Volume 26 (2), pp. 125–141
http://www.orssa.org.za

ORiON
ISSN 0529-191-X

A tabu-search for minimising the carry-over effects value of a round-robin tournament

MP Kidd*

Received: 26 August 2010; Revised: 21 October 2010; Accepted: 29 October 2010

Abstract

A player b in a round-robin sports tournament receives a *carry-over effect* from another player a if some third player opposes a in round i and b in round $i+1$. Let γ_{ab} denote the number of times player b receives a carry-over effect from player a during a tournament. Then the *carry-over effects value* of the entire tournament T on n players is given by $\Gamma(T) = \sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}^2$. Furthermore, let $\Gamma(n)$ denote the minimum carry-over effects value over all round-robin tournaments on n players. A strict lower bound on $\Gamma(n)$ is $n(n-1)$ (in which case there exists a round-robin tournament of order n such that each player receives a carry-over effect from each other player exactly once), and it is known that this bound is attained for $n = 2^r$ or $n = 20, 22$.

It is also known that round-robin tournaments can be constructed from so-called *starters*; round-robin tournaments constructed in this way are called *cyclic*. It has previously been shown that cyclic round-robin tournaments have the potential of admitting small values for $\Gamma(T)$, and in this paper a tabu-search is used to find starters which produce cyclic tournaments with small carry-over effects values. The best solutions in the literature are matched for $n \leq 22$, and new upper bounds are established on $\Gamma(n)$ for $24 \leq n \leq 40$.

Key words: Round-robin tournament, carry-over effects, starters, tabu-search.

1 Introduction

The scheduling of round-robin sports tournaments has given rise to a number of interesting optimisation problems in the theory of sports tournament scheduling, as recently summarised in the excellent annotation by Kendall *et al.* [12]. In the majority of the well-studied problems concerning sports tournament scheduling, the venues of the matches for a certain team throughout the tournament (often classified as *home* or *away*) play a significant role. Examples include the *minimum breaks* problem [3] (where a *break* in the tournament occurs when a team plays two consecutive home games or two consecutive away games) and the *travelling-tournament problem* [5], where the distances travelled between venues by the various teams are to be minimised.

*Department of Logistics, University of Stellenbosch, Private bag X1, Matieland, 7602, Republic of South Africa, email: 14623617@sun.ac.za

Another problem that is often considered is that of balancing so-called *carry-over effects* in tournaments, where a *carry-over effect* generally refers to the possible effect on the performance of a team at some stage of a sports tournament due to a specific event that occured during a previous stage in the tournament. For instance, the quality of the ground (such as a court, field or stadium) on which a match is played may have an effect on the perfomance of a team. The objective in this case is to balance the tournament in such a way that each team plays exactly once on each court, and a round-robin tournament satisfying this property is known as a *balanced tournament design* [4]. On the other hand, players may also have effects on one another. A strong player, for instance, may have a negative effect on an opponent in that his/her opponent may be physically exhausted and/or low in morale after playing him/her, in which case this effect may be carried over to future stages of the tournament where future opponents may benefit from this. The carry-over effect is then given from the strong player to one of these future players, and the objective is to balance the tournament in such a way that no player receives a carry-over effect from another player more than once. The idea of balancing carry-over effects in round-robin tournaments carried over from player to player was first introduced by Russel [16] in 1980, who was inspired by a schedule for a football tournament where 18 of the 21 carry-over effects received by teams were due to a single team.

In this paper the problem of balancing the effects carried over from player to player in a round-robin tournament is considered. In §2 the notions of a *round-robin tournament*, *carry-over effects* in a round-robin tournament, as well as the overall *carry-over effects value* of a round-robin tournament are defined formally, and a number of combinatorial equivalents of a round-robin tournament are described. In §3 the notion of a *starter* is introduced, and it is shown how a round-robin tournament may be constructed from a starter. Previous attempts at minimising the carry-over effects value of a round-robin tournament and constructing balanced tournaments are reviewed in §4. In §5 a tabu-search algorithm for scheduling round-robin tournaments by means of starters is presented, and the results obtained via this algorithm are reported in §6. The paper concludes with a summary of the work prersented and some ideas for future work in §7.

2 Preliminary definitions

Given the set $\mathbb{Z}_{2n} = \{0, 1, \ldots, 2n-1\}$ (the elements of which are called *players*), a *round-robin tournament of order* $2n$ is a partition into $2n-1$ parts (called *rounds*) of $\mathbb{Z}_{2n}$, each consisting of 2-subsets (called *matches*) so that each unordered pair in $\mathbb{Z}_{2n}^{(2)}$ (the set of all 2-subsets of $\mathbb{Z}_{2n}$) occurs in exactly one part[1]. Furthermore, the rounds of the tournament are ordered and by convention the first round is assumed to follow the last one, which is also not generally the case. In what follows the (ordered) rounds of a round-robin tournament T are given by $(T_0, T_1, \ldots, T_{2n-1})$. Note that a round-robin tournament over an odd number of players is equivalent to a round-robin tournament over an even number of players where playing against some fixed player is equivalent to receiving a bye,

[1]Note, however, that round-robin tournaments can be defined more generally; for instance, each player may oppose each other player more than once and the tournament does not need to last only for $2n-1$ rounds (see Rasmussen and Trick [15] for such a more general definition).

and therefore only round-robin tournaments over an even number of players need to be considered.

For instance, the partitions $T_0 = \{\{0,1\},\{2,3\}\}$, $T_1 = \{\{0,2\},\{1,3\}\}$ and $T_2 = \{\{0,3\},\{1,2\}\}$ of $\mathbb{Z}_4$ represent three rounds of a round-robin tournament of order 4. It is convenient to present a round-robin tournament in tabular form where the entry in row i and column j contains the player that opposes i in round j. For instance, the round robin tournament (T_0, T_1, T_2) above is given in Table 1 in this form.

		Round		
		0	1	2
Player	0	1	2	3
	1	0	3	2
	2	3	0	1
	3	2	1	0

Table 1: *A round-robin tournament of order 4 where the entry in row i and column j gives the player that opposes i in round j.*

It is interesting to note that a round-robin tournament of order n is (separately) equivalent to two combinatorial designs, namely a Latin square with a symmetric conjugate and a one-factorisation of the complete graph on n vertices. A *Latin square* is an array in which each symbol from some set occurs exactly once in each row and column. If the headings, lines and round numbers are removed from Table 1, the matrix

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

is obtained, which is a Latin square. Since each player opposes each other player exactly once and never opposes himself, and since each player plays exactly once in each round, a round-robin tournament represented in this tabular form will always produce a Latin square. Note also that a further property the Latin square will satisfy is that if the entry in row i and column j contains k, then the entry in row k and column j must contain i. This is equivalent to stating that the Latin square has a symmetric conjugate (see Keedwell [11] for more detail).

A *one-factor* in a complete graph K_{2n} on $2n$ vertices is a 1-regular subgraph of K_{2n} (*i.e.* a perfect matching of its vertices), and a *one-factorisation* of K_{2n} is a set of $2n-1$ edge-disjoint one-factors of K_{2n}. It is easy to see that if the vertices are taken as players, then the one-factors correspond to the rounds of a round-robin tournament. For example, Figure 1 shows the one-factorisation of K_4 that corresponds to the round-robin tournament given in Table 1. More detail on one-factorisations may be found in the survey by Mendelsohn and Rosa [13].

A *carry-over effect* in a round-robin tournament T of order $2n$ received by $b \in \mathbb{Z}_{2n}$ from $a \in \mathbb{Z}_{2n}$ occurs when $\{a,c\} \in T_i$ and $\{b,c\} \in T_{i+1}$ for some $c \in \mathbb{Z}_{2n}$ and some $i \in \{0,\ldots,2n-1\}$, where operations on the indices of T are performed modulo $2n-1$ (hence the round-robin tournaments considered wrap around in the sense that carry-over effects are carried over from the last to the first round). The notation $\gamma_{a,b}$ is used to denote the

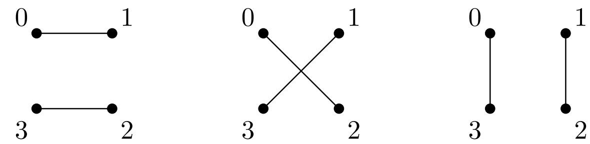

Figure 1: *A one-factorisation of the complete graph K_4.*

number of times a carry-over effect is received by b from a, and it is assumed that $\gamma_{a,a} = 0$ for all $a \in \mathbb{Z}_{2n}$. Consider, for example, the round-robin tournament shown in tabular form in Table 2. The reason why this representation is convenient is due to the fact that the carry-over effects are given by consecutive elements in the rows of this table. For instance, 1 opposes 5 in round 1, and 3 in round 2; therefore 3 receives a carry-over effect from 5.

		Round				
		0	1	2	3	4
Player	0	5	2	4	1	3
	1	4	5	3	0	2
	2	3	0	5	4	1
	3	2	4	1	5	0
	4	1	3	0	2	5
	5	0	1	2	3	4

Table 2: *A round-robin tournament of order 6.*

In order to record the carry-over effects observed in a round-robin tournament, a *carry-over effects matrix* is introduced, where the entry in row i and column j contains $\gamma_{i,j}$. The carry-over effects matrix for the round-robin tournament in Table 2, for example, is given in Table 3.

		Round					
		0	1	2	3	4	5
Player	0	0	1	3	0	0	1
	1	0	0	1	3	0	1
	2	0	0	0	1	3	1
	3	3	0	0	0	1	1
	4	1	3	0	0	0	1
	5	1	1	1	1	1	0

Table 3: *The carry-over effects matrix for the round-robin tournament in Table 2.*

As mentioned before, the objective is to construct a round-robin tournament in such a way that each player receives a carry-over effect from each other player exactly once. However, in cases where this is not possible it is necessary to be able to determine exactly how balanced the tournament is. A suitable measure was proposed by Russel [16], which consists of taking the variance of the numbers in each row of the carry-over effects matrix. Note that ideally there should be no variance in the numbers in a row, and therefore this measure is to be minimised.

Each row-sum of a carry-over effects matrix for a round-robin tournament of order $2n$ is $2n - 1$ since each player gives $2n - 1$ carry-over effects to other players throughout the tournament. Hence the mean value of each row is equal to 1. The variance of row i is therefore given by

$$\frac{1}{2n-1}\sum_{j=0}^{2n-1}\gamma_{i,j}^2 - 1.$$

Since the total variance of all rows of the carry-over effects matrix is to be minimsed, it is sufficient to simply minimise $\sum_{i=0}^{2n-1}\sum_{j=0}^{2n-1}\gamma_{i,j}^2$, henceforth referred to as the *carry-over effects value (COE-value)* of the round-robin tournament. For instance, the COE-value of the tournament in Table 2 is 60. The minimum value that the COE-value of a round-robin tournament of order $2n$ can attain is $2n(2n-1)$, which occurs when each non-diagonal element of the carry-over matrix contains a 1. In other words, each player gives a carry-over effect to each other player exactly once. A tournament attaining this lower bound is called *balanced with respect to carry-over effects*, or simply *balanced*. In what follows the notation $\Gamma(T)$ denotes the COE-value of the round-robin tournament T and $\Gamma(n)$ denotes the minimum COE-value over all round-robin tournaments of order n.

3 Constructions of round-robin tournaments using starters

A well-known method for constructing round-robin tournaments is the so-called *polygon method* [9, 14], which is actually a special case of a more general construction method using so-called *starters*. A *starter* in $\mathbb{Z}_{2n+1}$ is a set of n pairs from $\mathbb{Z}_{2n+1}^{(2)}$ so that each element of $\mathbb{Z}_{2n+1}$ appears in at most one pair, and so that for any $k \in \mathbb{Z}_{2n+1}\backslash\{0\}$ there exists exactly one pair $\{a, b\}$ in the starter for which either $a - b = k \pmod{2n+1}$ or $b - a = k \pmod{2n+1}$. There will always be some element of $\mathbb{Z}_{2n+1}$ that is not in any pair of a starter, henceforth referred to as the *residual element* of the starter.

An example of a starter in $\mathbb{Z}_9$ is $\{\{8, 0\}, \{2, 4\}, \{3, 6\}, \{1, 5\}\}$ where 7 is the residual element. The following theorem gives a simple construction of a round-robin tournament, where $s + 1$ refers to the starter obtained by adding 1 to each element of the starter s modulo $2n - 1$ (note that $s + 1$ is a starter if s is a starter).

Theorem 3.1 *If s is a starter in $\mathbb{Z}_{2n-1}$ then $s + i$ forms the i-th round of a round-robin tournament of order $2n$ where $\mathbb{Z}_{2n-1} \cup \{\infty\}$ represents the players and where the residual element of $s + i$ plays ∞ in round i.*

Proof: Since all the elements (including the residual) in the starter s are distinct, the same is true for $s + i$ (for any $i \in \mathbb{Z}_{2n}$), and therefore each player plays exactly once in each round of the tournament. Assume that the match $\{a, b\}$ (where $a \neq \infty \neq b$) is played twice in the tournament, namely in rounds r_1 and r_2. Hence $\{a - r_1, b - r_1\}$ and $\{a - r_2, b - r_2\}$ are both in s. However,

$$(a - r_1) - (b - r_1) = b - a \pmod{2n-1}$$

and

$$(a - r_2) - (b - r_2) = b - a \pmod{2n-1},$$

and hence $r_1 = r_2$ (by definition of a starter). Furthermore, the residual elements of $s + i$ and $s + j$ for $i \neq j$ are clearly different. This contradiction shows that each pair of players opposes each other exactly once throughout the tournament. ∎

For instance, the starter $\{\{8,0\},\{2,4\},\{3,6\},\{1,5\}\}$ produces the round-robin tournament shown in Table 4.

Round	Matches
0	$\{\{8,0\},\{2,4\},\{3,6\},\{1,5\},\{7,\infty\}\}$
1	$\{\{0,1\},\{3,5\},\{4,7\},\{2,6\},\{8,\infty\}\}$
2	$\{\{1,2\},\{4,6\},\{5,8\},\{3,7\},\{0,\infty\}\}$
3	$\{\{2,3\},\{5,7\},\{6,0\},\{4,8\},\{1,\infty\}\}$
4	$\{\{3,4\},\{6,8\},\{7,1\},\{5,0\},\{2,\infty\}\}$
5	$\{\{4,5\},\{7,0\},\{8,2\},\{6,1\},\{3,\infty\}\}$
6	$\{\{5,6\},\{8,1\},\{0,3\},\{7,2\},\{4,\infty\}\}$
7	$\{\{6,7\},\{0,2\},\{1,4\},\{8,3\},\{5,\infty\}\}$
8	$\{\{7,8\},\{1,3\},\{2,5\},\{0,4\},\{6,\infty\}\}$

Table 4: *A round-robin tournament of order 10 constructed from the starter* $\{\{8,0\},\{2,4\},\{3,6\},\{1,5\}\}$.

A round-robin tournament constructed in this way is called *cyclic* (see Anderson [1]). A starter is simply a one-factor of the complete graph, and the above construction method may be represented visually in a graph-theoretical manner as shown in Figure 2. Here the one-factor corresponding to the starter $\{\{8,0\},\{2,4\},\{3,6\},\{1,5\}\}$ is given by the black edges, and to obtain the remaining one-factors this configuration is rotated clockwise 8 times (with ∞ at the centre of the rotation). The first rotation of the configuration is given by the grey edges in Figure 2. This is the common manner in which the so-called polygon method is described, where the polygon method constructs a round-robin tournament from the starter which contains pairs of the form $\{i,-i\}$, known as the *patterned starter*.

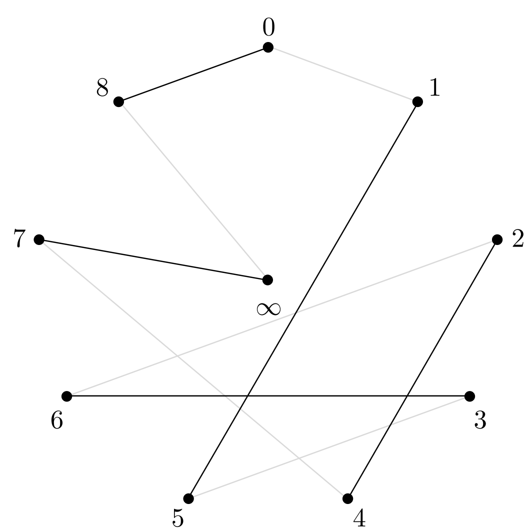

Figure 2: *A graph-theoretic visualisation of constructing a round-robin tournament of order 10 from the starter* $s=\{\{8,0\},\{2,4\},\{3,6\},\{1,5\}\}$. *The one-factor in black corresponds to* s, *while the one-factor in grey corresponds to* $s+1$.

4 Current upper bounds on $\Gamma(n)$

Constructions have also been given for tournaments balanced with respect to carry-over effects. In particular, constructions for when the number of players is a power of 2 have been proposed by Russel [16], Anderson [1] and Keedwell [11]. Their constructions rely on Galois fields and so-called R-sequenceable groups. The only other orders (except powers of two) for which balanced tournaments are known to exist are $n = 20$ and $n = 22$; these tournaments were found via a computer search for a special type of algebraic structure which guarantees the existence of a balanced tournament [11].

Another method that has proved to give good results is the construction of round-robin tournaments using starters, as noted by Anderson [1]. Anderson found that when a specific type of starter, which he called a *good starter* (to be defined in the next section), was used to construct a round-robin tournament, a tournament was obtained that is balanced with respect to carry-over effects. In [1] a computer search for good starters in $\mathbb{Z}_{n-1}$ for odd $n \leq 24$ was employed, and again balanced tournaments were only found where n was a power of two and for $n = 20, 22$.

A number of attempts at finding round-robin tournaments with small COE-values for orders other than 2^m, 20 or 22 have been published. Russel [16] constructed round-robin tournaments of order $2n = q + 1$ (with q a prime power) from starters in the Galois field $GF(q)$ for $q = 5, 11, 13, 17, 19, 23$, and reordered the rounds of the tournaments in an attempt to minimise their COE-values. His results are shown in Table 5 together with his results for tournaments of order $2n = 2^m$ (note that 22 is omitted since it is not one more than a prime number).

Anderson [1] found starters via a computer search that produce round-robin tournaments with COE-values even smaller than those found by Russel, as shown in Table 5. Furthermore, constraint programming applications to round-robin tournament scheduling problems were investigated by Trick [17] and by Henz *et al.* [10]. Trick obtained COE-values for $n = 6, 10$ (and proved that 60 is optimal for $n = 6$), while Henz *et al.* also obtained COE-values for tournaments of orders $n = 6, 10$ as well as $n = 12$. Their results are also shown in Table 5. Miyashiro and Matsui [14] used the round-robin tournament obtained from the patterned starter and randomly reordered its rounds a large number of times in order to obtain the COE-values shown in Table 5. Finally, Guedes and Ribeiro [9] used a hybridised iterated local search to solve a generalisation of the problem called *the weighted carry-over effects problem*, where a cost is introduced for a carry-over given by a specific player to another. For the special case where the costs are all equal to 1, Guedes and Ribeiro found the COE-values shown in Table 5.

It should be noted, however, that the only improvement on the results of Anderson [1] are due to Guedes and Ribeiro, for the case $n = 12$. For all other orders the use of starters outperformed the other methods, and for this reason the search for good starters is continued in this paper. In the next section a tabu-search is described, which uses starters as trial solutions in an attempt to find starters which produce round-robin tournaments with small COE-values.

n	$n(n-1)$	[16]	[1]	[17]	[10]	[14]	[9]
4	12	**12**	–	–	–	–	**12**
6	30	60	–	60	60	–	60
8	56	**56**	**56**	–	–	–	**56**
10	90	138	108	122	128	108	108
12	132	196	176	–	188	176	160
14	182	260	234	–	–	254	254
16	240	**240**	**240**	–	–	–	**240**
18	306	428	340	–	–	400	–
20	380	520	**380**	–	–	488	–
22	462	–	**462**	–	–	–	–
24	552	684*	644	–	–	–	–

Table 5: *Best upper bounds on $\Gamma(n)$ found for round-robin tournaments of even orders $n \leq 24$ by various authors (references are given in chronological order). Cases for which the lower bound $n(n-1)$ is attained are shown in boldface. *This COE-value was found by Anderson [1] using the method of Russel [16].*

5 A tabu-search algorithm for tournament construction

The *tabu-search* is a well-known member of the class of search-algorithms called *meta-heuristics*, which uses short term memory in order to guide the search away from local optima and cycles. The search moves from trial-solution to trial-solution by means of transformations often simply referred to as *moves*, and a so-called *tabu-list* is maintained which contains information on previous moves applied and/or trial-solutions visited. The tabu-list then governs which moves may be applied when, and it prevents the search from cycling within a locally optimal region. For more extensive details on the tabu-search methodology, see Glover [6, 7].

5.1 The COE-value of a starter

The *COE-value* of a starter is defined to be equal to the COE-value of the round-robin tournament constructed from it. In order to calculate the COE-value of a starter, the following notion, called the *type* of a starter, is introduced. Let x_i denote the element of $\mathbb{Z}_{2n-1}$ that is paired with i in a starter s. The *type* of a starter s in $\mathbb{Z}_{2n-1}$ with residual element k is given by $1^{d_1}2^{d_2}\ldots(2n-3)^{d_{2n-3}}$, where d_i differences are each repeated i times in the set of differences $\{x_i - x_{i-1} \,(\mathrm{mod}\ 2n-1) \mid i \in \mathbb{Z}_{2n-1}\backslash\{k, k+1\}\}$ for all i, and where the terms containing $d_i = 0$ in the superscript of any value of i are omitted. For example, from Table 6 it follows that the starter $\{\{0,8\},\{1,2\},\{3,5\},\{4,9\},\{6,10\}\}$ in $\mathbb{Z}_{11}$ is of type $1^2 2^2 3^1$.

i	0	1	2	3	4	5	6	7	8	9	10
x_i	8	2	1	5	9	3	10	–	0	4	6
$x_i - x_{i-1}$	2	5	10	4	4	7	7	–	–	4	2

Table 6: *Information used to obtain the type for the starter $\{\{0,8\},\{1,2\},\{3,5\},\{4,9\},\{6,10\}\}$ in $\mathbb{Z}_{11}$, which is $1^2 2^2 3^1$ since there are two differences repeated only once (namely 5 and 10), two differences repeated twice (namely 2 and 7) and one difference repeated three times (namely 4).*

The type $1^{d_1}2^{d_2}\dots(2n-3)^{d_{2n-3}}$ of a starter represents an integer partition of $2n-3$ since $\sum_{i=1}^{2n-3} id_i = 2n-3$, as can be seen in the example in Table 6. Hence the number of possible types of starters in $\mathbb{Z}_{2n-1}$ is equal to the number of distinct integer partitions of $2n-3$ (see Biggs [2] for more detail on integer partitions). It follows from the next result that it is not necessary to construct the entire tournament in order to obtain the COE-value of a starter.

Theorem 5.1 *The COE-value of a starter s in $\mathbb{Z}_{2n-1}$ of type $1^{d_1}2^{d_2}\dots(2n-3)^{d_{2n-3}}$ is*

$$\gamma(s) = (2n-1)\left(3 + \sum_{i=1}^{2n-3} i^2 d_i\right).$$

Proof: Consider counting the number of carry-over effects due to $a \in \mathbb{Z}_{2n-1}$. Notice that if a gives a carry-over effect to ∞ from round i to round $i+1$, then $a-j$ gives a carry-over effect to ∞ from round $i-j$ to round $i-j+1$ for each $j \in \mathbb{Z}_{2n-1}$. Hence a gives a carry-over effect to ∞ exactly once.

Furthermore, if a gives a carry-over effect to b and $b-a=1$, then the match $\{b,k\}$ follows the match $\{a,k\}$ for a fixed k, which can only occur if $k=\infty$. Hence a gives exactly one carry-over effect to b if $b-a=1$.

Finally, consider counting the number of times a gives a carry-over effect to b, where $b-a=k\neq 1$. If $x_i - x_{i-1} = 1-k$ for some $i \in \mathbb{Z}_{2n-1}$ (where x_i once again denotes the player that opposes i in s), then x_i gives a carry-over effect to $x_{i-1}+1$, and $x_i+(a-x_i)=a$ gives a carry-over effect to $x_{i-1}+1+(a-x_i) = a+1-(x_i-x_{i-1}) = a+1-1+k = b$. Hence the number of times a gives a carry-over effect to b is equal to the number of elements $i \in \mathbb{Z}_{2n-1}$ for which $x_i - x_{i-1} = 1-k$, and this number is recorded in the type of the starter.

Hence each player (except ∞) contributes $1^2 + 1^2 + \sum_{i=1}^{2n-3} i^2 d_i$ to the COE-value of the tournament. Notice that if ∞ gives a carry-over effect to a, then ∞ gives a carry-over effect to $a+i$ for any $i \in \mathbb{Z}_{2n-1}$, and hence ∞ gives a carry-over effect to each other player exactly once. The player ∞ therefore contributes $\sum_{i=0}^{2n-1} 1^2 = 2n-1$ to the COE-value of the tournament. Taking the sum of the contributions over all players delivers the desired result. ■

For example, the starter $\{\{0,8\},\{1,2\},\{3,5\},\{4,9\},\{6,10\}\}$ in $\mathbb{Z}_{11}$ of type $1^2 2^2 3^1$ has COE-value

$$(2n-1)\left(3 + \sum_{i=1}^{2n-3} i^2 d_i\right) \quad = \quad 11(3 + 1^2(2) + 2^2(2) + 3^2(1)) = 242.$$

A *good starter* is a starter of type 1^{2n-3}, which delivers a round-robin tournament that is balanced with respect to carry-over effects, since

$$(2n-1)\left(3 + \sum_{i=1}^{2n-3} i^2 d_i\right) \quad = \quad (2n-1)\left(3 + 1^2(2n-3)\right) = (2n-1)2n.$$

While $2n(2n-1)$ is a lower bound on the COE-value of a starter, an upper bound is reached when the starter is of type $(2n-3)^1$ (the patterned starter is of this type), in which case the COE-value of the starter is

$$(2n-1)\left(3+\sum_{i=1}^{2n-3} i^2 d_i\right) \quad = \quad (2n-1)\left(3+(2n-3)^2\right).$$

5.2 Moves in the starter solution space

In order to better explain the tabu-search moves that were applied to starters, a new notation for starters is introduced. Let s_i denote the element which is paired with the element[2] $s_i + i$ in the starter s for each $i \in \{1, 2, \ldots, (2n-2)/2\}$. Using this notation a starter can be fully represented simply by the sequence $(s_1, s_2, \ldots, s_{(2n-2)/2})$. For instance, the starter $\{\{0,8\},\{1,2\},\{3,5\},\{4,9\},\{6,10\}\}$ may be represented in this way by the sequence $(1,3,8,6,4)$, and note that there is a unique one-to-one correspondance between the two representations.

A move is performed on the starter $(s_1, s_2, \ldots, s_{(2n-2)/2})$ by either replacing s_i by $s_i + i$ or by $s_i - i$ for some $i \in \{1, 2, \ldots, (2n-2)/2\}$, which will not necessarily result in another starter. In fact, if $s_i + i$ ($s_i - i$, resp.) is not the residual element of s, then the resulting sequence will not represent a starter if s_i is replaced by $s_i + i$ ($s_i - i$, resp.). For instance, consider replacing 8 in the starter $(1,3,8,6,4)$ by $8+3=0$, resulting in $(1,3,0,6,4)$, or $\{\{1,2\},\{3,5\},\{0,3\},\{6,10\},\{4,9\}\}$ (as in the usual representation for starters). This is not a starter since 3 appears in two distinct pairs.

This infeasibility may be repaired, however, by a finite, deterministic sequence of moves similar to the *ejection chains* for travelling salesman problems discussed by Glover [8]. If s_i is replaced by $s_i \pm i$ (and this element is not the residual element of s), then there exists some j so that either $s_j = s_i \pm i$ or $s_j + j = s_i \pm i$. If $s_j = s_i \pm i$, then this duplication may be removed by replacing s_j by $s_j + 1$, or if $s_j + j = s_i \pm i$ it may be removed by replacing s_j by $s_j - j$. Once again some element may appear twice, and this process is repeated until a starter is obtained.

Consider, for example, replacing 3 by 5 in the starter $t = (8,3,4,6)$. Table 7 contains in column i the pair $(t_i, t_i + i)$, and the sequence of moves applied to t in order for a starter to be obtained again is given in boldface. For instance, in the first step 3 is replaced by 5, and consequently 7 is repeated. In order to remove this duplication, 4 is replaced by 1, and since this causes another duplication, 6 is replaced by 2. Since 2 is the residual element of t, it is not repeated, and the resulting set of pairs form a starter.

5.3 Choice of the tabu-list structure

A number of different tabu-list structures were considered for implementation, but most of them exhibited difficulties. The most common approach is to either list moves or inverses of moves in the tabu-list, as discussed in detail in Glover [6]. If moves are to be listed in

[2] In what follows it is assumed that all operations are performed modulo the order of the set over which a starter is defined.

1	2	3	4
$(8,0)$	$(3,\mathbf{5})$	$(4,7)$	$(6,1)$
$(8,0)$	$(\mathbf{5},7)$	$(\mathbf{4},7)$	$(6,1)$
$(8,0)$	$(5,7)$	$(1,\mathbf{4})$	$(\mathbf{6},1)$
$(8,0)$	$(5,7)$	$(1,4)$	$(2,\mathbf{6})$

Table 7: *Column i in this table contains the pair $(t_i, t_i + i)$ where $t = (8,3,4,6)$. Replacing 3 by 5 in this starter results in the sequence of moves given in boldface in order for the resulting set of pairs to be a starter again.*

the tabu-list, the entire chain of moves resulting from a single move (as discussed in the previous section) should be listed as one entry in the tabu-list. However, it often happens that the chains are extremely long compared to the length of the starter, and the lengths of these chains also turn out to be rather unpredictable. For instance, replacing 14 by 6 in the starter $(0,8,2,7,4,12,14)$ is followed by a sequence of 32 moves before another starter is obtained, while replacing 0 by 1 is followed by a sequence of only 2 moves.

Another difficulty is that there are often two distinct moves that map one starter to another. For instance, replacing 0 by 14 in the starter $(0,7,3,10,8,11,5)$ results in the starter $(0,5,9,10,3,11,6)$ after 13 moves, while replacing 5 by 13 results in the same starter after 6 moves. This implies that moves do not have unique inverses, and it is therefore of little use to list inverses of moves in the tabu-list. What is observed in this case is that the search is concentrated in a region around a local optimum even though a large number of previous moves are listed in the tabu-list, and this is due to the fact that the search is able to find alternative paths through the same set of solutions.

The best alternative is therefore to list previous solutions in the tabu-list instead of moves or inverses of moves, and impirical testing showed that this structure for the tabu-list indeed outperformed the other alternatives, as was expected.

5.4 Initialisation and termination

The initial solution for the tabu-search algorithm was generated randomly by using a backtracking tree-search approach for constructing a starter, where branches are selected at random on each level of the tree. If a terminating node in the tree is reached, the search terminates if a starter has been constructed and restarts otherwise. As stopping criterium the tabu-search terminates if the best solution found so far has not been improved upon for a predefined number of iterations.

6 Numerical results

In order to make use of a number of different combinations of the two parameters (the tabu list length N and the number of iterations allowed without improvement I), the following experiment was conducted for various orders. The intial values of $(N, I) = (100, 100)$ were chosen, and for each combination of N and I the tabu-search was run 500 times (a number which was chosen after empirical testing to represent a sufficient trade-off between diversity of initial starting solutions and computing time required), and after these 500 runs N was doubled if $N < I$, while I was doubled and N set to 100 if $N = I$ (hence N was restricted not to be larger than I). Finally, for each $4 \leq n \leq 40$ (where n is the order of the round-robin tournament) this expermiment was terminated if an optimal solution was not reached whithin six hours.

n	IP	LB	UB	Best found
4	1	12	12	**12**
6	3	30	60	**60**
8	7	56	196	**56**
10	15	90	90	*108
12	30	132	924	**176
14	56	182	1612	**234
16	101	240	2580	**240**
18	176	306	3876	*340
20	297	380	5548	**380**
22	490	462	7644	**462**
24	792	552	10212	*598
26	1255	650	13300	*700
28	1958	756	16956	*810
30	3010	870	21228	*928
32	4565	992	26164	*1054
34	6842	1122	31812	**1254
36	10143	1260	38220	****1540
38	14883	1406	45436	***1628
40	21637	1560	53508	****1872

Table 8: *Results obtained for round-robin tournaments of orders $4 \leq n \leq 40$. COE-values that attain the lower bound are shown in boldface, while * indicates that the COE-value attains the second best possible solution value (using starters), ** indicates that the starter attains the third best possible solution value,* etc. *The number of integer partitions of $n-3$* (i.e. *the number of possible COE-values of round-robin tournaments of order n constructed from starters) are also given, together with the lower bound $n(n-1)$ (corresponding to a balanced tournament) as well as the upper bound $(n-1)\left(3+(n-3)^2\right)$.*

The results obtained are shown in Table 8. Where the lower bound was attained (*i.e.* where a good starter was found), the COE-values are given in boldface. It is also indicated where solution values are second best, third best, *etc.* Notice that, while the best starter (a good starter) is of type $1^{(n-3)}$, the second best starter is of type $1^{(n-5)}2^1$, the third best starter is of type $1^{(n-7)}2^2$ and the fourth best starter is of type $1^{(n-9)}2^3$.

Up to and including order 22, the same solutions were found as by Anderson [1], according to whom the COE-values for orders 12 and 14 are best possible. Anderson was not sure, however, whether the solutions found for orders 18 and 24 are best possible, and here the COE-value of 644 for $n = 24$ found by Anderson was improved upon. For orders larger than 24, however, no previous COE-values were published for orders which are not powers of two, and the COE-values shown in Table 8 are therefore the best known upper bounds for $24 \leq n \leq 40$ and $n \neq 32$. The starters corresponding to the COE-values for $24 \leq n \leq 40$ given in Table 8 are given in Table 9 (starters for $n \leq 22$ may be found in Anderson [1]).

n	Best starter found
24	$(7, 16, 20, 1, 21, 11, 6, 4, 10, 22, 14)$
26	$(24, 7, 16, 22, 8, 4, 14, 15, 11, 2, 17, 18)$
28	$(7, 19, 0, 1, 12, 25, 15, 18, 11, 6, 13, 2, 23)$
30	$(8, 2, 17, 11, 23, 24, 25, 6, 10, 16, 7, 0, 21, 13)$
32	$(4, 10, 20, 27, 14, 7, 11, 29, 24, 22, 17, 3, 8, 16, 25)$
34	$(24, 15, 6, 3, 8, 21, 28, 22, 10, 1, 20, 14, 32, 4, 23, 0)$
36	$(8, 24, 27, 16, 7, 32, 33, 15, 22, 0, 17, 1, 6, 11, 34, 2, 4)$
38	$(20, 10, 8, 27, 18, 35, 32, 6, 16, 28, 29, 24, 17, 5, 7, 34, 9, 15)$
40	$(29, 9, 19, 24, 32, 8, 34, 12, 35, 16, 7, 31, 36, 38, 0, 17, 23, 3, 6)$

Table 9: *The best starters found for round-robin tournaments of orders* $24 \leq n \leq 40$.

Unfortunately, the lower bound is not reached for any new orders; however, up to and including order 32 the second best solution was found for all orders greater than 22, whereafter the solution quality began to decrease slightly. This is partly due to the procedure used to generate random starters, which, for orders 34 and upwards, was the stage of the experiment that required the largest amount of computing time. For this reason the experiment was not run for orders larger than 40.

7 Conclusion

In this paper a tabu-search algorithm was implemented in order to find a starter that produces a round-robin tournament with a small COE-value. The best previously published solutions were validated for round-robin tournaments of orders $4 \leq n \leq 22$, and for $n = 24$ the best previously published solution found was improved. For orders larger than 24 no previous solutions were published, except where n is a power of two, and in this paper new upper bounds for orders $26 \leq n \leq 40$ and $n \neq 32$ were obtained, where most of these attain the second lowest COE-value possible when using starters.

Starters provide a means of implementing fast algorithms for finding round-robin tournaments satisfying certain requirements for two reasons. Firstly, when working with starters, only a one-dimensional array of length $(n-2)/2$ is necessary in order to represent a starter in $\mathbb{Z}_{n-1}$, while representing an entire round-robin tournament of order n requires an $n \times (n-1)$ two-dimensional array. Secondly, applying moves to starters results in much smaller neighbourhoods compared to the moves applied to round-robin tournaments (as in Guedes and Ribeiro [9]). Moves in round-robin tournaments of order n include the so-called *partial round swaps* (more than $n-1$ possible moves in total), *partial team swaps* (more than n possible move in total), as well as reordering of the rounds ($n!$ possible moves in total), which results in very large neighbourhoods for large n, whereas the neighbourhoods for starters are simply of size $n-2$.

The drawback of using starters is that not all round-robin tournaments are cyclic, and that only a small portion of the class of all round-robin tournaments of any given order is considered. For instance, the best COE-value found for $n = 12$ is 160, which was found by Guedes and Ribeiro [9]. It is, however, impossible for a round-robin tournament constructed from a starter to have a COE-value of 160.

The greatest drawback of the tabu-search algorithm is the decrease in speed due to the large amount of time spent generating initial solutions for orders $n \geq 34$. The generation procedure for random starters is essentially a brute force method, and is therefore ineffective for large n. Future considerations should therefore certainly include a more efficient method for generating random starters.

References

[1] Anderson I, 1999, *Balancing carry-over effects in round robin tournaments*, pp. 1–16 in Holroyd FC, Quinn KAS, Rowley C & Webb BS (Eds.), *Combinatorial designs and their applications*, CRC Press, Boca Raton (FL).

[2] Biggs NL, 1985. *Discrete mathematics*, Oxford University Press, New York (NY).

[3] De Warra D, 1980, *Geography, games and graphs*, Discrete Applied Mathematics, **2**, pp. 327–337.

[4] Dinitz JH, Froncek D, Lamken ER & Wallis WD, 2007, *Scheduling a tournament*, pp. 591–606 in Colbourn CJ & Dinitz JH (Eds.), *The handbook of combinatorial designs*, 2nd edition, CRC Press, Boca Raton (FL).

[5] Easton K, Nemhauser G, Trick MA, 2001, *The travelling tournament problem: Description and benchmarks*, pp. 580–584 in Walsh T (Ed.), Principles and practice of constraint programming, Lecture Notes in Computer Science, **2239**, Springer, Berlin.

[6] Glover F, 1989, *Tabu search — Part I*, Operations Research Society of America Journal on Computing, **1(3)**, pp. 190–206.

[7] Glover F, 1990, *Tabu search — Part II*, Operations Research Society of America Journal on Computing, **2(1)**, pp. 4–32.

[8] Glover F, 1996, *Ejection chains, reference structures and alternating path methods for traveling salesman problems*, Discrete Applied Mathematics, **65**, pp. 223–253.

[9] Guedes ACB & Ribeiro CC, 2009, *A hybrid heuristic for minimising the weighted carry-over effects in round robin tournaments*, Proceedings of the 4th Multidisciplinary International Conference on Scheduling Theory and Applications, Dublin, pp. 115–129.

[10] Henz M, Müller T & Thiel S, 2004, *Global constraints for round robin tournament scheduling*, European Journal of Operational Research, **153**, pp. 92–101.

[11] Keedwell AD, 2000, *Designing tournaments with the aid of Latin squares: A presentation of old and new results*, Utilitas Mathematica, **58**, pp. 65–85.

[12] Kendall G, Knust S, Ribeiro CC & Urrutia S, 2010, *Scheduling in sports: An annotated bibliography*, Computers and Operations Research, **37(1)**, pp. 1–19.

[13] Mendelsohn E & Rosa A, 1985, *One-factorisations of the complete graph — A survey*, Journal of Graph Theory, **9**, pp. 43–65.

[14] Miyashiro R & Matsui T, 2006, *Minimizing the carry-over effects value in a round robin tournament*, pp. 460–463 in Burke EK & Rudová H (Eds.), Proceedings of the 6th International Conference on the Practice and Theory of Automated Timetabling, Brno.

[15] Rasmussen RV & Trick MA, 2008, *Round robin scheduling — A survey*, European Journal of Operational Research, **188**, pp. 617–636.

[16] Russel KG, 1980, *Balancing carry-over effects in round robin tournaments*, Biometrika, **67**, pp. 127–131.

[17] Trick MA, 2000, *A schedule-then-break approach to sports timetabling*, pp. 324–343 in Burke E & Erben W (Eds.), *Practice and theory of automated timetabling III*, Springer-Verlag, Berlin.

Appendix: Round-robin tournament schedules

Round-robin tournaments of orders $4 \leq n \leq 30$ with COE-values attaining the current best upper bounds are given here in tabular form. Recall that in tabular form, the opponent of player i in round j is given by the entry in row i and column j. In the upper left hand corner of each table the number of players present in the corresponding tournament is given in boldface.

4	0	1	2
0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

6	0	1	2	3	4
0	5	2	4	1	3
1	4	5	3	0	2
2	3	0	5	4	1
3	2	4	1	5	0
4	1	3	0	2	5
5	0	1	2	3	4

8	0	1	2	3	4	5	6
0	1	5	4	2	7	3	6
1	0	2	6	5	3	7	4
2	5	1	3	0	6	4	7
3	7	6	2	4	1	0	5
4	6	7	0	3	5	2	1
5	2	0	7	1	4	6	3
6	4	3	1	7	2	5	0
7	3	4	5	6	0	1	2

10	0	1	2	3	4	5	6	7	8
0	4	3	8	1	7	5	2	6	9
1	9	5	4	0	2	8	6	3	7
2	8	9	6	5	1	3	0	7	4
3	5	0	9	7	6	2	4	1	8
4	0	6	1	9	8	7	3	5	2
5	3	1	7	2	9	0	8	4	6
6	7	4	2	8	3	9	1	0	5
7	6	8	5	3	0	4	9	2	1
8	2	7	0	6	4	1	5	9	3
9	1	2	3	4	5	6	7	8	0

12	0	1	2	3	4	5	6	7	8	9	a
0	6	3	9	4	2	5	a	1	b	8	7
1	8	7	4	a	5	3	6	0	2	b	9
2	a	9	8	5	0	6	4	7	1	3	b
3	b	0	a	9	6	1	7	5	8	2	4
4	5	b	1	0	a	7	2	8	6	9	3
5	4	6	b	2	1	0	8	3	9	7	a
6	0	5	7	b	3	2	1	9	4	a	8
7	9	1	6	8	b	4	3	2	a	5	0
8	1	a	2	7	9	b	5	4	3	0	6
9	7	2	0	3	8	a	b	6	5	4	1
a	2	8	3	1	4	9	0	b	7	6	5
b	3	4	5	6	7	8	9	a	0	1	2

14	0	1	2	3	4	5	6	7	8	9	a	b	c
0	b	4	2	5	a	6	d	3	c	1	9	8	7
1	8	c	5	3	6	b	7	d	4	0	2	a	9
2	a	9	0	6	4	7	c	8	d	5	1	3	b
3	c	b	a	1	7	5	8	0	9	d	6	2	4
4	5	0	c	b	2	8	6	9	1	a	d	7	3
5	4	6	1	0	c	3	9	7	a	2	b	d	8
6	9	5	7	2	1	0	4	a	8	b	3	c	d
7	d	a	6	8	3	2	1	5	b	9	c	4	0
8	1	d	b	7	9	4	3	2	6	c	a	0	5
9	6	2	d	c	8	a	5	4	3	7	0	b	1
a	2	7	3	d	0	9	b	6	5	4	8	1	c
b	0	3	8	4	d	1	a	c	7	6	5	9	2
c	3	1	4	9	5	d	2	b	0	8	7	6	a
d	7	8	9	a	b	c	0	1	2	3	4	5	6

16	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e
0	f	c	9	4	3	a	8	d	6	2	5	e	1	7	b
1	c	f	d	a	5	4	b	9	e	7	3	6	0	2	8
2	9	d	f	e	b	6	5	c	a	0	8	4	7	1	3
3	4	a	e	f	0	c	7	6	d	b	1	9	5	8	2
4	3	5	b	0	f	1	d	8	7	e	c	2	a	6	9
5	a	4	6	c	1	f	2	e	9	8	0	d	3	b	7
6	8	b	5	7	d	2	f	3	0	a	9	1	e	4	c
7	d	9	c	6	8	e	3	f	4	1	b	a	2	0	5
8	6	e	a	d	7	9	0	4	f	5	2	c	b	3	1
9	2	7	0	b	e	8	a	1	5	f	6	3	d	c	4
a	5	3	8	1	c	0	9	b	2	6	f	7	4	e	d
b	e	6	4	9	2	d	1	a	c	3	7	f	8	5	0
c	1	0	7	5	a	3	e	2	b	d	4	8	f	9	6
d	7	2	1	8	6	b	4	0	3	c	e	5	9	f	a
e	b	8	3	2	9	7	c	5	1	4	d	0	6	a	f
f	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e

18	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f	g
0	6	e	h	d	3	9	7	4	g	1	c	b	f	8	2	5	a
1	b	7	f	h	e	4	a	8	5	0	2	d	c	g	9	3	6
2	7	c	8	g	h	f	5	b	9	6	1	3	e	d	0	a	4
3	5	8	d	9	0	h	g	6	c	a	7	2	4	f	e	1	b
4	c	6	9	e	a	1	h	0	7	d	b	8	3	5	g	f	2
5	3	d	7	a	f	b	2	h	1	8	e	c	9	4	6	0	g
6	0	4	e	8	b	g	c	3	h	2	9	f	d	a	5	7	1
7	2	1	5	f	9	c	0	d	4	h	3	a	g	e	b	6	8
8	9	3	2	6	g	a	d	1	e	5	h	4	b	0	f	c	7
9	8	a	4	3	7	0	b	e	2	f	6	h	5	c	1	g	d
a	e	9	b	5	4	8	1	c	f	3	g	7	h	6	d	2	0
b	1	f	a	c	6	5	9	2	d	g	4	0	8	h	7	e	3
c	4	2	g	b	d	7	6	a	3	e	0	5	1	9	h	8	f
d	g	5	3	0	c	e	8	7	b	4	f	1	6	2	a	h	9
e	a	0	6	4	1	d	f	9	8	c	5	g	2	7	3	b	h
f	h	b	1	7	5	2	e	g	a	9	d	6	0	3	8	4	c
g	d	h	c	2	8	6	3	f	0	b	a	e	7	1	4	9	5
h	f	g	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e

20	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f	g	h	i
0	c	6	a	5	8	g	f	7	3	j	4	9	i	1	d	b	h	e	2
1	3	d	7	b	6	9	h	g	8	4	j	5	a	0	2	e	c	i	f
2	g	4	e	8	c	7	a	i	h	9	5	j	6	b	1	3	f	d	0
3	1	h	5	f	9	d	8	b	0	i	a	6	j	7	c	2	4	g	e
4	f	2	i	6	g	a	e	9	c	1	0	b	7	j	8	d	3	5	h
5	i	g	3	0	7	h	b	f	a	d	2	1	c	8	j	9	e	4	6
6	7	0	h	4	1	8	i	c	g	b	e	3	2	d	9	j	a	f	5
7	6	8	1	i	5	2	9	0	d	h	c	f	4	3	e	a	j	b	g
8	h	7	9	2	0	6	3	a	1	e	i	d	g	5	4	f	b	j	c
9	d	i	8	a	3	1	7	4	b	2	f	0	e	h	6	5	g	c	j
a	j	e	0	9	b	4	2	8	5	c	3	g	1	f	i	7	6	h	d
b	e	j	f	1	a	c	5	3	9	6	d	4	h	2	g	0	8	7	i
c	0	f	j	g	2	b	d	6	4	a	7	e	5	i	3	h	1	9	8
d	9	1	g	j	h	3	c	e	7	5	b	8	f	6	0	4	i	2	a
e	b	a	2	h	j	i	4	d	f	8	6	c	9	g	7	1	5	0	3
f	4	c	b	3	i	j	0	5	e	g	9	7	d	a	h	8	2	6	1
g	2	5	d	c	4	0	j	1	6	f	h	a	8	e	b	i	9	3	7
h	8	3	6	e	d	5	1	j	2	7	g	i	b	9	f	c	0	a	4
i	5	9	4	7	f	e	6	2	j	3	8	h	0	c	a	g	d	1	b
j	a	b	c	d	e	f	g	h	i	0	1	2	3	4	5	6	7	8	9

22	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f	g	h	i	j	k
0	5	j	a	2	h	e	k	1	4	f	d	c	7	b	i	6	g	3	8	l	9
1	a	6	k	b	3	i	f	0	2	5	g	e	d	8	c	j	7	h	4	9	l
2	l	b	7	0	c	4	j	g	1	3	6	h	f	e	9	d	k	8	i	5	a
3	b	l	c	8	1	d	5	k	h	2	4	7	i	g	f	a	e	0	9	j	6
4	7	c	l	d	9	2	e	6	0	i	3	5	8	j	h	g	b	f	1	a	k
5	0	8	d	l	e	a	3	f	7	1	j	4	6	9	k	i	h	c	g	2	b
6	c	1	9	e	l	f	b	4	g	8	2	k	5	7	a	0	j	i	d	h	3
7	4	d	2	a	f	l	g	c	5	h	9	3	0	6	8	b	1	k	j	e	i
8	j	5	e	3	b	g	l	h	d	6	i	a	4	1	7	9	c	2	0	k	f
9	g	k	6	f	4	c	h	l	i	e	7	j	b	5	2	8	a	d	3	1	0
a	1	h	0	7	g	5	d	i	l	j	f	8	k	c	6	3	9	b	e	4	2
b	3	2	i	1	8	h	6	e	j	l	k	g	9	0	d	7	4	a	c	f	5
c	6	4	3	j	2	9	i	7	f	k	l	0	h	a	1	e	8	5	b	d	g
d	h	7	5	4	k	3	a	j	8	g	0	l	1	i	b	2	f	9	6	c	e
e	f	i	8	6	5	0	4	b	k	9	h	1	l	2	j	c	3	g	a	7	d
f	e	g	j	9	7	6	1	5	c	0	a	i	2	l	3	k	d	4	h	b	8
g	9	f	h	k	a	8	7	2	6	d	1	b	j	3	l	4	0	e	5	i	c
h	d	a	g	i	0	b	9	8	3	7	e	2	c	k	4	l	5	1	f	6	j
i	k	e	b	h	j	1	c	a	9	4	8	f	3	d	0	5	l	6	2	g	7
j	8	0	f	c	i	k	2	d	b	a	5	9	g	4	e	1	6	l	7	3	h
k	i	9	1	g	d	j	0	3	e	c	b	6	a	h	5	f	2	7	l	8	4
l	2	3	4	5	6	7	8	9	a	b	c	d	e	f	g	h	i	j	k	0	1

24	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f	g	h	i	j	k	l	m
0	k	a	5	3	e	l	h	2	n	b	g	f	6	9	d	m	1	7	j	8	i	c	4
1	5	l	b	6	4	f	m	i	3	n	c	h	g	7	a	e	0	2	8	k	9	j	d
2	e	6	m	c	7	5	g	0	j	4	n	d	i	h	8	b	f	1	3	9	l	a	k
3	l	f	7	0	d	8	6	h	1	k	5	n	e	j	i	9	c	g	2	4	a	m	b
4	c	m	g	8	1	e	9	7	i	2	l	6	n	f	k	j	a	d	h	3	5	b	0
5	1	d	0	h	9	2	f	a	8	j	3	m	7	n	g	l	k	b	e	i	4	6	c
6	d	2	e	1	i	a	3	g	b	9	k	4	0	8	n	h	m	l	c	f	j	5	7
7	8	e	3	f	2	j	b	4	h	c	a	l	5	1	9	n	i	0	m	d	g	k	6
8	7	9	f	4	g	3	k	c	5	i	d	b	m	6	2	a	n	j	1	0	e	h	l
9	m	8	a	g	5	h	4	l	d	6	j	e	c	0	7	3	b	n	k	2	1	f	i
a	j	0	9	b	h	6	i	5	m	e	7	k	f	d	1	8	4	c	n	l	3	2	g
b	h	k	1	a	c	i	7	j	6	0	f	8	l	g	e	2	9	5	d	n	m	4	3
c	4	i	l	2	b	d	j	8	k	7	1	g	9	m	h	f	3	a	6	e	n	0	5
d	6	5	j	m	3	c	e	k	9	l	8	2	h	a	0	i	g	4	b	7	f	n	1
e	2	7	6	k	0	4	d	f	l	a	m	9	3	i	b	1	j	h	5	c	8	g	n
f	n	3	8	7	l	1	5	e	g	m	b	0	a	4	j	c	2	k	i	6	d	9	h
g	i	n	4	9	8	m	2	6	f	h	0	c	1	b	5	k	d	3	l	j	7	e	a
h	b	j	n	5	a	9	0	3	7	g	i	1	d	2	c	6	l	e	4	m	k	8	f
i	g	c	k	n	6	b	a	1	4	8	h	j	2	e	3	d	7	m	f	5	0	l	9
j	a	h	d	l	n	7	c	b	2	5	9	i	k	3	f	4	e	8	0	g	6	1	m
k	0	b	i	e	m	n	8	d	c	3	6	a	j	l	4	g	5	f	9	1	h	7	2
l	3	1	c	j	f	0	n	9	e	d	4	7	b	k	m	5	h	6	g	a	2	i	8
m	9	4	2	d	k	g	1	n	a	f	e	5	8	c	l	0	6	i	7	h	b	3	j
n	f	g	h	i	j	k	l	m	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e

26	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
0	o	1	h	4	i	g	m	c	b	3	8	7	k	f	9	j	n	5	2	p	d	6	e	a	l
1	m	0	2	i	5	j	h	n	d	c	4	9	8	l	g	a	k	o	6	3	p	e	7	f	b
2	c	n	1	3	j	6	k	i	o	e	d	5	a	9	m	h	b	l	0	7	4	p	f	8	g
3	h	d	o	2	4	k	7	l	j	0	f	e	6	b	a	n	i	c	m	1	8	5	p	g	9
4	a	i	e	0	3	5	l	8	m	k	1	g	f	7	c	b	o	j	d	n	2	9	6	p	h
5	i	b	j	f	1	4	6	m	9	n	l	2	h	g	8	d	c	0	k	e	o	3	a	7	p
6	p	j	c	k	g	2	5	7	n	a	o	m	3	i	h	9	e	d	1	l	f	0	4	b	8
7	9	p	k	d	l	h	3	6	8	o	b	0	n	4	j	i	a	f	e	2	m	g	1	5	c
8	d	a	p	l	e	m	i	4	7	9	0	c	1	o	5	k	j	b	g	f	3	n	h	2	6
9	7	e	b	p	m	f	n	j	5	8	a	1	d	2	0	6	l	k	c	h	g	4	o	i	3
a	4	8	f	c	p	n	g	o	k	6	9	b	2	e	3	1	7	m	l	d	i	h	5	0	j
b	k	5	9	g	d	p	o	h	0	l	7	a	c	3	f	4	2	8	n	m	e	j	i	6	1
c	2	l	6	a	h	e	p	0	i	1	m	8	b	d	4	g	5	3	9	o	n	f	k	j	7
d	8	3	m	7	b	i	f	p	1	j	2	n	9	c	e	5	h	6	4	a	0	o	g	l	k
e	l	9	4	n	8	c	j	g	p	2	k	3	o	a	d	f	6	i	7	5	b	1	0	h	m
f	n	m	a	5	o	9	d	k	h	p	3	l	4	0	b	e	g	7	j	8	6	c	2	1	i
g	j	o	n	b	6	0	a	e	l	i	p	4	m	5	1	c	f	h	8	k	9	7	d	3	2
h	3	k	0	o	c	7	1	b	f	m	j	p	5	n	6	2	d	g	i	9	l	a	8	e	4
i	5	4	l	1	0	d	8	2	c	g	n	k	p	6	o	7	3	e	h	j	a	m	b	9	f
j	g	6	5	m	2	1	e	9	3	d	h	o	l	p	7	0	8	4	f	i	k	b	n	c	a
k	b	h	7	6	n	3	2	f	a	4	e	i	0	m	p	8	1	9	5	g	j	l	c	o	d
l	e	c	i	8	7	o	4	3	g	b	5	f	j	1	n	p	9	2	a	6	h	k	m	d	0
m	1	f	d	j	9	8	0	5	4	h	c	6	g	k	2	o	p	a	3	b	7	i	l	n	e
n	f	2	g	e	k	a	9	1	6	5	i	d	7	h	l	3	0	p	b	4	c	8	j	m	o
o	0	g	3	h	f	l	b	a	2	7	6	j	e	8	i	m	4	1	p	c	5	d	9	k	n
p	6	7	8	9	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	0	1	2	3	4	5

28	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q
0	3	j	6	g	d	k	p	i	2	8	m	h	7	f	b	5	9	r	e	q	1	a	n	l	o	c	4
1	5	4	k	7	h	e	l	q	j	3	9	n	i	8	g	c	6	a	r	f	0	2	b	o	m	p	d
2	e	6	5	l	8	i	f	m	0	k	4	a	o	j	9	h	d	7	b	r	g	1	3	c	p	n	q
3	0	f	7	6	m	9	j	g	n	1	l	5	b	p	k	a	i	e	8	c	r	h	2	4	d	q	o
4	p	1	g	8	7	n	a	k	h	o	2	m	6	c	q	l	b	j	f	9	d	r	i	3	5	e	0
5	1	q	2	h	9	8	o	b	l	i	p	3	n	7	d	0	m	c	k	g	a	e	r	j	4	6	f
6	g	2	0	3	i	a	9	p	c	m	j	q	4	o	8	e	1	n	d	l	h	b	f	r	k	5	7
7	8	h	3	1	4	j	b	a	q	d	n	k	0	5	p	9	f	2	o	e	m	i	c	g	r	l	6
8	7	9	i	4	2	5	k	c	b	0	e	o	l	1	6	q	a	g	3	p	f	n	j	d	h	r	m
9	n	8	a	j	5	3	6	l	d	c	1	f	p	m	2	7	0	b	h	4	q	g	o	k	e	i	r
a	r	o	9	b	k	6	4	7	m	e	d	2	g	q	n	3	8	1	c	i	5	0	h	p	l	f	j
b	k	r	p	a	c	l	7	5	8	n	f	e	3	h	0	o	4	9	2	d	j	6	1	i	q	m	g
c	h	l	r	q	b	d	m	8	6	9	o	g	f	4	i	1	p	5	a	3	e	k	7	2	j	0	n
d	o	i	m	r	0	c	e	n	9	7	a	p	h	g	5	j	2	q	6	b	4	f	l	8	3	k	1
e	2	p	j	n	r	1	d	f	o	a	8	b	q	i	h	6	k	3	0	7	c	5	g	m	9	4	l
f	m	3	q	k	o	r	2	e	g	p	b	9	c	0	j	i	7	l	4	1	8	d	6	h	n	a	5
g	6	n	4	0	l	p	r	3	f	h	q	c	a	d	1	k	j	8	m	5	2	9	e	7	i	o	b
h	c	7	o	5	1	m	q	r	4	g	i	0	d	b	e	2	l	k	9	n	6	3	a	f	8	j	p
i	q	d	8	p	6	2	n	0	r	5	h	j	1	e	c	f	3	m	l	a	o	7	4	b	g	9	k
j	l	0	e	9	q	7	3	o	1	r	6	i	k	2	f	d	g	4	n	m	b	p	8	5	c	h	a
k	b	m	1	f	a	0	8	4	p	2	r	7	j	l	3	g	e	h	5	o	n	c	q	9	6	d	i
l	j	c	n	2	g	b	1	9	5	q	3	r	8	k	m	4	h	f	i	6	p	o	d	0	a	7	e
m	f	k	d	o	3	h	c	2	a	6	0	4	r	9	l	n	5	i	g	j	7	q	p	e	1	b	8
n	9	g	l	e	p	4	i	d	3	b	7	1	5	r	a	m	o	6	j	h	k	8	0	q	f	2	c
o	d	a	h	m	f	q	5	j	e	4	c	8	2	6	r	b	n	p	7	k	i	l	9	1	0	g	3
p	4	e	b	i	n	g	0	6	k	f	5	d	9	3	7	r	c	o	q	8	l	j	m	a	2	1	h
q	i	5	f	c	j	o	h	1	7	l	g	6	e	a	4	8	r	d	p	0	9	m	k	n	b	3	2
r	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	0	1	2	3	4	5	6	7	8	9

30	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s
0	c	o	f	j	7	6	5	t	d	q	k	i	3	a	p	l	e	h	4	9	s	1	b	8	g	r	m	2	n
1	o	d	p	g	k	8	7	6	t	e	r	l	j	4	b	q	m	f	i	5	a	0	2	c	9	h	s	n	3
2	4	p	e	q	h	l	9	8	7	t	f	s	m	k	5	c	r	n	g	j	6	b	1	3	d	a	i	0	o
3	p	5	q	f	r	i	m	a	9	8	t	g	0	n	l	6	d	s	o	h	k	7	c	2	4	e	b	j	1
4	2	q	6	r	g	s	j	n	b	a	9	t	h	1	o	m	7	e	0	p	i	l	8	d	3	5	f	c	k
5	l	3	r	7	s	h	0	k	o	c	b	a	t	i	2	p	n	8	f	1	q	j	m	9	e	4	6	g	d
6	e	m	4	s	8	0	i	1	l	p	d	c	b	t	j	3	q	o	9	g	2	r	k	n	a	f	5	7	h
7	i	f	n	5	0	9	1	j	2	m	q	e	d	c	t	k	4	r	p	a	h	3	s	l	o	b	g	6	8
8	9	j	g	o	6	1	a	2	k	3	n	r	f	e	d	t	l	5	s	q	b	i	4	0	m	p	c	h	7
9	8	a	k	h	p	7	2	b	3	l	4	o	s	g	f	e	t	m	6	0	r	c	j	5	1	n	q	d	i
a	j	9	b	l	i	q	8	3	c	4	m	5	p	0	h	g	f	t	n	7	1	s	d	k	6	2	o	r	e
b	f	k	a	c	m	j	r	9	4	d	5	n	6	q	1	i	h	g	t	o	8	2	0	e	l	7	3	p	s
c	0	g	l	b	d	n	k	s	a	5	e	6	o	7	r	2	j	i	h	t	p	9	3	1	f	m	8	4	q
d	r	1	h	m	c	e	o	l	0	b	6	f	7	p	8	s	3	k	j	i	t	q	a	4	2	g	n	9	5
e	6	s	2	i	n	d	f	p	m	1	c	7	g	8	q	9	0	4	l	k	j	t	r	b	5	3	h	o	a
f	b	7	0	3	j	o	e	g	q	n	2	d	8	h	9	r	a	1	5	m	l	k	t	s	c	6	4	i	p
g	q	c	8	1	4	k	p	f	h	r	o	3	e	9	i	a	s	b	2	6	n	m	l	t	0	d	7	5	j
h	k	r	d	9	2	5	l	q	g	i	s	p	4	f	a	j	b	0	c	3	7	o	n	m	t	1	e	8	6
i	7	l	s	e	a	3	6	m	r	h	j	0	q	5	g	b	k	c	1	d	4	8	p	o	n	t	2	f	9
j	a	8	m	0	f	b	4	7	n	s	i	k	1	r	6	h	c	l	d	2	e	5	9	q	p	o	t	3	g
k	h	b	9	n	1	g	c	5	8	o	0	j	l	2	s	7	i	d	m	e	3	f	6	a	r	q	p	t	4
l	5	i	c	a	o	2	h	d	6	9	p	1	k	m	3	0	8	j	e	n	f	4	g	7	b	s	r	q	t
m	t	6	j	d	b	p	3	i	e	7	a	q	2	l	n	4	1	9	k	f	o	g	5	h	8	c	0	s	r
n	s	t	7	k	e	c	q	4	j	f	8	b	r	3	m	o	5	2	a	l	g	p	h	6	i	9	d	1	0
o	1	0	t	8	l	f	d	r	5	k	g	9	c	s	4	n	p	6	3	b	m	h	q	i	7	j	a	e	2
p	3	2	1	t	9	m	g	e	s	6	l	h	a	d	0	5	o	q	7	4	c	n	i	r	j	8	k	b	f
q	g	4	3	2	t	a	n	h	f	0	7	m	i	b	e	1	6	p	r	8	5	d	o	j	s	k	9	l	c
r	d	h	5	4	3	t	b	o	i	g	1	8	n	j	c	f	2	7	q	s	9	6	e	p	k	0	l	a	m
s	n	e	i	6	5	4	t	c	p	j	h	2	9	o	k	d	g	3	8	r	0	a	7	f	q	l	1	m	b
t	m	n	o	p	q	r	s	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f	g	h	i	j	k	l