - \frac{q}{D} = 0 \quad (6)$$

Equation (6) is the non-dimensionalised governing differential equation of rectangular plate on elastic foundation.

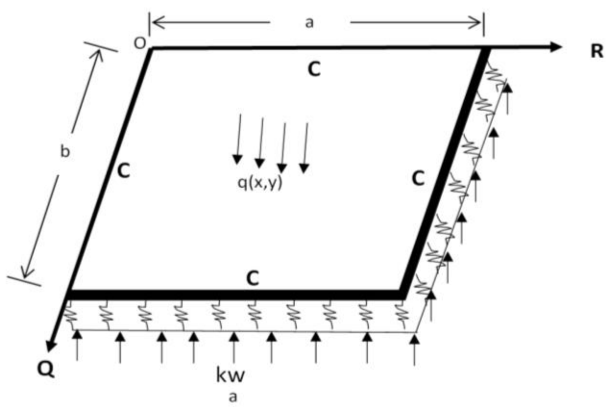

Figure 1: An all clamped rectangular thin plate on elastic foundation

3. CHARACTERISTIC ORTHOGONAL POLYNOMIAL SHAPE FUNCTION OF ALL CLAMPED RECTANGULAR PLATE

Oguaghamba [13] used direct integration method to derive the deflection polynomial function of a rectangular thin isotropic plate subjected to uniformly distributed load. However, Ogunjiofor [14] extended the integration approach to solving plate equation of plate on Winkler's foundation. The deflection in this non-dimensional coordinates satisfied equation (6) which is given as

$$w = \Lambda \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} i_m R^m j_n Q^n \tag{7}$$

Λ is the consolidated coefficient of deflection, i_m and j_n are coefficients to be determined using the boundary conditions of the plate,.

Truncating Equation (7) at the fourth term gives the deflection for the plate as:

$$w = \Lambda[(i_4R^4 + i_3R^3 + i_2R^2 + i_1R^1 + i_0R^0)(j_4Q^4 + j_3Q^3 + j_2Q^2 + j_1Q^1 + j_0Q^0)] \tag{8}$$

$$w = \Lambda w_x . w_y \tag{9}$$

3.1 Boundary Conditions

For an all clamped rectangular plate on Winkler foundation, the prescribed boundary conditions along R and Q directions are given as

For R – directions

$$W_R = 0 \; for \; R = 0 \; or \; 1 \tag{10}$$

$$\frac{\partial W_R}{\partial R} = 0 \; for \; R = 0 \; or \; 1 \tag{11}$$

Therefore,

$$i_0 = 0, \; i_1 = 0, \; i_2 = i_4, \; i_3 = -2\, i_4, \; i_4 = i_4 \tag{12}$$

For Q – directions

$$W_Q = 0 \; for \; Q = 0 \; or \; 1 \tag{13}$$

$$\frac{\partial W_Q}{\partial Q} = 0 \; for \; Q = 0 \; or \; 1 \tag{14}$$

Therefore,

$$j_0 = 0, \; j_1 = 0, \; j_2 = j_4, \; j_3 = -2\, j_4, \; j_4 = j_4 \tag{15}$$

Hence, the deflection function of all round clamped rectangular plate on Winkler foundation is obtained by substituting the coefficients in equations (12 and (15) into equation (8). The substitution gives

$$W(R,Q) = \Lambda i_4 j_4 (R^2 - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4) \tag{16}$$

$$W(R,Q) = w_{uv}(R^2 - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4) \tag{17}$$

Where,

$$w_{uv} = \Lambda i_4 j_4 \tag{18}$$

$$w = W_{uv} H \tag{19}$$

$$H = (R^2 - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4) \tag{20}$$

4. WORK PRINCIPLE APPLICATION FOR ALL CLAMPED PLATE ON WINKLER FOUNDATION

Application of work principle according to Ibearugbulem, Ezeh and Ettu [15] to equation (6) gives:

$$\int_0^1\int_0^1 \left(\frac{{W_{uv}}^2}{a^4}\frac{\partial^4 H}{\partial R^4}.H + 2\frac{{W_{uv}}^2}{a^2b^2}\frac{\partial^4 H}{\partial R^2 \partial Q^2}.H + \frac{{W_{uv}}^2}{b^4}\frac{\partial^4 H}{\partial Q^4}.H + \frac{k{W_{uv}}^2H^2}{D}\right)\partial R \partial Q - ab\frac{q}{D}W_{uv}\int_0^1\int_0^1 H\,\partial R\partial Q = 0 \tag{21}$$

Rearranging Equation (21) gives,

$${W_{uv}}^2 \int_0^1\int_0^1 \left(\frac{1}{a^4}\frac{\partial^4 H}{\partial R^4}.H + 2\frac{1}{a^2b^2}\frac{\partial^4 H}{\partial R^2 \partial Q^2}.H + \frac{1}{b^4}\frac{\partial^4 H}{\partial Q^4}.H + \frac{kH^2}{D}\right)\partial R \partial Q - ab\frac{q}{D}W_{uv}\int_0^1\int_0^1 H\,\partial R\partial Q = 0 \tag{22}$$

Further rearrangement of Equation (22) gives (23) which is at the bottom of this page.

Integration of Shape function matrix H stated in Equation (20) to their various powers in Equation (23) within the boundary limits of 0 to 1 gives,

$$\int_0^1\int_0^1 H\,\partial R\partial Q = 0.001111 \tag{24}$$

$$\int_0^1\int_0^1 H^2\,\partial R\partial Q = 0.00000252 \tag{25}$$

$$\int_0^1\int_0^1 \frac{\partial^4 H}{\partial R^4}.H\,\partial R\partial Q = 0.00127 \tag{26}$$

$$2\int_0^1\int_0^1 \frac{\partial^4 H}{\partial R^2\partial Q^2}.H\,\partial R\partial Q = 0.00073 \tag{27}$$

$$\int_0^1\int_0^1 \frac{\partial^4 H}{\partial Q^4}.H\,\partial R\partial Q = 0.00127 \tag{28}$$

Substituting equations 24 through 28, gives (29) which is at the bottom of this page.

$$W_{uv} = \frac{q\int_0^1\int_0^1 H\,\partial R\partial Q}{D\int_0^1\int_0^1\left(\frac{1}{a^4}\frac{\partial^4 H}{\partial R^4}.H + 2\frac{1}{a^2b^2}\frac{\partial^4 H}{\partial R^2\partial Q^2}.H + \frac{1}{b^4}\frac{\partial^4 H}{\partial Q^4}.H + \frac{kH^2}{D}\right)\partial R\partial Q} \tag{23}$$

$$W_{uv} = \frac{0.001111q}{D\left(0.00127\frac{1}{a^4} + (0.00073\frac{1}{a^2b^2}) + 0.00127\frac{1}{b^4}\right) + 0.00000252k} \tag{29}$$

Table 1: Non dimensional central deflections for the clamped supported plate with uniformly distributed load

			$100q/D$		
K	Ozgan and Daloglu [7]	Mishra and Chakrabarti [8]	Present study	%difference with O&D	%difference with M&C
0	0.1369	0.1360	0.1327	0.000042	0.000033
1	0.1367	0.1350	0.1315	0.000052	0.000035
2	0.1350	0.1340	0.1307	0.000043	0.000033
3	0.1277	0.1270	0.1288	-0.000011	-0.000018
4	0.1114	0.1110	0.1182	-0.000068	-0.000072
5	0.0874	0.0870	0.0873	0.000001	-0.000003
6	0.0622	0.0620	0.0623	0.000001	-0.000003
7	0.0414	0.0410	0.0413	0.000001.	-0.000003
8	0.0267	0.0270	0.0268	-0.000001	0.000002
9	0.0172	0.0170	0.0172	0.000000	-0.000002
10	0.0112	0.0110	0.0111	0.000001	-0.000001

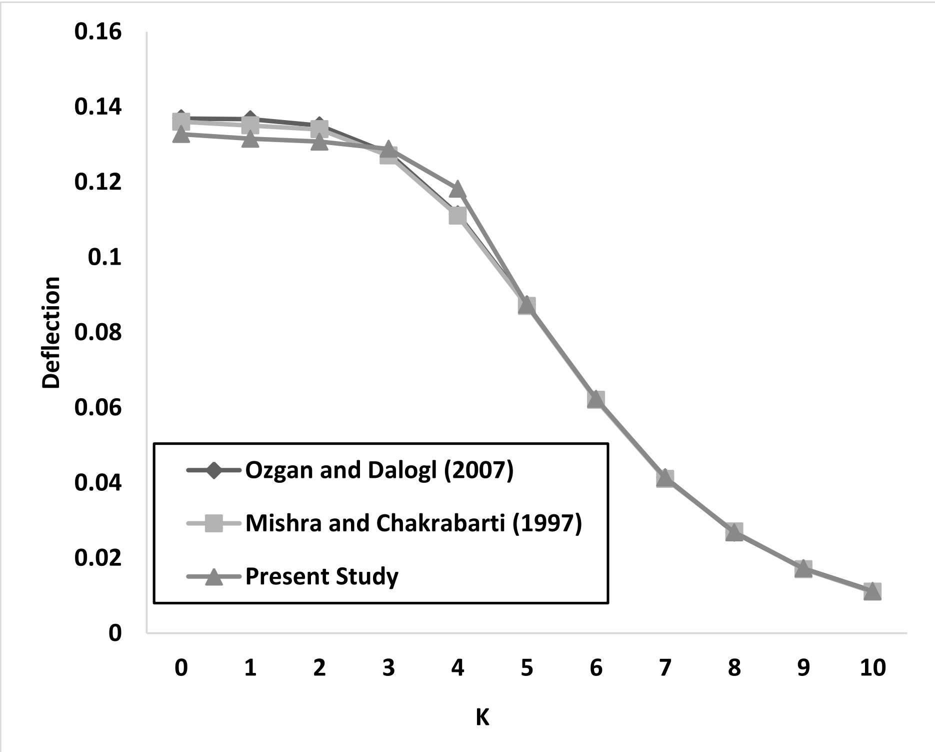

Fig. 2 Variation of the non dimensional central deflection with K for clamped supported plate subject to concentrated load

5. RESULTS AND DISCUSSIONS

The comparison of the results for the non-dimensional central deflections for all clamped supported plate with uniformly distributed load was presented in table 1.0. The K values range:$(0 \leq K \leq 10)$. When the value of soil subgrade K is zero then the structure is equivalent to an ordinary plate, for which the exact solution is found to be in good agreement with the result obtained in Ozgan and Daloglu and Mishra and Chakrabarti. Where K is non dimensional subgrade reaction.

6. CONCLUSION

In this study, an all clamped rectangular plate element on Winkler foundation was analysed using characteristic orthogonal polynomials. The solution was tested using an all clamped plate with uniformly distributed load on Winkler elastic foundation and it gives satisfactory results comparing with other methods available in the literature. It is seen that the approach is very good for analysis of rectangular plate on elastic foundation. Only simple definite integration is needed to obtain a solution for this problem and the governing differential equation can be satisfied throughout the domain of the plate. It is a known shortcoming of solution of some methods that is not possible to select shape functions which exactly satisfy the clamped edge conditions. The problem is completely eliminated here. The mathematical technique described here is applicable to other complicated problems, such as point-supported plates etc.

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