STABILITY MATRICES FOR LATERAL BUCKLING ANALYSIS OF BEAMS

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ABSTRACT

In the present work, a new formulation for lateral buckling of beams comprising bi-symmetric sections has been proposed. The formulation employs a coupled lateral buckling functional to investigate the lateral buckling behaviour of a class of beams comprising bi-symmetric sections. While retaining the coupled modes of displacements at buckling, the formulation focuses attention on the need to reduce the number of degrees of freedom per element so *that the solution process can be carried out on small microcomputers. This effort will be of interest to design offices, which have investments in microcomputers to ease their routine designs. The results obtained from the present formulation compare well with those published in the literature. There is also a gain in the fact that the formulation can be programmed either in basic or FORTRAN and with efficient programming, a saving in computer time can be made.*

1.0 INTRODUCTION

Lateral buckling of beams has been found to pose many engineering challenges both to the researchers and the design engineer. However, following the pioneering analytical works of Michell [1] and Prandtl [2], each working independently of the other and each working on the stability of long beams under transverse forces, there followed numerous analytical contributions to the theory of lateral buckling of beams in the literature (e.g. see refs [3-6]).

However, with the development in computer technology (cheaper and more efficient computers are now available), emphasis has shifted from both analytical and experimental to the numerical solution of the lateral buckling problem. In this direction names that quickly come to mind are: Powell and Klingner [7] have reported their findings on lateral buckling analysis of steel I-section beams. They employed a two equilibrium state and sate formulation in which warping element displacement was included. In the present work, it is assumed that warping is restrained. This is done so as to reduce the size of the problem so that microcomputers can be used. Further more, uncoupled formulation is that employed to test the accuracy of the two equilibrium states formulation first proposed by Powell and Klingner [7],Jiki [9] and Jiki [10]. With this reduction, any method well-written and portable (micro) computer program for lateral buckling analysis such as the one reported by Jiki [8] can be employed to extract the eigenvalues of interest. Nethercot and Rockey [11] have also reported their findings on the finite element solutions for buckling of columns and beams in which they investigated the effect of lateral restraint and partial yielding of the beam section. The present effort employs elastic analysis and focuses attention on lateral buckling displacements and angles of twist of the beam.

One of the most recent works on lateral buckling of beams by the finite element method has been reported by Attard [12] in which he investigated the behaviour of a general thin-walled section in the derivation of his stability matrices. However, such stability matrices need not be used directly to solve bi-symmetric problems because both monosymmetric and asymmetric parameters have already been included in the integration of elements of the matrices. Another recent work on lateral buckling of beams employing the use of the finite element method has come from Kitipornchai and Chan [13]. However, they too have employed mono symmetric and asymmetric parameters to derive elements of their stability matrices. Therefore use of their stability matrices directly to solve bi-symmetric problems would introduce errors due to inclusion of above-mentioned parameters.

It can be seen from the above review that although bi-symmetric beams are among the most widely used beams in Engineering construction, stability matrices employing the finite element method and the concept of two equilibrium states to study the behaviour of such beams using micro-computers are lacking. Thus the purpose of the present effort is to provide such matrices

2.0 ASSUMPTIONS

The following assumptions are made in the present effort

1 The beams used for the present work are prismatic
2 Loads are applied at the vertical axis plane and at the top flange of the beam element only.
3 Effect of load height on lateral buckling is not considered.
4 Only beams of bi-symmetric sections are considered.
5 Uniform torsion of the beam element has been assumed.
6 Uniform bending of beams and cantilevers is assumed.

3.0 TOTAL POTENTIAL ENERGY FUNCTIONAL

The total potential Energy of a bi- symmetric thin-walled element shown in fig. 1 is given as: $\pi_e = U - W$ (1)

In which U is the strain energy of the element and W is the work done by the loads. For a thin-walled beam element considered herein, the strain energy is given as:

$$U = \frac{1}{2}\int_0^L \left[EI_X \frac{d^2 v}{dz^2} + EI_Y \frac{d^2 u}{dz^2} + GJ\left(\frac{d\theta}{dz}\right)^2 \right] dz \quad (2)$$

In which E and G= Youngs modulus and shear modulus respectively, I_x and I_Y =.The second moments of area about the principal X and Y axes. J is St. Venant torsion constant. Equation (2) gives rise to element elastic stiffness matrix [14].

The work done W by the applied loads during deformation of the element is given as: $W = W_1 + W_2$ (3)

In which W_I is due to linear nodal deformations and W_2 is the nonlinear part arising from changes of geometry.

For the present work, there is no axial effect and loads are applied in the y-z plane only (assumptions: Loads and shears are applied in the y-z plane only, prismatic sections are used and we assume that warping torsion is zero i.e. $EI_W = 0$). Therefore the product of applied loads gives W_1 with their respective displacements as:

$$W_1 = [F_{y1}U_1 + M_{z1}\theta_{z1} + M_{y1}\theta_{y1} + F_{y2}U_2 + M_{z2}\theta_{z2} + M_{y2}\theta_{y2}]^T \quad (4)$$

In which F, M, are shears and moments and u, θ are lateral displacements and rotations respectively at the end nodes. The subscripts 1 and 2 refer to the near and far nodes. M, is torsional moment and the angle of twist θ is equal to θ_{z2} - θ_{z1}.

The geometric stiffness matrix

The usual expression for geometric stiffness matrix for a continuum beam element is given as:

$$W_2 = \int_{vol} \sigma_{ij}\varepsilon_{ij}\,dv \quad (5)$$

In which σ_{ij} and ε_{ij} are the stress and quadratic strain tensor respectively. For the present work it is sufficient to employ only direct stress σ_{zz} and ε_{yz} (assumption 2) such that W_2 is given as:

$$W_2 = \int_{vol} \left(\sigma_{zz}\varepsilon_{zz} + 2l_{yz}\varepsilon_{yz}\right)dv \quad (6)$$

For any arbitrary applied. loads in the yz plane the stresses c and t are given as:

$$\sigma_{zz} = \frac{M_z Y}{J} + \frac{M_Y X}{I_y} \quad (7)$$

$$\tau_{yz} = \frac{F_y}{A} \quad (8)$$

In which M, is twisting moment along the element. M_y is moment causing lateral (out of plane) bending, F is shear force and A is area of section. Also for a bi-symmetric section, X and Yare distances from the extreme fibres to the centroidal axes of the section.

The corresponding quadratic strain tensor is given as:

$$\varepsilon_{zz} = \frac{1}{2}\left[\left(\frac{dv_i}{dz}\right)^2 + \left(\frac{du_i}{dz}\right)^2\right] \quad (9)$$

$$\varepsilon_{zz} = \frac{1}{2}\left(\frac{du_i}{dy}.\frac{du_i}{dx} + \frac{dv_i}{dy}.\frac{dv_i}{dx}\right) \quad (10)$$

In which v_i and u_i= the displacement of an arbitrary point on the cross section in the y and x directions respectively Substitution of equations (7),(8),(9) and (10) into equation (6), the continuum work done by W_2 in deforming the element is given

$$W_2 = \frac{1}{2}\int vol\left(\frac{M_{zy}}{I_x} + \frac{M_{yx}}{I_y}\right)\left[\left(\frac{dv_i}{dz}\right)^2 + \left(\frac{du_i}{dz}\right)^2\right]$$

$$dv + \int_{vol} \frac{F_y}{A}\left(\frac{dv_i}{dy}.\frac{dv_i}{dz} + \frac{du_i}{dy}.\frac{du_i}{dz}\right)dv \quad (11)$$

The moments M_z and M_y can be expressed in their nodal values by assuming linear variation as:

$$M_z = M_{z1}\gamma_1 - M_{z2}\gamma 2 \quad (12)$$

$$M_y = M_{x1}\phi\gamma_1 - M_{x2}\phi\gamma 2 \quad (13)$$

In which ϕ is angle of twist (i. e. θ_{z1}-θ_{z2})

$$\gamma_1 = 1 - \frac{z}{L} \quad (14)$$

$$\gamma 2 = \frac{z}{L} \quad (15)$$

The shear force in the horizontal plane F_y is obtained by simple statics

Consider the deformations to be small and finite; we have v_i and u_i in terms of angle of twist θ_z as:

$$V_i = v - \theta_z X \quad (16)$$

$$u_i = u - \theta_z y \quad (17)$$

Substitution of equations (12)-(17) into equation (11) and simplifying gives:

$$W_2 = \frac{1}{2}\int_L \left(M_{z1}\gamma_1 - M_{z2}\gamma_2\right)\left(2\frac{du}{dz}.\frac{d\theta_z}{dz}\right)dz$$

$$+\frac{1}{2}\int_L \left(-M_x\phi\gamma_1 + M_{x2}\phi\gamma_2\right)$$

$$\left(2\frac{dv}{dz}.\frac{d\theta_z}{dz}\right)dz + \int_L F_y\theta_z\frac{du}{dz}dz \quad (18)$$

A set of displacement functions id now proposed as:

$$\begin{Bmatrix} v \\ u \\ \theta_z \end{Bmatrix} = \begin{pmatrix} a_1 a_2 a_3 a_4 \\ b_1 b_2 b_3 b_4 \\ c_1 c_2 c_3 b_4 \end{pmatrix} \begin{Bmatrix} 1 \\ z \\ z^2 \\ z^3 \end{Bmatrix} \quad (19)$$

The displacement functions in equations (19)are usually transformed to element nodal variables r. Therefore in terms of the nodal displacement variables, W_2 is written as:

$$W_2 = \frac{1}{2}r^T\left[K_G\right]r \quad (20)$$

In which $[K_G]$=element geometric stiffness matrix, r is nodal displacement vector and is given here as:

$$r^T = \{u_1\theta_{z1}\theta_{y1}u_2\theta_{z2}\theta_{y2}\}^T \quad (21)$$

In which the components of the vector r^T corresponding to $\{V_1\ q_{xI}\ v_2\ q_{X2}\}$. As shown in figure 1 have been suppressed from equation (21) to focus attention on lateral buckling behaviour only [9, 10]. The element geometric stiffness matrix k_G is presented in appendix A of the present work.

4.0 EXAMPLES ON BIFURCATION ANALYSIS

The validity of the present formulation is tested against three examples employing bifurcation analysis. The software code used for the stability calculation is that reported by Jiki [8].The first of the three examples is a simply supported beam of I -section, The beam is loaded with a constant moment M at both ends. A known closed form solution for this type of problem exists and can be found in Attard [12] and is given here as:

$$M_{cr} = \frac{\pi}{L}\left(EI_y GJ\right)^{1/2} \quad (22)$$

A finite element solution of the beam in question has been carried out employing the present formulation and bifurcation analysis for which a convergence study is presented in fig.2 and compares well with results of studies by Jiki [10] and Attard [12]. The comparison is good. Convergence has been achieved with only 4 elements.

The second example employs a cantilever beam of rectangular cross- section and a bifurcation analysis to test the validity of the proposed formulation is presented in fig.3. Again the software code developed by Jiki [8] has been used in the bifurcation calculations. It can be seen that even after 16 elements convergence has not been achieved. However, the comparison with known solutions by Jiki [10] and Kitipornchai and Chan [13] is good. Infact the present formulation has produced better results with fewer elements and tends to stabilize as the number of elements increases. A closed form solution for the problem can be found in Kitipomchai and Chan [13] and is given as:

$$P_{cr} = \frac{4.013}{L}\left(EI_y GJ\right)^{1/2} \quad (23)$$

The third and last of the examples considered for validation of the proposed theory employs a simply supported beam of rectangular cross-section. The beam is loaded with constant moments M_o at the supports. Once more a bifurcation analysis has produced a set of results, which is presented in Fig.4.The results compare well with known solutions by Jiki [10] and Kitipornchai and Chan [13]. Convergence for the present formulation was achieved with 8 elements while Kitipornchai and Chan [13] achieved convergence with only 6 elements and Jiki [10] has achieved convergence will 12 elements employing the sane software tool reported by Jiki [8]. A closed form solution employing the classical Rayleigh-Ritz method was carried out by Jiki [9] and is reproduced here as:

$$M_{ocr} = \frac{3.46}{L}\left(EI_y GJ\right)^{1/2} \quad (24)$$

It can be seen from the three examples considered here that the present theory is valid for beams of bi-symmetric sections in which uniform (warping torsion is not considered

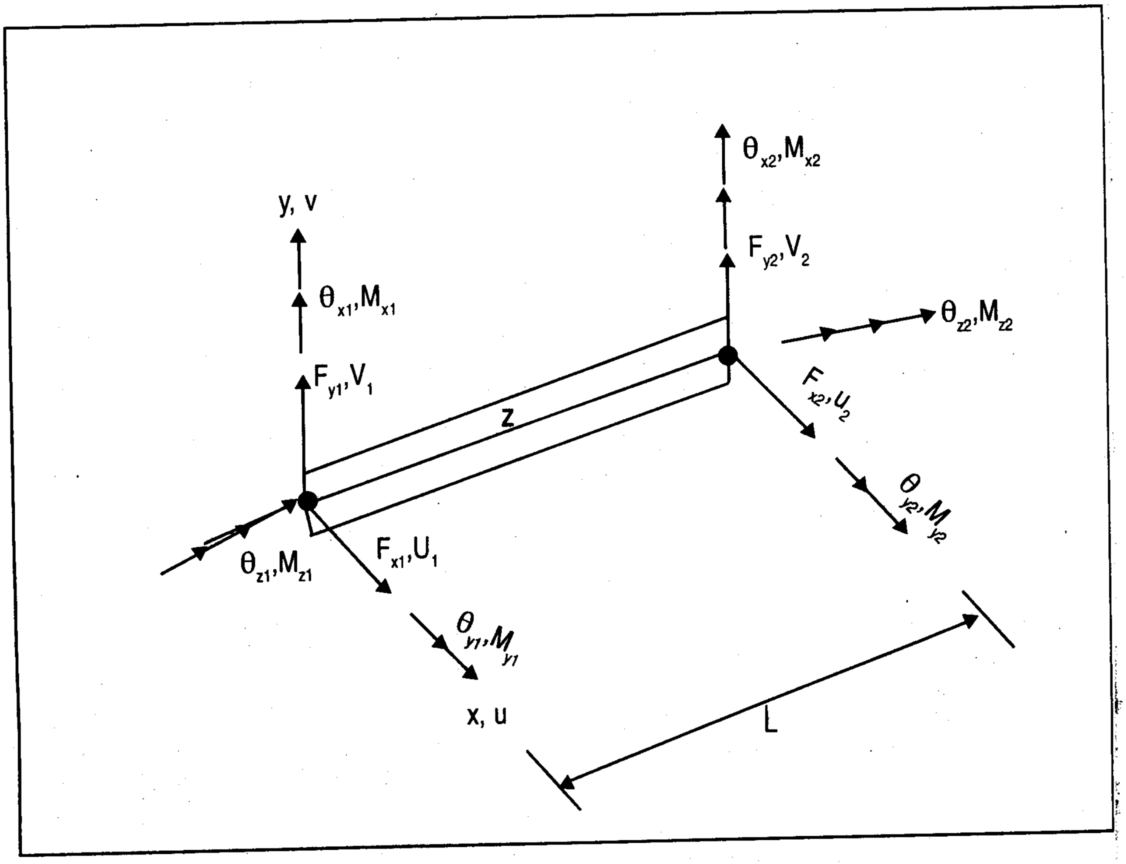

Fig. 1: Prismatic beam element for present Formulation.

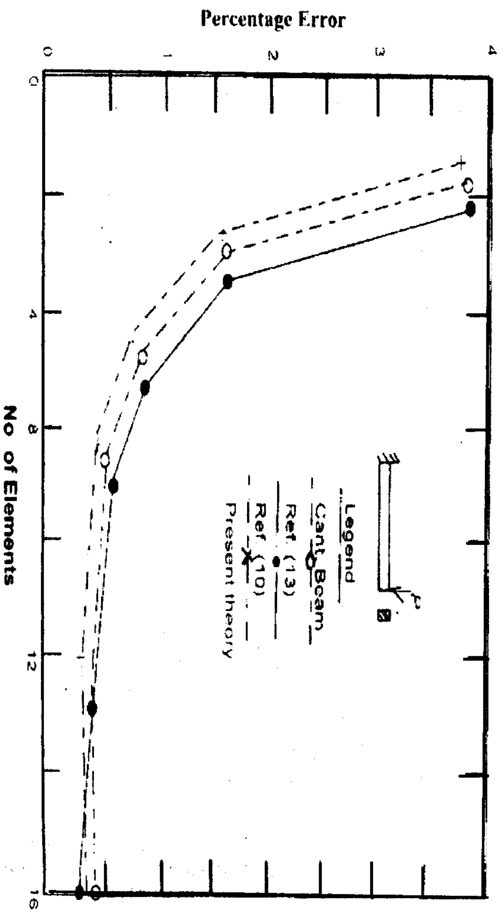

Fig. 3 Convergence of the Finite Element Solutions

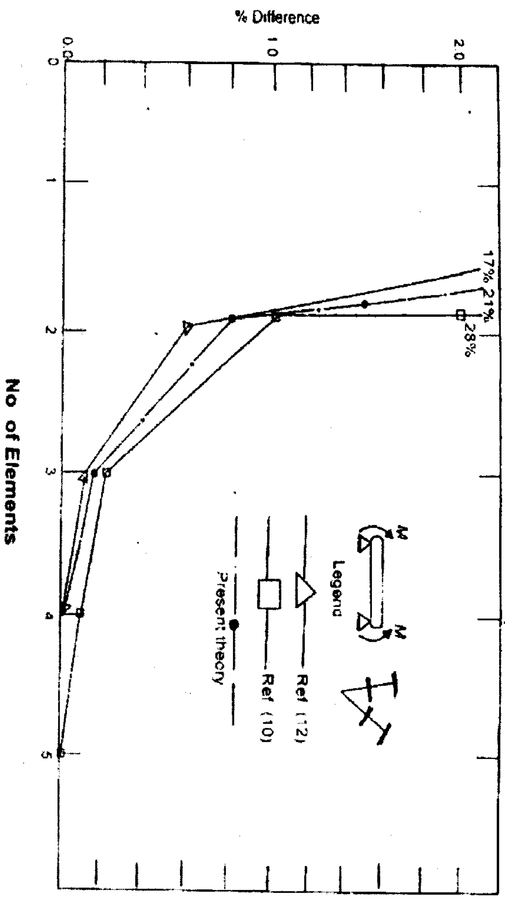

Fig. 2 Convergence of the Finite Element Solutions

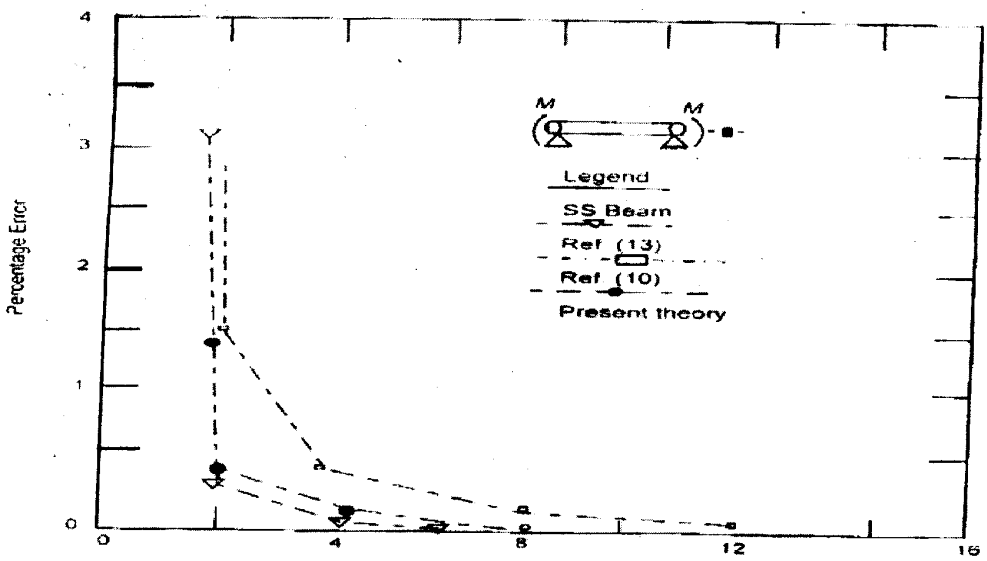

Fig. 4 Convergence of the Finite Element Solutions

5.0 CONCLUSION

The present formulation, which has paid particular attention to a general behaviour of a flexural torsional buckling of bi-symmetric thin-walled beams in the absence of uniform torsion, has produced results, which lead to the following conclusions.

(1) The use of stiffness matrices derived from bi-symmetric sections will produce better results over the use of stiffness matrices derived using asymmetric

sections to solve bi- symmetric problems.

(2) For some problems omission of uniform torsion may not introduce large errors in the solution. Therefore for preliminary analysis (for design) warping may be omitted for bi-symmetric sections.

(3) The use of two-stage equilibrium formulation reduces a number of degrees of freedom and saves computer costs and it has been shown here by convergence studies in figures 2,3 and 4 that accuracy of the solution is well within the requirement of design offices.

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APPENDIX A
Stability Matrices for Lateral-Torsional Buckling Analysis of Bi-Symmetric Beams.

$$k_G = \frac{M_y}{L}\begin{pmatrix} 0 & 0 & \frac{-6}{5} & 0 & 0 & \frac{6}{5} \\ 0 & 0 & \frac{-11L}{10} & 0 & 0 & \frac{L}{10} \\ \frac{-6}{5} & \frac{-11}{10} & 0 & \frac{6}{5} & \frac{-L}{10} & 0 \\ 0 & 0 & \frac{6}{5} & 0 & 0 & \frac{-6}{5} \\ 0 & 0 & \frac{-L}{10} & 0 & 0 & \frac{11L}{10} \\ \frac{6}{5} & \frac{L}{10} & 0 & \frac{-6}{5} & \frac{11L}{10} & 0 \end{pmatrix}$$

$$+\ F_y\begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{7L}{20} & 0 & 0 & \frac{3L}{20} \\ \frac{1}{2} & \frac{7L}{20} & 0 & \frac{-1}{2} & \frac{3L}{20} & 0 \\ 0 & 0 & \frac{-1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{3L}{20} & 0 & 0 & \frac{-1}{2} \\ \frac{1}{2} & \frac{3L}{20} & 0 & \frac{-1}{2} & \frac{7L}{20} & 0 \end{pmatrix}$$

$$+\ \frac{M_z}{L}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 & -1.0 \\ 0 & 1.0 & 0 & 0 & -1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.0 & 0 & 0 & 1.0 \\ 0 & -1.0 & 0 & 0 & 1.0 & 0 \end{pmatrix}$$