A NOTE ON STRESS FUNCTION DISCONTINUITIES IN PLANE PLASTIC BENDING AND TORSION

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ABSTRACT

The variational and differential equation formulations of the stress function problem for combined plane bending and torsion of a fully plastic material are discussed. The nature of discontinuities to be expected as well as the actual degree of correspondence between the two formulations is determined. A modification of the variational integrand reveals the pattern of discontinuity which is then completely determined. This disposes of some mathematical issues raised by Steele and Imegwu in their relaxation numerical analysis of the problem.

1. INTRODUCTION

The formulation of problems of perfectly plastic structures in terms of variational principles as well as differential equations is well known [1-8]. For a Levy-Mises material which is rigid/plastic and non-hardening, the formulation leads to a stress function satisfying the Handelmann equation, a non-linear partial differential equation of the second order. Steele [1] and Imegwu [2, 3] using quasi-harmonic relaxation methods obtained approximate numerical solutions of the equations subject to natural boundary conditions. Incidentally Piechnik [4, 5] found that it is much more convenient to solve the same problem in terms of displacement rather than stress function. This is presumably to avoid certain discontinuities which the stress-wise formulation entails. The nature of these discontinuities which tend naturally to vitiate the accuracy near the neutral, axis in the case of combined flexure and torsion, needs clarification. This is because of some questionable assumptions made in Steele's paper. This note provides a clarification and the true relationship between the variational and differential formulations of the particular problem. A short while before his death, Dr. Imegwu+ introduced the author to the issue of investigating the mathematical relationships of the various formulations and analysis of discontinuities.

2. THE ORIGIN AND INTRINSIC NATURE OF THE PROBLEM

The general case of two dimensional fully plastic state leads to consideration of the variational integral

$$\iint \left[(ax+by+c)(1-\phi_x^2)^{\frac{1}{2}}+\mu\phi\right]dx\,dy.$$

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$$(ax+by+c)(1-\phi_x^2)^{\frac{1}{2}}$$

positive. When properly written, the expression should read

$$\frac{W}{kL}=\iint \left|(ax+by+c)(1-\phi_x^2)^{\frac{1}{2}}+\mu\phi\right|dx\,dy.$$

with $dxdy=0$ on the contours. The innovation here is the modulus. Steele [1] considered Handelmann's equation in the form

$$\frac{\partial}{\partial x}\left[X\frac{\partial\phi}{\partial x}\right]+\frac{\partial}{\partial y}\left[X\frac{\partial\phi}{\partial y}\right]+u=0$$

where $X=y(1-\phi_x^2-\phi_y^2)^{\frac{1}{2}}$ and $\phi=0$ on the boundary, and claimed that when $y=0$, $x=0$ unless $\phi_x^2+\phi_y^2=1$ in which case X is intermediate. He in fact is further stated that if the ϕ-surface is smooth at $y=0$, then $\phi_y=0$ in which case $\phi_x=1$ and X is again intermediate. From an examination of the expanded form of the Handelmann equation it appeared likely that the ϕ-surface comes to a cusp along the y-axis or part of it. His numerical solutions had been approached with this point in mind. These assumptions are examined closely in the next section. Steele noted quite rightly that a numerical procedure which would essentially smooth out any ridges in the ϕ-surface may be then in considerable error near the bending axis. The error will show up most

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severely in the stresses τ_{yz} and σ_z near the x-axis but their effect on the corresponding moments and torques was likely to be negligible. In contrast, however, later in the same reference Steele asserted that the shape of the ϕ -surfaces indicates the nature of the singularities on the x-axis. The surfaces, he submitted, form a cusp on the x-axis and this shows up in the numerical solution in spite of the fact that its existence was ignored in obtaining the solution. He concluded that the ridge or cusp on the x-axis, however cannot be ignored in computing derivatives. Similar remarks occur in reference [3]. In each solution obtained, no particular assumptions were made about the nature of the function ϕ and X on the neutral axis. However, the use of the finite difference method was thought to imply the assumption of smoothness of ϕ and X everywhere on the section and no attempt was made to justify that.

3. RESOLUTION OF THE DILEMMA

The first fundamental thing is that the difficulties with the numerical treatment of the Handelmann equation occur as a result of the essentially incomplete formulation of the original problem. It is usually the case that there is a faithful two-way equivalence between the variational formulation of a field problem and its differential formulation by way of the related Euler-Lagrange equation. This equivalence holds in most of the problems of classical linear field theory subject to certain conditions which are fulfilled in this plasticity situation. The Euler-Lagrange equation is in the first instance only a necessary condition but even then it does not exist at all points in general, its behaviour depending on the smoothness properties of the integrand in the variational integral. In this particular case, it is therefore clear that the Handelmann's equation is the Euler-Lagrange equation of the variational formulation only in so far as the integrand is continuously differentiable. But here it is not even differentiable. Since $|z|$ is not differentiable at z =0, it may be concluded that the Handelmann's equation is inapplicable when $(ax+by+c)(1-\phi_x^2-\phi_y^2)^{\frac{1}{2}}=0$. Hill [7] stated that ϕ may have discontinuities in its derivatives. It is in fact very easy to show that there exists no solution ϕ which is continuously differentiable everywhere on the domain. If there exists such a ϕ in any closed and bounded (compact) subset of the plane, then there must be at least one stationary point, and thereby hypothesis $\phi_x=\phi_y=0$. Even if $(ax+by+c)\neq 0$ at any such point, Handelmann's equation cannot be satisfied for $\mu\neq 0$. Since $(1-\phi_x^2-\phi_y^2)\neq 0$ at any of these stationary points, this manifestly leads to a contradiction. Further by Wierstrass's maxima and minima theorem, it can be deduced that ϕ must attain its greatest or least values either on *y=0* as in the case considered by Steele (a=c=O, b=l) or on the boundary of the domain. Since $(\phi_x^2+\phi_y^2)<1$, ϕ is bounded and bounds must be attained in a closed finite domain. Thus, at least one of the derivatives ϕ_x , ϕ_y must be discontinuous on y=O. The function ϕ thus has a line cusp precisely of the sort with finite one-sided derivatives. This determines fully the stress discontinuity pattern which could have been absolutely predicted prior to numerical computation. Since the desired function turns out piecewise smooth and since the finite-difference method is capable of handling such situations in addition to globally smooth functions, no error is introduced by virtue of choice of the finite-difference or relaxation methods. An application of the finite-element technique would of course prove interesting.

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