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FIRST PRINCIPLES DERIVATION OF A STRESS FUNCTION FOR AXIALLY SYMMETRIC ELASTICITY PROBLEMS, AND APPLICATION TO BOUSSINESQ PROBLEM

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ABSTRACT
In this work, a stress function is derived from first principles to describe the behaviour of three dimensional axially symmetric elasticity problems involving linear elastic, isotropic homogeneous materials. In the process, the fifteen governing partial differential equations of linear isotropic elasticity were reduced to the solution of the biharmonic problem involving the stress function. $\phi(r,z)$;thus simplifying the solution process. The stress function derived was found to be identical with the Love stress function. The stress function was then applied to solve the axially symmetric problem of finding the stress fields, strain fields and displacement fields in the semi-infinite linear elastic, isotropic homogeneous medium subject to a point load P acting at the origin of coordinates also called the Boussinesq problem. The results obtained in this study for the stresses and displacements were exactly identical with those from literature, as obtained by Boussinesq.

Keywords:_stress function, stress field, strain field, displacement field, axisymmetric problem, Boussinesq problem._

1. INTRODUCTION

Elasticity theory deals with the "exact" formulation of the behaviour of materials – subject to certain fundamental hypotheses when the materials are subject to forces or other disturbing effects [1]. It has extensive applications in the formulation of the behaviour of bars, beams, plates and semi-finite bodies like soils. The general elasticity problem is governed by the material constitutive laws, the geometric or kinematic relations, and the differential equations of equilibrium. Elasticity theory can be used to describe static and dynamic behaviours.

The general elasticity problem is a system of partial differential equations in terms of 15 unknown fields namely: 3 displacement fields u, v, w, six strain fields $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}$ and six stress fields $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}$, and σ_{zx} in addition to suitable displacement and traction boundary conditions [2 - 4].

The general problem of three-dimensional elasticity is very difficult and in most cases impossible to solve mathematically, resulting in closed form analytical solutions being possible in a few cases [5]. The general elasticity problem is solved mathematically by reducing the set of 15 governing partial differential equations to fewer equations and unknowns by a process of elimination. Depending on the primary unknowns of the resulting simplified equations, two formulation approaches are identified namely: displacement formulation and stress formulation.

In the displacement formulation, stresses and strains are eliminated from the general elasticity equations, and a reduced set of equations obtained in terms of displacements (u, v, w) as the primary unknowns. This formulation is useful when the boundary conditions are specified in terms of displacements which are known or prescribed on the boundary. Navier-Lamé presented the displacement formulation of elasticity as the vector equation

$$(\lambda + G)\nabla\cdot(\nabla u) + G\nabla^2 u + f = 0 \qquad (1)$$

In (1), $\lambda$is theLamé's constant, G is the shear modulus, u the displacement vector, f is the body force vector.

In the stress formulation, displacements and strains are eliminated to obtain a reduced set of equations where stresses are the only unknowns. This formulation is useful when the boundary conditions are given in terms of tractions, which are specified on the boundary.

Axisymmetric elasticity problems are elasticity problems involving symmetry with respect to geometry and loading about an axis of the body. Examples of axisymmetric three dimensionalelasticity problems are soil masses of semi-infinite extent subjected to point loads, soil masses of semi-infinite extent subjected to circular footing loads, thick walled pressure vessels.

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In axisymmetricsolid elasticity problems, the radial displacements develop circumferential strains that cause stresses $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}$ and τ_{rz} where $\sigma_{rr}, \sigma_{\theta\theta}and\sigma_{zz}$are normal stresses in the radial, circumferential and longitudinal directions, respectively; τ_{rz} is the shear stress, and r, θ and z represent the radial, circumferential and longitudinal coordinate directions [6 – 8].

2. RESEARCH AIM AND OBJECTIVES

The general aim and objective of this study is to derive from first principles a suitable stress function for axially symmetric three dimensionalproblems of elasticity. The specific objectives are:

(i) to derive stress function for axially symmetric problems of elasticity(or axisymmetric problems of elastic space)

(ii) to apply the stress function so derived in solving the Boussinesq problem of point load acting at the origin of an isotropic homogeneous soil mass of semi-infinite extent.

3. THEORETICAL FRAMEWORK

Elasticity problems must satisfy three basic requirements, namely, the kinematic relations, the material constitutive laws and the differential equations of equilibrium, subject to the boundary conditions of loading and supports[4, 9-16].

These governing equations of the theory of elasticity for three dimensional axisymmetric conditions are given as [2, 8].

$$\varepsilon_{rz} = \frac{\partial u}{\partial r} \tag{2}$$

$$\varepsilon_{\theta\theta} = \frac{u}{r} \tag{3}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \tag{4}$$

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \tag{5}$$

$$\gamma_{\theta z} = 0 \tag{6}$$

$$\gamma_{\theta r} = 0 \tag{7}$$

where$u$, v, and w are displacement components in the r, θ, and z coordinate directions respectively;$\varepsilon_{\theta\theta}, \varepsilon_{rr}, \varepsilon_{zz}$, are normal strains, $\gamma_{\theta z}$, $\gamma_{\theta r}$, γ_{rz} are shear strains.

The stress-strain law for axially symmetric elasticity solidproblems can be written, for isotropic, homogeneous materials as:

$$\varepsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - \mu(\sigma_{\theta\theta} + \sigma_{zz})\right) \tag{8}$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - \mu(\sigma_{rr} + \sigma_{zz})\right) \tag{9}$$

$$\varepsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \mu(\sigma_{\theta\theta} + \sigma_{rr})\right) \tag{10}$$

$$\gamma_{\theta z} = 0 \tag{11}$$

$$\gamma_{\theta r} = 0 \tag{12}$$

$$\gamma_{zr} = \frac{2(1+\mu)}{E}\tau_{rz} = \frac{\tau_{rz}}{G} \tag{13}$$

Where E is the Young's modulus of elasticity, G is the shear modulus, μ is the Poisson's ratio, and

$$G = \frac{E}{2(1+\mu)} \tag{14}$$

The stress-strain law can be expressed in terms of Lamé's constants as follows

$$\sigma_{rr} = \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz},) + 2G\,\varepsilon_{rr} \tag{15}$$

$$\sigma_{\theta\theta} = \lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz},) + 2G\,\varepsilon_{\theta\theta} \tag{16}$$

$$\sigma_{zz} = \lambda(\varepsilon_{rr} + \sigma_{\theta\theta} + \varepsilon_{zz},) + 2G\,\varepsilon_{zz} \tag{17}$$

$$\tau_{\theta z} = 0 \tag{18}$$

$$\tau_{\theta r} = 0 \tag{19}$$

$$\tau_{rz} = G\,\gamma_{rz} \tag{20}$$

where λ is the Lamé's constant, and

$$\lambda = \frac{E\mu}{(1+\mu)(1-2\mu)} \tag{21}$$

The differential equations of equilibrium for axially symmetric deformation of a three dimensional body in the absence of body forces, are obtained from the differential equations of equilibrium of three dimensional elastic bodies when terms involving the derivatives $\partial/\partial\theta$ and$\partial^2/\partial\theta^2$ are made to vanish, in order to fulfill the conditions of axisymmetry as:

For the r direction

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\tau_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \tag{22}$$

For the z direction

$$\frac{\partial\tau_{rz}}{\partial z} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\tau_{rz}}{r} = 0 \tag{23}$$

4. DERIVATION OF STRESS FUNCTION FOR AXISYMMETRIC ELASTICITY PROBLEMS

In axisymmetric elasticity problems, the deformations do not depend on the θ coordinate, hence a stress function for axisymmetric solid elasticity problems is derived as a scalar field of the radial and depth coordinate variables i.e. $\phi(r,z)$; where obviously ϕ is not dependent on the θ coordinate. Following the principles in deriving Airy's stress function for plane strain elasticity, it is assumed that the shear stress, τ_{rz}could be derived from the stress functionsfrom$\phi(r,z)$ as follows:

$$\tau_{rz} = \frac{-\partial^2\phi}{\partial r\partial z} \tag{24}$$

τ_{rz} is required to produce the desired stress function $\phi(r,z)$froman equilibrating stress field, hence it must satisfy all the differential equations of equilibrium. Considering the differential equation of equilibrium in the z direction (in the absence of body forces) given by Equation (23) we have:

$$-\frac{\partial^3\phi}{\partial r^2\partial z} + \frac{\partial\sigma_{zz}}{\partial z} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial z} = 0 \tag{25}$$

Solving for σ_{zz} we obtain:

$$\sigma_{zz} = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} \quad (26)$$

where integration constants are omitted since $\phi(r, z)$ is an arbitrary function of the spatial derivatives with respect to r and z and includes all such integration constants.

From the kinematic relations,

$$u = r\varepsilon_{\theta\theta} \quad (27)$$

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} = \frac{\partial}{\partial r}(r\varepsilon_{\theta\theta}) \quad (28)$$

$$\varepsilon_{rr} = \frac{\partial}{\partial r}\left(r\frac{1}{E}(\sigma_{\theta\theta} - \mu(\sigma_{rr} + \sigma_{zz})\right) \quad (29)$$

Using Equations(8), and (29), we obtain:

$$\sigma_{rr} - \mu(\sigma_{\theta\theta} + \sigma_{zz}) = \frac{\partial}{\partial r}\big(r(\sigma_{\theta\theta} - \mu(\sigma_{rr} + \sigma_{\theta\theta})\big) \quad (30)$$

Upon simplification, we have:

$$\sigma_{rr} - \sigma_{\theta\theta} = \frac{1}{1+\mu} r \frac{\partial}{\partial r}\big(\sigma_{\theta\theta} - \mu(\sigma_{rr} + \sigma_{zz})\big) \quad (31)$$

Let
$$\sigma_{rr} = \frac{\partial^2 \phi}{\partial z^2} + A \quad (32)$$

where A is an unknown we seek to find.

Then, substitution of Equations (32), (26), and (24) into the differential equation of equilibrium in the radial r direction will yield upon simplification,

$$(1+\mu)\frac{\partial A}{\partial r} + \frac{\partial}{\partial r}\big(\sigma_{\theta\theta} - \mu(\sigma_{rr} + \sigma_{zz})\big) = 0 \quad (33)$$

let
$$\sigma_{\theta\theta} = \mu\nabla^2\phi - A \quad (34)$$

wherein
$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial}{\partial z^2} \quad (35)$$

∇^2 is the Laplacian operator in axisymmetric coordinates.

Then,

$$\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz} = (1+\mu)\nabla^2\phi \quad (36)$$

Since

$$\nabla^2(\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz}) = 0 \quad (37)$$

It follows that

$$\nabla^2(1+\mu)\nabla^2\phi = 0 \quad (38)$$

Thus $\phi(r, z)$ is a biharmonic function of r and z

Then,

$$u = r\varepsilon_{\theta\theta} = \frac{-r}{E}(1+\mu)A \quad (39)$$

$$\gamma_{rz} = - = \frac{\partial^2 \phi}{\partial r\partial z} = \frac{E}{2(1+\mu)}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \quad (40)$$

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = -\frac{2(1+\mu)}{E}\frac{\partial^2 \phi}{\partial r\partial z} \quad (41)$$

$$\frac{\partial w}{\partial r} = \frac{-2(1+\mu)}{E}\frac{\partial^2 \phi}{\partial r\partial z} - \frac{\partial u}{\partial z} \quad (42)$$

$$\frac{\partial w}{\partial r} = \frac{-2(1+\mu)}{E}\frac{\partial^2 \phi}{\partial r\partial z} + \frac{r(1+\mu)}{E}\frac{\partial A}{\partial z} \quad (43)$$

$$\frac{\partial w}{\partial r} = \varepsilon_{zz} = \frac{1}{E}\left((1-\mu^2)\nabla^2\phi - (1+\mu)\frac{\partial^2 \phi}{\partial z^2}\right) \quad (44)$$

For compatibility of the expressions for $\frac{\partial w}{\partial r}(r, z)$ and $\frac{\partial w}{\partial z}(r, z)$ we require that:

$$\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial w}{\partial z}\right) \quad (45)$$

Hence, upon simplification,

$$r\frac{\partial^2 A}{\partial z^2} = (1-\mu)\frac{\partial}{\partial r}\nabla^2\phi + \frac{\partial^3 \phi}{\partial r\partial z^2} \quad (46)$$

Let
$$rA = \frac{\partial \phi}{\partial r} + \frac{\partial \beta}{\partial r} \quad (47)$$

Then
$$\frac{\partial^2 \beta}{\partial z^2} = (1-\mu)\nabla^2\phi \quad (48)$$

Then we obtain from the equations for $\partial w/\partial r$ and $\partial w/\partial z$,

$$\frac{\partial w}{\partial r} = -\frac{(1+\mu)}{E}\frac{\partial^2 \phi}{\partial r\partial z} + \frac{1+\mu}{E}\frac{\partial^2 \beta}{\partial r\partial z} \quad (49)$$

$$\frac{\partial w}{\partial z} = \frac{(1+\mu)}{E}\left(\frac{\partial^2 \beta}{\partial z^2} - \frac{\partial^2 \phi}{\partial z^2}\right) \quad (50)$$

By integration with respect to r,

$$w = \frac{-(1+\mu)}{E}\left(\frac{\partial \phi}{\partial z} - \frac{\partial \beta}{\partial z}\right) \quad (51)$$

Similarly,

$$u = \frac{-(1+\mu)}{E}\left(\frac{\partial \phi}{\partial r} + \frac{\partial \beta}{\partial r}\right) = -\frac{(1+\mu)}{E}\frac{\partial}{\partial r}(\phi + \beta) \quad (52)$$

$$\text{Let}\ \varphi(r, z) = \phi(r, z) + \beta(r, z) \quad (53)$$

Then

$$\sigma_{rr} = \frac{\partial^2 \phi}{\partial z^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r}\frac{\partial \beta}{\partial r} \quad (54)$$

$$\sigma_{rr} = \frac{\partial^2}{\partial z^2}(\varphi - \beta) + \frac{1}{r}\frac{\partial}{\partial r}(\varphi - \beta) + \frac{1}{r}\frac{\partial \beta}{\partial r} \quad (55)$$

$$\sigma_{rr} = \frac{\partial^2 \varphi}{\partial z^2} - \frac{\partial^2 \beta}{\partial z^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} - \frac{1}{r}\frac{\partial \beta}{\partial r} + \frac{1}{r}\frac{\partial \beta}{\partial r} \quad (56)$$

$$\sigma_{rr} = \nabla^2\varphi - \frac{\partial^2 \varphi}{\partial r^2} - (1-\mu)\nabla^2\phi \quad (57)$$

$$\sigma_{rr} = \mu\nabla^2\varphi - \frac{\partial^2 \varphi}{\partial r^2} \quad (58)$$

$$\sigma_{\theta\theta} = \mu\nabla^2\varphi - \frac{1}{r}\frac{\partial \varphi}{\partial r} \quad (59)$$

$$\sigma_{zz} = (2-\mu)\nabla^2\phi - \frac{\partial^2 \varphi}{\partial z^2} \quad (60)$$

$$\text{Let}\ \varphi = \frac{\partial}{\partial z}\Phi \quad (61)$$

$$\sigma_{rr} = \frac{\partial}{\partial z}\left[\mu\nabla^2\Phi - \frac{\partial^2}{\partial r^2}\Phi\right] \quad (62)$$

$$\sigma_{\theta\theta} = \frac{\partial}{\partial z}\left[\mu\nabla^2\Phi - \frac{1}{r}\frac{\partial}{r\partial}\Phi\right] \quad (63)$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left[(2-\mu)\nabla^2\phi - \frac{\partial^2 \Phi}{\partial z^2}\right] \quad (64)$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left[(1-\mu)\nabla^2\Phi - \frac{\partial^2 \Phi}{\partial z^2}\right] \quad (65)$$

We observe that upon substitution of the stresses into the differential equations of equilibrium for body forces absent, Φ satisfies the biharmonic equation, given by Equation (66)

$$\nabla^4\Phi = 0 \quad (66)$$

The displacements are obtained as:

$$u(r,z)=\frac{-(1+\mu)}{E}\frac{\partial^2\Phi}{\partial r\partial z}=-\frac{1}{2G}\frac{\partial^2\Phi}{\partial r\partial z} \quad (67)$$

$$w(r,z)=\frac{1+\mu}{E}\left[(1-2\mu)\nabla^2\phi+\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}\right] \quad (68)$$

$$w(r,z)=\frac{1}{2G}\left[2(1-\mu)\nabla^2\phi-\frac{\partial^2\Phi}{\partial z^2}\right] \quad (69)$$

The solution to the axially symmetric solidelasticity problem then simplifies to finding the biharmonic scalar function of the axially symmetric spatial coordinates *r*, *z*that satisfies the particular boundary conditions of the problem. We observe that Equations (62), (65) and Equations (67 – 69) are identical with Love's stress function, also called Love's displacement function (or Love's displacement potential).

5. APPLICATION OF THE STRESS FUNCTION TO THE BOUSSINESQ PROBLEM OF POINT LOAD ON ELASTIC HALF SPACE SOIL

The Boussinesq problem which is a classical problem in soil mechanics is the problem of determining stress fields and displacement fields due to a point load P acting on the surface of a linear elastic, homogeneous isotropic half space soil [17]. The point load is assumed to be acting at the origin of the three dimensional coordinate system, as shown in Figure 1, rendering the problem an axisymetric problem of elasticity, with the axis of load application being the axis of symmetry.

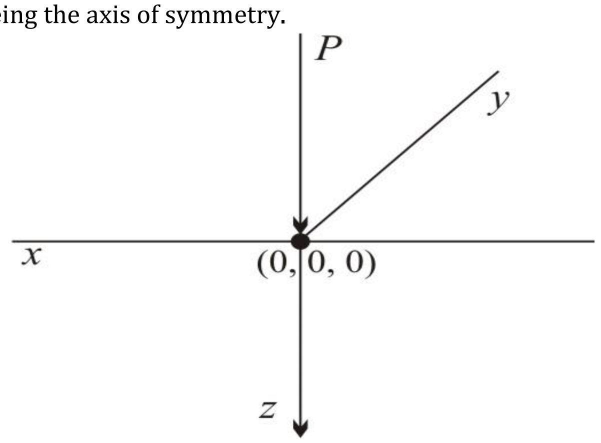

Figure1: Point load on elastic half space soil

We consider the biharmonic function:

$$\Phi(r,z)=c_0PR \quad (70)$$

where $R=(r^2+z^2)^{1/2}$, where c_0 is a constant, $P=$ point load. The stresses are:

$$\sigma_z=-c_0P\left((1-2\mu)\frac{z}{R^3}+\frac{3z^3}{R^5}\right) \quad (71)$$

$$\tau_{rz}=-c_0P\left((1-2\mu)\frac{r}{R^3}+\frac{3z^2r}{R^5}\right) \quad (72)$$

$$\sigma_z(z=0,r\neq 0)=0 \quad (73)$$

$$\tau_{rz}(z=0,r\neq 0)=\frac{-c_0P(1-2\mu)}{r^2} \quad (74)$$

The non-zero value of $\tau_{zr}(z=0,r\neq 0)$ shows that the shear traction boundary condition on the *xy* coordinate plane ($z=0$) cannot be satisfied by the chosen biharmonic function. We adjust the solution, using a superposition principle to ensure that shear traction boundary conditions are satisfied on the surface $z=0$. The singularity of the problems at the point of application of the point load (origin) suggests that we assume the Lamé strain potential function as the logarithmic function:

$$\Phi_l=c_1P\ln(R+z) \quad (75)$$

where c_1 is a constant, and ln is the natural logarithm. Then displacements and stresses due to this Lamé's potential are:

$$2Gu_r=\frac{\partial\phi_l}{\partial r}=\frac{\partial}{dr}c_1P\ln(R+z) \quad (76)$$

$$2Gw=\frac{\partial\phi_l}{\partial z}=\frac{\partial}{\partial z}c_1P\ln(R+z) \quad (77)$$

$$\sigma_{rr}=\frac{\partial^2\phi_l}{\partial r^2}=\frac{\partial^2}{\partial r^2}\big(c_1P\ln(R+z)\big) \quad (78)$$

$$\sigma_{\theta\theta}=\frac{1}{r}\frac{\partial\phi_l}{\partial r}=\frac{1}{r}\frac{\partial}{dr}\big(c_1P\ln(R+z)\big) \quad (79)$$

$$\tau_{rz}=\frac{\partial^2\phi_l}{\partial r\partial z}=\frac{\partial^2}{\partial r\partial z}\big(c_1P\ln(R+z)\big) \quad (80)$$

$$\sigma_{zz}=\frac{\partial^2\phi_l}{\partial z^2}=\frac{\partial^2}{\partial z^2}\big(c_1P\ln(R+z)\big) \quad (81)$$

$$\text{Then }\ \sigma_{\theta\theta}=\frac{c_1P}{R(R+z)} \quad (82)$$

$$\sigma_{rr}=c_1P\left(\frac{z}{R^3}-\frac{1}{R(R+z)}\right) \quad (83)$$

$$\sigma_{zz}=-c_1P\ \frac{z}{R^3} \quad (84)$$

$$\tau_{zr}=-\frac{c_1Pr}{R^3} \quad (85)$$

The normal and shear stresses on the boundary ($z=0$) of the semi-infinite soil are:

$$\sigma_{zz}(z=0,r\neq 0)=0 \quad (86)$$

$$\tau_{zr}(z=0,r\neq 0)=-\frac{c_1P}{r^2} \quad (87)$$

The boundary conditions are:

$$\sigma_{zz}(z=0,r\neq 0)=0 \quad (88)$$

$$\tau_{zr}(z=0,r\neq 0)=0 \quad (89)$$

$$\iint \sigma_{zz}(r,a)r\,dr\,dz+P=0 \quad (90)$$

Hence from the boundary condition on$\tau_{zr}(z=0,r\neq 0)=0$ we have:

$$-c_0\frac{P(1-2\mu)}{r^2}-\frac{c_1P}{r^2}=0 \quad (91)$$

$$\therefore c_1=-c_0(1-2\mu) \quad (92)$$

$$\int_0^\infty 2\pi r\sigma_{zz}(r,a)\,dr+P=0 \quad (93)$$

$$-2\pi c_0P\int_0^\infty\left(\frac{(1-2\mu)ar}{(r^2+a^2)^{3/2}}+\frac{3a^3r}{(r^2+a^2)^{5/2}}\right)dr-2\pi c_1P\int_0^\infty\frac{ar}{(r^2+a^2)^{3/2}}dr+P=0 \quad (94)$$

But,using Equation (92), we obtain:

$$-2\pi c_0 P\int_0^\infty\left(\frac{(1-2\mu)ar}{(r^2+a^2)^{3/2}}+\frac{3a^3r}{(r^2+a^2)^{5/2}}\right)dr$$

$$+2\pi c_0(1-2\mu)P\int_0^\infty\frac{ar}{(r^2+a^2)^{5/2}}dr+P=0 \quad (95)$$

$$-2\pi c_0 P(1-2\mu)\int_0^\infty\frac{ar}{(r^2+a^2)^{3/2}}dr$$

$$-2\pi c_0 P\int_0^\infty\frac{3a^3r}{(r^2+a^2)^{5/2}}dr$$

$$+2\pi c_0(1-2\mu)P\int_0^\infty\frac{ar}{(r^2+a^2)^{3/2}}dr+P=0 \quad (96)$$

$$P=2\pi c_0 P\int_0^\infty\frac{3a^3r}{(r^2+a^2)^{5/2}}dr \quad (97)$$

$$\text{Let } (r^2+a^2)=u \quad (98)$$

$$\frac{du}{dr}=2r \quad (99)$$

$$2rdr=du \quad (100)$$

$$rdr=\frac{du}{2} \quad (101)$$

$$P=2\pi c_0 P\int_0^\infty\frac{3a^3}{u^{5/2}}\frac{du}{2}=3\pi c_0 P\int\frac{a^3du}{u^{5/2}} \quad (102)$$

$$P=3\pi c_0 Pa^3\int_0^\infty u^{-5/2}du \quad (103)$$

$$P=3\pi c_0 Pa^3\left[\frac{u^{-3/2}}{-3/2}\right]_0^\infty \quad (104)$$

$$P=-\frac{2}{3}3\pi c_0 Pa^3\left[u^{-3/2}\right]_0^\infty \quad (105)$$

$$P=-2\pi c_0 Pa^3\left[(r^2+a^2)^{-3/2}\right]_0^\infty \quad (106)$$

$$P=-2\pi c_0 Pa^3(-a^{-3}) \quad (107)$$

$$P=+2\pi c_0 P \quad (108)$$

$$c_0=\frac{1}{2\pi} \quad (109)$$

$$c_1=-\frac{(1-2\mu)}{2\pi} \quad (110)$$

Then by superposition, the displacement and stresses are found as:

$$u_r=\frac{(1+\mu)P}{2\pi ER}\left[\frac{rz}{R^2}-\frac{(1-2\mu)r}{R+z}\right] \quad (111)$$

$$w=\frac{(1+\mu)P}{2\pi ER}\left(2(1-\mu)+\frac{z^2}{R^2}\right)=u_z \quad (112)$$

$$u_\theta=0 \quad (113)$$

$$\sigma_{rr}=\frac{P}{2\pi R^2}\left(\frac{(1-2\mu)R}{R+z}-\frac{3r^2z}{R^3}\right) \quad (114)$$

$$\sigma_{\theta\theta}=\frac{(1-2\mu)P}{2\pi R^2}\left[\frac{z}{R}-\frac{R}{R+z}\right] \quad (115)$$

$$\sigma_{zz}=\frac{3P}{2\pi}\frac{z^3}{R^5} \quad (116)$$

$$\tau_{rz}=-\frac{3Prz^2}{2\pi R^5} \quad (117)$$

6. DISCUSSIONS

A stress function$\Phi(r,z)$ has been successfully derived for axially symmetric problems of elasticity of linearly elastic, homogeneous isotropic materials. The derivation was based on first principles, and the stress function was derived to satisfy identically, the differential equations of equilibrium for axially symmetric problems of elasticity; as well as the material constitutive laws of axially symmetric elasticity. The stress function $\Phi(r,z)$ derived was presented as Equations (62) – (65) in terms of the stresses, and the displacements were presented in terms of the stress functions as Equations (67) and (68), or (67) and (69). It was found that the stress function was identical with the Love stress function for elasticity. The stress function was then applied to the Boussinesq problem to find the solution of stresses and displacement fields in an axisymmetric half space soil due to a point load acting at the origin. The solutions of the Boussinesq problem found using the derived stress function were presented as Equations (111) – (117). The solutions obtained in this work agreed exactly with Boussinesq solutions obtained using the Boussinesq potentials.

It can be further observed that the stresses become infinite and singular at the origin ($z=0$, $r=0$) which is the point of application of the point load, P. Both displacement components u and w tend to zero as the distance R becomes larger; for $r\to 0$ at the point of application of the concentrated load, the displacement becomes infinitely large. This again is as a result of the singularity in the surface load, since in the origin, the stress is singular.

7. CONCLUSIONS

The following conclusions can be made from this work:

(i) the stress function derived satisfies all the differential equations of equilibrium for axisymmetric elastic space (three dimensional axisymmetric elasticity problems).

(ii) the stress function derived satisfies the stress strain laws for axisymmetric elastic solid problems.

(iii) the stress function derived satisfies the strain-displacement relations for linear small displacement elasticity problems.

(iv) the stress function derived was identical with the Love stress function.

(v) the stress function derived was successfully applied to solve the Boussinesq problem and results obtained agree with results from literature.

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