APPLICATION OF LAPLACE TRANSFORMATION TO THE ANALYSIS OF AN UNDERGROUND CIRCULAR CYLINDRICAL RESERVOIR

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ABSTRACT

Analysis of underground circular cylindrical shell is carried out in this work. The forth order differential equation of equilibrium, comparable to that of beam on elastic foundation, was derived from static principles on the assumptions of P. L Pasternak. Laplace transformation was used to solve the governing differential equation at critical condition. The Greenwich contour formula was successfully used to determine the inverse transform of the obtained subsidiary equation. The final solution was found to agree with that obtained using the classical method. The Laplace transformation appeared to be less tedious and more time saving than the classical method in the solution of the aforementioned differential equation.

Keywords: Laplace transformation, underground reservoir, cylindrical shell.

1. INTRODUCTION

Thin shells as structural elements occupy a leadership position in engineering and, in particular in civil, mechanical, architectural, aeronautical, and marine engineering [1].

The wide application of shell structures in engineering is conditioned by their following advantages:

- Efficiency of load-carrying behavior
- High degree of reserved strength and structural integrity
- High strength / weight ratio
- Very High stiffness
- Containment of space

Shells are for most part the deep-seated structures in manufacturing submarines, missiles, tanks and their roofs, and fluid reservoirs [2]. Circular cylindrical shells are used in a large variety of civil engineering structures; e.g. off-shore platforms, chimneys, silos, pipelines, bridge arches or wind turbine towers [3].

The objective of this study is to carry out the analysis of an underground circular cylindrical shell using Laplace transformation and compare the results with those obtained by means of the classical method.

Shell theories of varying degrees of accuracy were derived, depending on the degree to which the elasticity equations were simplified. The approximations necessary for the development of an adequate theory of shells have been subject of numerous discussions among the researchers in the field.

Love [4] was the first investigator to present a successful approximation of shell theory based on the analogy of plates due to Kirchoff [5]. This theory known as moment theory, in spite of its attractive accuracy, is seldom applied by an average engineer due to the rigorous mathematics involved. Subsequently Finsterwalder [6]; Vlasov [7] and Pasternak [8], by ignoring the effects of longitudinal bending moment, shear forces and torques arrived at the so-called semi-moment theory. This method which has been experimentally verified is found to give acceptable results for cylindrical shells whose ratio of length to diameter ranges between 2 and 8 [9].

Shells of revolution, a very important class of thin shells, have many technical applications in engineering. [1]. H. Reissner [10] presented a classical formulation of the bending problems for a shell of revelation and studied a spherical shell under axisymmetric bending. He reduced the differential equations of a spherical shell to a convenient form and then applied the asymptotic method for their integration.

Pasternak [8] later showed that when the load on a circular cylindrical shell is axisymmetric, the stresses and strains are functions of only one variable along the axis of the cylinder. The differential equation of equilibrium of the shell in this case reduces to a forth order differential equation equivalent to that of Bean on Elastic Foundation (BEF). This latter equation will be analyzed here by means of Laplace transformation to obtain the displacement function.

Laplace transformation is widely used in Engineering to solve various differential equations. Few applications of this method are the solution of the equation that describes the variation of the charge in a capacitor, the equation of variation of concentration of solids in sewage sludge etc. [11].

An abundant literature is available on the use of numerical methods, specially the finite element method, for the analysis of cylindrical shells [12]; but recent studies have shown the great dangers of using numerical modeling without a sufficient deep understanding of the effects of choosing different analysis options [13]. Thus the use of exact analytical methods, such as Laplace transformation, in the analysis of cylindrical shells cannot be overemphasized.

2. Differential Equation of Equilibrium

The equations of equilibrium of a cylindrical shell according to the semi-moment theory are [9]:

$$\frac{\partial^2 u}{\partial x^2}+\frac{1-u}{2}\cdot\frac{\partial^2 u}{\partial y^2}+\frac{1+\mu}{2}\cdot\frac{\partial^2 v}{\partial x \partial y}+$$
$$\frac{u}{R}\cdot\frac{\partial w}{\partial x}+\frac{1-\mu^2}{Eh}\cdot X=0 \qquad (1)$$

$$\frac{1+\mu}{2}\cdot\frac{\partial^2 u}{\partial x \partial y}+\frac{1-\mu}{2}\cdot\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+$$
$$\frac{\partial}{\partial y}\left(\frac{w}{R}\right)+\frac{h^2}{12R}\cdot\left[\frac{\partial^2}{\partial y^2}\left(\frac{v}{R}\right)-\frac{\partial^2 w}{\partial y^3}-\mu\cdot\frac{\partial^3 w}{\partial x^2 \partial y}\right]$$
$$+\frac{1-\mu^2}{Eh}\cdot Y=0 \qquad (2)$$

$$-\frac{\partial^4 w}{\partial x^4}+\frac{\partial}{\partial y}\left[\frac{u}{R}\cdot\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2}{\partial y^2}\left(\frac{v}{R}\right)\right]$$
$$-\frac{\partial^4 w}{\partial y^4}-2\mu\cdot\frac{\partial^4 w}{\partial x^2 \partial y^2}-\frac{12}{Rh^2}*$$
$$\left[\frac{\partial v}{\partial y}+\frac{w}{R}+\mu\cdot\frac{\partial u}{\partial x}\right]+\frac{12(1-\mu)}{Eh^3}\cdot Z=0 \qquad (3)$$

where x, y and z axes at a given point O of the middle surface are taken in the directions of the axis of the cylinder, the tangent to the circumference, and the normal to the middle surface of the shell respectively; X, Y, and Z are the components of the transverse distributed load in x, y, and z directions respectively; u, v, and w are the displacement components in x, y, and z directions respectively;
h is the thickness of the shell;
E is Young modulus of the material;
R is the radius of the cylinder;
and μ is Poisson's ratio.

On the assumptions that (i) the middle surface of the shell is inextensible in y – direction such that v =0, and (ii) the normal force N_x acting on the transverse section of the shell is neglected, Pasternak arrived at the following relations:

$$\frac{du}{dx}=u'=\mu\frac{w}{R} \qquad (4)$$

$$N_\varphi=N=\frac{Ehw}{R} \qquad (5)$$

$$S=0 \qquad (6)$$

$$M_x=M=-Dw'' \qquad (7)$$

$$M_\varphi=\mu M \qquad (8)$$

Where $w''=\frac{d^2 w}{dx^2}$;

$D=\frac{Eh^3}{12(1-\mu^2)}$;

N_φ is the hoop tension;
S is the membrane shearing force;
M_x is the longitudinal moment; and
M_φ is the transverse moment.

Substituting the above relations into equation (3), we obtain:

$$\frac{d^4w}{dx^4}+4\beta^4 w=\frac{q}{D} \qquad (9)$$

Where: $\beta^4=\frac{3\left(1-\mu^2\right)}{R^2h^2}$

q is the intensity of transverse load.

Equation (9) is due to Pasternak [8] and is only applicable to cylindrical shell subject to axisymmetric loading i.e. stresses and strains are constant along the circumferential section. One of the critical conditions examined during the design of buried reservoirs, is when they are empty. In this case, the transverse load q will consist of only the earth pressure. It follows that:

$q=K_a\gamma(L-x)$

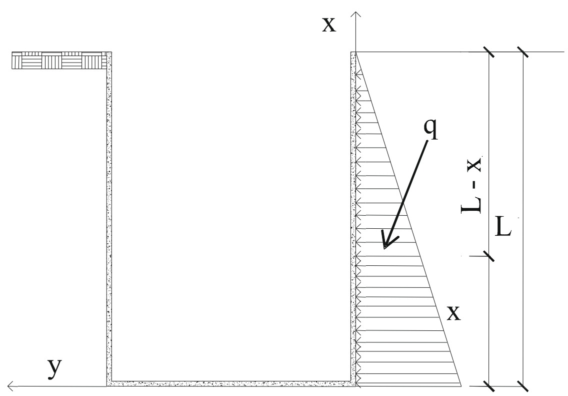

Fig. 1: Underground reservoir subjected to earth pressure

where the distance x is measured vertically from the base, while L stands for the height of the reservoir;

K_a = (1 – sinφ) / (1 + sinφ) is the Rankine active earth pressure coefficient; φ is the angle of internal friction of the soil; γ is specific weight of the retained soil.

Equation (9) can thus be written as:

$$\frac{d^4w}{dx^4}+4\beta^4 w=\frac{K_a\gamma(L-x)}{D} \qquad (10)$$

3. Representation of the ODE inTerms of "s" Parameter.

Laplace transformation is next applied to the various terms in equation (10):

$$L\left\{\frac{d^4w}{dx^4}\right\}=s^4\overline{w}(s)-s^3w(0)-s^2w^i(0) \qquad (11)$$
$$-sw^{ii}(0)-w^{iii}(0)$$

$$L\{w(x)\}=\overline{w}(s) \qquad (12)$$

$$L\{K_a\gamma(L-x)/D\}=AL/s-A/s^2 \qquad (13)$$

where $L\{f(x)\}$ is Laplace transformation a function of x which equals $\overline{f}(s)$;

A = $K_a\gamma$/D;

w^i, w^{ii} and w^{iii} are the 1st, 2nd, and 3rd derivatives of w with respect to x respectively.

Substituting where appropriate equations (11), (12) and (13) and collecting like terms, we have that:

$$\overline{w}(s)[s^4+4\beta^4]=c_1s^3+c_2s^2+c_3s+c_4$$
$$+AL/s-A/s^2 \qquad (14)$$

where $c_1=w(0)$; $c_2=w^i(0)$; $c_3=w^{ii}(0)$ and $c_4=w^{iii}(0)$.

Making $\overline{w}(s)$ subject of the formula in the equation above, we obtain:

$$\overline{w}(s)=\frac{c_1s^3+c_2s^2+c_3s+c_4}{s^4+4\beta^4}+$$
$$\frac{AL}{s(s^4+4\beta^4)}-\frac{A}{s^2(s^4+4\beta^4)} \qquad (15)$$

4. Application of Greenwich Contour Theorem to the Determination of the Laplace Inverse.

Having a close look at equation (15), it is found out that the governing equation has been reduced to a function of "s" parameter. Equation (15) is known as subsidiary equation. The inverse transform of the subsidiary equation will yield the solution w(x). In the absence of any boundary conditions we shall utilize the Greenwich contour formula for this purpose.

In general, the inverse of a Laplace Transformation is defined by means of Greenwich contour or complex integral as: [10]

$$w(x)=\frac{1}{2\pi i}\oint_c w(s)e^{sx}ds$$

where c is the Greenwich contour region.

$\oint_c ds$ is the close integral.

From residue theorem, the close integral equals the summation of the residues obtained within the Greenwich contour region.

Let equation (15) be written as:

$$\overline{w}(s)=\overline{w_1}(s)+\overline{w_2}(s)+\overline{w_3}\ (s) \qquad (16)$$

where $\overline{w_1}(s)=\frac{c_1s^3+c_2s^2+c_3s+c_4}{s^4+4\beta^4}$

$$\overline{w_2}(s)=\frac{AL}{s(s^4+4\beta^4)}$$

$$\overline{w_3}(s)=-\frac{A}{s^2(s^4+4\beta^4)}$$

Applying inverse transformation to equation (16) gives:

$$w(x)=w_1(x)+w_2(x)+w_3(x) \qquad (17)$$

where $w_1(x)=L^{-1}\{\overline{w_1}(s)\}$ (18)

$$w_2(x)=L^{-1}\{\overline{w_2}(s)\} \qquad (19)$$

$$w_3(x)=L^{-1}\{\overline{w_3}(s)\} \qquad (20)$$

Equation (17) will be analyzed term by term.

$$w_1(x)=L^{-1}\left\{\frac{c_1s^3+c_2s^2+c_3s+c_4)}{(s^4-4\beta^4)}\right\} \qquad (21)$$

Owing to $s^4+4\beta^4=(s-\beta\sqrt{2i})\,(s+\beta\sqrt{2i})\,(s-\sqrt{-2i})\,(s+\sqrt{-2i})$

equation (21) becomes:

$$w_1(x)=L^{-1}\left\{\frac{c_1s^3+c_2s^2+c_3s+c_4}{(s-\beta\sqrt{2i})\,(s+\beta\sqrt{2i})\,(s-\sqrt{-2i})\,(s+\beta\sqrt{-2i})}\right\} \qquad (22)$$

By the residue theorem, $w_1(x)$ equals the sum of the residues at the poles of $e^{sx}w_1(s)$ within the contour. The residue is obtained by the evaluation of the expression:

$$\frac{1}{(k-1)!}\lim_{s\to s_0}\frac{d^{k-1}}{ds^{k-1}}\left[(s-s_0)^k f(s)\right] \qquad (23)$$

where k = order of pole; f(s) is the function and s_o is the pole or point of discontinuity. For equation (22), the poles are:

$\beta\sqrt{2i}$; $-\beta\sqrt{2i}$; $\beta\sqrt{-2i}$ and $-\beta\sqrt{-2i}$.

For $s_o=\beta\sqrt{2i}$, the residue R_1 is given by:

$$R_1=\lim_{s\to\beta\sqrt{2i}}\left\{e^{sx}\frac{c_1s^3+c_2s^2+c_3s+c_4}{(s+\beta\sqrt{2i})\,(s-\beta\sqrt{-2i})\,(s+\beta\sqrt{-2i})}\right\}$$

$$\Rightarrow R_1=\frac{(\beta^3\sqrt{-8i}\ c_1+2i\beta^2c_2+\beta\sqrt{2i}\ c_3+c_4)e^{\beta x\sqrt{2i}}}{8i\beta^3\sqrt{2i}} \qquad (24)$$

For $s_o=-\beta\sqrt{2i}$, the residue R_2 is:

$$R_2=\lim_{s\to-\beta\sqrt{2i}}\left\{e^{sx}\frac{c_1s^3+c_2s^2+c_3s+c_4}{(s-\beta\sqrt{2i})\ (s-\beta\sqrt{-2i})\,(s+\beta\sqrt{-2i})}\right\}$$

$$\Rightarrow R_2=\frac{(-\beta^3\sqrt{-8i}\,c_1+2i\beta^2c_2-\beta\sqrt{2i}\,c_3+c_4)e^{-\beta x\sqrt{2i}}}{-8i\,\beta^3\sqrt{2i}} \qquad (25)$$

For $s_o=\beta\sqrt{-2i}$, the residue R_3 will be:

$$R_3=\lim_{s\to\beta\sqrt{-2i}}\left\{e^{sx}\frac{c_1s^3+c_2s^2+c_3s+c_4}{(s-\beta\sqrt{2i})\,(s+\beta\sqrt{2i})\,(s+\beta\sqrt{-2i})}\right\}$$

$$\Rightarrow R_3=\frac{(\beta^3\sqrt{8i}\,c_1-2i\beta^2c_2+\beta\sqrt{-2i}\ c_3+c_4)e^{\beta x\sqrt{-2i}}}{-8i\beta^3\sqrt{-2i}} \qquad (26)$$

For $s_o=-\beta\sqrt{-2i}$, the residue R_4 is given by:

$$R_4=\lim_{s\to-\beta\sqrt{2i}}\left\{e^{sx}\frac{c_1s^3+c_2s^2+c_3s+c_4}{(s-\beta\sqrt{2i})\,(s+\beta\sqrt{2i})\,(s-\beta\sqrt{-2i})})\right\}$$

$$\Rightarrow R_4=\frac{(-\beta^3\sqrt{8i}\,c_1-2i\beta^2c_2-\beta\sqrt{-2i}\ c_3+c_4)e^{-\beta x\sqrt{-2i}}}{8i\beta^3\sqrt{-2i}} \qquad (27)$$

The sum of R, R_2, R_3 and R_4 gives $w_1(x)$:

W_1 (x) = R_1 + R_2 + R_3 + R_4 (28)

Substituting R_1, R_2, R_3 and R_4 into equation (28), we obtain:

$$w_1(x)=e^{\beta x}(E_1\cos\beta x+E_2\sin\beta x)$$
$$+e^{-\beta x}(E_3\cos\beta x+E_4\sin\beta x) \qquad (29)$$

Where: $E_1=D_1-i(D_2+D_3)$; $E_2=D_4+i(D_2-D_3)$

$E_3=D_1+i(D_2+D_3)$; $E_4=-D_4+i(D_2-D_3)$

$D_1=\frac{c_1}{2}$; $D_2=\frac{c_2}{4\beta}$; $D_3=\frac{c_4}{8\beta^3}$ and $D_4=\frac{c_3}{4\beta^2}$

In a similar manner, making use of the residue theorem, we obtain the expressions for $w_2(x)$ and $w_3(x)$ as follows:

$$w_2(x)=\frac{AL[2-(\cos\beta x)(e^{\beta x}+e^{-\beta x})]}{8\beta^4} \qquad (30)$$

$$w_3=\frac{-Ax}{4\beta^4}+\frac{AL(\cos\beta x)(e^{\beta x}+e^{-\beta x})}{8\beta^4} \qquad (31)$$

Substituting equations (29); (30) and (31) into equation (17) we obtain:

$$W(x)=e^{\beta x}(E_1\cos\beta x+E_2\sin\beta x)+$$
$$e^{-\beta x}(E_3\cos\beta x+E_4 Sin\beta x)+$$
$$\frac{A(L-x)}{4\beta^4} \qquad (32)$$

Noting that:

$e^{\beta x}=\cosh\beta x+\sinh\beta x$;

$e^{-\beta x}=\cosh\beta x-\sinh\beta x$;

$A=\frac{k_a\gamma}{D}$; $4\beta^4=\frac{Eh}{R^2D}$; equation (32) becomes.

$$W(x)=(E_1-E_3)\cosh\beta x\cos\beta x+$$
$$(E_2+E_4)\cosh\beta x\sin\beta x+(E_1-E_3)\sinh\beta x\cos\beta x$$
$$+(E_2-E_4)\sinh\beta x\sin\beta x+\frac{K_a\gamma R^2}{Eh}(L-x)$$

which can be better written as:

$$W(x)=A_1\cosh\beta x\cos\beta x+$$
$$A_2\cosh\beta x\sin\beta x\sin\beta x+A_3\sinh\beta x\cos\beta x$$
$$+A_4\sinh\beta x\sin\beta x+\frac{K_a\gamma R^2}{Eh}(L-x) \qquad (33)$$

where $A_1=E_1+E_3$; $A_2=E_2+E_4$;

$A_3=E_1-E_3$; and $A_4=E_2-E_4$.

Equation (33) above represents the solution of equation (10) thus the displacement function which we are solving for.

5. Discussion

Previous works [14, 15] have used the classical method to solve the fourth order differential equation (10). They arrived at the following solution:

$$w(x)=C_1\cosh\beta x\cos\beta x+C_2\cosh\beta x\sin\beta x$$
$$C_3\sinh\beta x\cos\beta x+C_4\sinh\beta x\sin\beta x+$$
$$\frac{K_a\gamma a^2(L-x)}{Et} \qquad (34)$$

where K_a, L, γ and E denote the same parameters as in equation (33); a stands for the radius of the cylinder and t for its thickness; C_1, C_2, C_3 and C_4 are constants to be determined using the boundary conditions.

It is obvious that equations (33) and (34) are identical. It follows that the results obtained using the Laplace transformation method are in order with those given by the classical method.

The Laplace transformation has the merit to be less tedious and more time saving than the classical approach in the sense that it is direct and does not implicate any splitting of the solution into homogeneous and particular solutions as the classical approach does.

The Laplace transformation method has advantages over the numerical methods such as finite element and finite difference in the sense that it gives an exact solution at every point along the height of the reservoir.

Other parameters such as bending moment, shear force, and hoop tension very essential for the design of circular cylindrical shell can be obtained by using the well known relationships between these parameters and the displacement function obtained in this work.

5. Conclusion

This study showed the agreement between the results obtained using Laplace transformation and those achieved by means of the classical method in the analysis of an underground circular cylindrical reservoir.

The benefit of using the semi-moment theory in the analysis of thin cylindrical shells rather than employing strictly the membrane theory cannot be over-emphasized as it provides more accurate equations for design purposes than the membrane theory.

Finally Laplace Transformation which is very convenient in handling differential equations involving Dirac delta function can be recommended for the analysis of ring-stiffened buried circular cylindrical reservoir.

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