Rwanda Journal of Engineering, Science, Technology and Environment, Volume 7, Issue 1, March 2025
eISSN: 2617-233X | print ISSN: 2617-2321

Mathematical Modeling of Methane Gas Extraction from Lake Kivu

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Abstract

Lake Kivu, located on the border of Rwanda and the Democratic Republic of Congo, contains vast reserves of dissolved methane gas, offering significant opportunities for energy generation. However, improper extraction could destabilize the lake's stratification, potentially triggering catastrophic gas eruptions. The challenge lies in developing an efficient and sustainable method for methane extraction that maximizes energy recovery while maintaining the stability of the lake. This study develops a mathematical model to describe the process of methane gas extraction from Lake Kivu. The model focuses on key aspects of extraction while omitting detailed considerations of the lake's stratification physics and gas-water chemistry. It accounts for the rate of methane gas generation in the resource zone, the extraction of methane from deep water via bubble-driven tubes, and processes occurring in the separation and washing zones. Furthermore, the model incorporates the rate at which methane is released into the deep layers from these zones. It also examines the effects of extraction rates, well placement, and operational strategies on lake stability. From the results, the extraction system will be stable when the yearly average extraction of methane gas from Resource Zone is 0.1704 $km^3/year$, 0.1672 $km^3/year$ in the separation zone and 0.1464 $km^3/year$ in the washing zone. Numerical simulations were conducted to explore various scenarios related to methane gas extraction rates. To better understand the conversion of extracted gas into electrical power, a separate two-compartment model was developed, comprising the production zone and the electricity zone. Overall, the mathematical model serves as a valuable tool for analyzing and optimizing methane gas extraction from Lake Kivu, offering insights to enhance efficiency and sustainability in resource utilization

Keywords: Methane gas, Lake Kivu, mathematical modeling, gas extraction, parameters

1. Introduction

Lake Kivu, located between the Republic of Rwanda and the Democratic Republic of Congo (DRC), is unique among similar East African Rift lakes with an unusual thermal structure. The lake has a surface area of approximately 2,370 km^2 on average depth of 240 m, a maximum width of 48 km, a total length of 89 km, and a total lake volume of about 560 km^3. It also lies at an elevation of 1,460 m above sea level and has approximately two million people in its surrounding areas (Schmid et al., 2002). Lake Kivu water contains big quantities of dissolved

https://dx.doi.org/10.4314/rjeste.v7i1.10

carbon dioxide (CO_2) of 300 km^3, with high concentrations between 55 to 60 km^3 of methane gas (CH_4). Following the catastrophic gas eruptions of the two Cameroonian lakes Monoun and Nyos, whose eruptions caused massive animal and human deaths in Cameroon in 1984 and 1986, respectively, it was feared that a similar disaster could occur in Lake Kivu (Descy et al., 2012; Doevenspeck, 2007).

Most authors agree that the CO_2 has a primarily geogenic origin, whereas the CH_4 is biogenic (Schoell et al., 1988; Tietze, 1978; Tietze et al., 1980, Wüest, A et al., 2012). Schoell et al., 1988 reported that two thirds of CH_4 is produced by reduction of geogenic CO_2 of which the chemical reaction is represented in Equation (1)

$$CO_2 + 4H_2 \rightarrow \mathrm{CH}_4 + 2H_2O \quad (1)$$

and one third by acetoclastic methano-genesis of sedimentary organic material (biogenic material).

Three major risks that could cause a gas eruption: (a) a relatively small uplift of water by a strong internal wave; (b) a volcanic event could produce sufficient thermal energy that would lift water with high gas concentrations to a level where it is oversaturated then bubbles could form; (c) a large amount of gas could be injected into the lake, e.g., by a gas release from the sediments triggered by intruding magma. Catastrophic eruption predicted within 100-200 years (Schmid et. al., 2002).

The governments of Rwanda and DRC took the decision to award concessions for methane extraction to private companies. The 2 Governments required those companies to meet some specific management prescription requirements. The first requirement is to ensure the safety of the population, the second requirement is to maintain the overall stratification of the lake, and last requirement is to maximize the methane harvest by minimizing the amount of the methane loss to the atmosphere and to the toxic surface water of the lake.

Currently, four concessions are awarded; KivuWatt, operational (current 26 MW, total 100 MWe); SPLK (Shema Power Lake Kivu -former Symbion; 56 MW for 25 years) and GasMeth (50 MWe) are under design and construction; and EPPM (KivuPower) in DRC (30 MW). Previously, the capacity of Lake Kivu for electricity generation was believed to be around 700 MW over a duration of 55 years, to be distributed equally over Rwanda and DRC with 350 MW each (Esiara, 2016; Mwai, 2019) - at present Rwanda has awarded 265 MW, and DRC 30 MW. However, recent insights indicate that the capacity for electricity generation may be more limited.

In general, many researchers used different approaches to optimize the methane gas extraction. The first approach for the maximum risk reduction is to extract all CH_4 rapidly and remove all CO_2. The second approach which is the best for conserving the lake's ecological integrity is to

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use the CH_4 from Resource Zone (RZ) and reinject the extracted deep water back into the same RZ without removing the CO_2 in order to maintain the lake stratification. Deep water would then be minimized in CH_4 maintaining properties of all other water in each stratified zone. The last approach which is most economical is to take the deep water entirely down to RZ and to return the reinject-water into the ground of the lake (Biozone) to prevent any dilution of the CH_4-containing deep-water and to remove completely all CH_4 contained in the deep-water.

The RZ located in the deep part of the lake (deeper than 260 m), as shown in Figure 1, contains a substantial amount of methane gas. To extract the gas, a vertical tube is inserted into the RZ, and the deep water flows upwards in the tube due to decreasing pressure, causing bubbles to form and rise. The bubbles are continuously and self-sustainably generated in the tube, and the extracted gases are separated from the deep-water using a separator near the lake surface. The raw gases are washed before the refined gases are transferred through underwater pipelines to the shore, where they can be utilized for distribution, bottling, or electricity. The water without gas is then re-injected into the lake at a depth of around 90 m.

An important factor in determining the capacity for electricity generation is the efficiency of the process of converting the methane to electrical power. The higher the rate of extraction in the short term, the shorter the time until the resource is depleted. If an efficiency of 32% is achieved, a generation of 1,563 MW would deplete the methane in 10 years; a generation of 391 MW in 40 years. If the earlier assumed 700 MW would be realized in the short term, depending on efficiency it would take only between 12-20 years before the resource would be depleted, rather than the previously believed 55 years. In order to further substantiate the total exploitable quantity of methane in the lake additional research and measurements should be conducted to verify the resource potential of Lake Kivu (Stichting Deltares, 2021).

Many researchers were interested in modeling different scenarios in Lake Kivu regarding gas stored in the deep waters. For instance, (Schmid et al., 2003) described vertical mass flow in the lake using a 1-dimensional model based on the κ-ε model for turbulent diffusion. The 1-dimensional model makes the crucial and fundamental assumption that, regardless of the density differences between the lake and the source, any water introduced into the lake at a specific level will stay there. As a result, the model predicts that such a source will continue to exist there, perpetuating it by blending with and diluting water as it rises. Hence, the observation that the water intrusions stay at a fixed depth (Schmid et al., 2010) is really an artifact that was stated in the underlying model.

Due to the exclusion of double diffusive processes, (Osborn, 1980) argues that the application of the κ-ε model to a body of water with diffusively stratified structure like Lake Kivu is problematic. The k-ε model is commonly employed to simulate turbulence in various homogeneous fluid flows, but it might not be the best tool for capturing the complex dynamics of Lake Kivu with high stratification. Additionally, in the diffusively stratified Lake Kivu, it is

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crucial to comprehend the vertical mixing and transport of heat, nutrients, and other substances across the stratification layers. Vertical transport processes might not be accurately modeled using the κ-ε model.

Current research on Lake Kivu asserts that because of the permanent stratification, the methane gas cannot escape to the atmosphere and the quantity is still increasing. It is stated that almost 50% of the newly generated CH_4 is accumulating within the lake, according to a comparison of the rates of creation and oxidation. 0.14 km^3 are stored at this rate per year. According to measurements made by Schmid et al., 2005, CH_4 concentrations had increased by up to 15% since those made by Tietze, 1978, which equates to an increasing rate of 0.5% per year. This means that we anticipate the current formation to be greater than the steady-state. Enormous amounts of Methane gas in Lake Kivu are both beneficial and dangerous to the population around the lake. In terms of benefits, the government of Rwanda is supporting projects to extract the methane gas for electricity generation. However, this process of methane gas extraction may in the long term disturb the stability of the lake, probably resulting in uncontrollable gas inflows and outflow. Therefore, there is a need for a mathematical model to mathematically understand the extraction process and predict the quantity of electricity to be produced annually for decision-makers to prevent the destabilization of the lake.

The main objective of this paper is to formulate a mathematical model describing the dynamics of methane gas from its formation deep down in the sediments. This model takes into account the contributions from hydro-thermal and volcanic activities confined into the resource zone layer by using a compartmental model, its extraction through bubbles inside the pipe injected into the deep waters. This model also takes into consideration the separation and washing zones to maximize the methane gas harvest. The variables that are involved in the problem were defined after the problem was identified. We were able to discover the relationship between the variables and develop a mathematical model similar to the known compartment model to depict the issue as a result of the transfer diagram. Thanks to the transfer diagram, we created a system of ordinary differential equations to represent the relationships between the variables. Then, we ran numerical simulations using the model's predicted parameters to test the equations by examining various model characteristics. Finally, we put the equations to the test by using them to forecast simulated data. In order to comprehend the conversion of gathered gas into power, a separate model with two compartments, namely the production zone and the electricity zone, is developed and analysed as well.

The rest of the work is structured as follows: Section 2 develops a mathematical model from a transfer diagram. In this part, the model's properties and parameters estimation are also listed. In-depth numerical simulations are provided in Section 3 together with discussions of the key findings. Section 4 ends with a final observation and a suggestion for further research.

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2. Model mathematical formulation

Mathematical models are built using mathematical equations that describe the behavior of the system being studied. These equations are ordinary differential equations depending on the nature of the problem being modeled, which is modeling the extraction of gas from deep waters. We refer to the specific techniques and tools used to construct the model, its variables, parameters as well as assumptions to inform the model.

2.1 Model variables and parameters

The dynamics of CH_4 in Lake Kivu from its formation, extraction to its utilization for power production is modeled using a compartmental model. State variables in the mathematical model are variables that represent the state of the system and change over time. They are used to describe the behavior of the system over time and are used to predict the future behavior of the system. In our mathematical model, the state variables are represented by a set of differential equations which describe how the state variables change over time. The initial values of the state variables, or the initial conditions, are also included in the model to fully define the system's behavior. Our model has 4 state variables. These are the quantity of methane gas in the resource zone (Q_{RZ}), the quantity of methane gas extracted from resource zone to the separation zone (Q_{SZ}), the quantity of methane gas in the washing zone (Q_{WZ}) and the quantity of methane gas in the production zone (Q_{PZ}). These states are linked by constant parameters that describe various aspects of the system being modeled. These parameters are values that are used in the equations of the model to represent real-world phenomena. The values of these parameters can have a significant impact on the behavior of the model and its predictions. The specific parameters required for the mathematical model represented by Figure 1 depend on the nature of the system being modeled and the equations used to describe it.

A model in the form of a compartment model is built. The compartments correspond to the extraction processes. Let RZ be the resource zone, SZ separation zone and WZ washing zone. Let Q the quantity of methane gas at each state. The model flow diagram in Figure 1 is drawn explaining the gas extraction process. The compartments are connected by the parameters and in addition to that, the model uses a system of ordinary differential Equations in relation with Figure 1.

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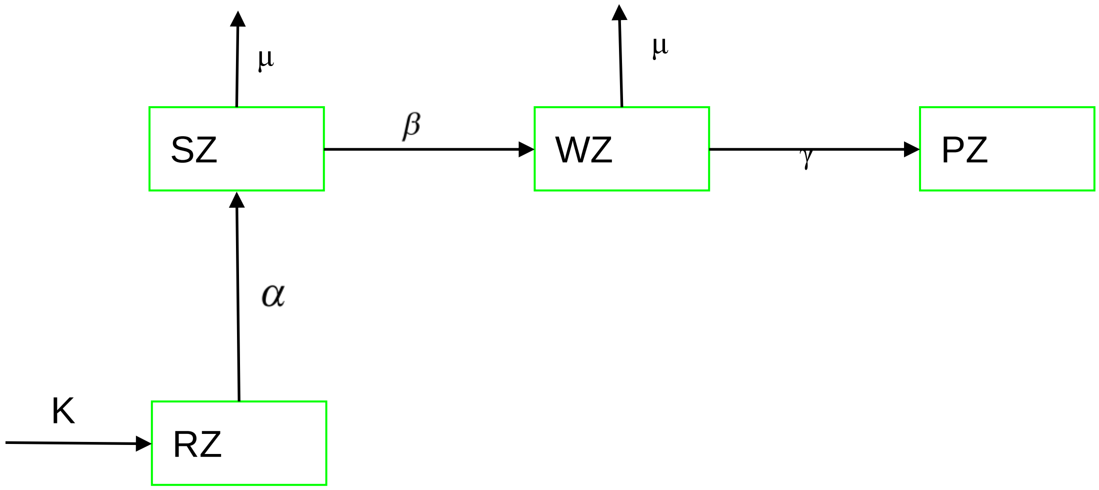

Figure 1: Model Flow chart describing the methane gas extraction process

Assumptions are made about the system being modeled in order to simplify the problem and make it more tractable. These assumptions may include things like ignoring certain variables or assuming that certain relationships hold true. From the proposed model, below are the suggested assumptions: (1) the inflow methane gas K into RZ is assumed to be continuously constant (Fowkes, N. & Mason, D., 2019), (2) all the refined CH_4 is used for production of energy, (3) the returned CH_4 in the lake is reversed into PRZ and (4) the proportion of methane gas returned in the lake from the Separation zone and the proportion of methane gas returned in the lake during the washing process are the same represented by μ and almost negligible. It is important to optimize the exploitation system so as to minimize the quantity of methane discharged in deep water during the separation and washing stages. We assume this loss of methane to be approximately zero or very small.

From the above model flow chart (Figure 1), the following are the ordinary differential equations describing the dynamics of the state variables shown by the system (2)

$$
\begin{aligned}
\frac{dQ_{RZ}}{dt} &= K - \alpha Q_{RZ}\,, \\
\frac{dQ_{SZ}}{dt} &= \alpha Q_{RZ} - (\beta + \mu) Q_{SZ}\,, \\
\frac{dQ_{WZ}}{dt} &= \beta Q_{SZ} - (\gamma + \mu) Q_{WZ}, \\
\frac{dQ_{PZ}}{dt} &= \gamma Q_{WZ}\,,
\end{aligned}
\qquad (2)
$$

with $Q_{RZ}(t), Q_{SZ}(t), Q_{WZ}(t)$ and $Q_{PZ}(t)$ all greater or equal to zero.

2.2 Qualitative Analysis

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Qualitative analysis of a mathematical model involves studying the behavior and properties of the model without explicitly solving it. This can provide insights into the dynamics and behavior of the system described by the model.

The analysis starts by computing the Equilibrium solutions which are the values of the variables that do not change over time. To find equilibrium solutions, set the derivatives of the variables in the systems (2) to zero by decoupling the first three equations from the last one (Ndanguza et al., 2013) and solve the resulting equations. The solution is obtained by equating the right hand side of the model (2) to zero. After simple calculation the only one equilibrium point for the remaining 3 equations is found to be

$$({Q_{RZ}}^{*}, {Q_{SZ}}^{*}, {Q_{WZ}}^{*}) = \left(\frac{K}{\alpha}, \frac{K}{\beta+\mu}, \frac{K}{\gamma+\mu}\right).$$

Another important model property is the stability analysis which determines whether the system will approach or diverge from the equilibrium solutions. Linear stability analysis involves examining the eigenvalues of the Jacobian matrix evaluated at each equilibrium solution. If all eigenvalues have negative real parts, the equilibrium solution is stable. If any eigenvalue has a positive real part, the equilibrium solution is unstable.

From the System (2), let $X = (Q_{RZ}, Q_{SZ}, Q_{WZ})$ be the vector of the state variables. Then, the system can be written in the form, $\frac{dX}{dt} = AX + B$. Computing the eigenvalues of the Jacobian matrix J around the critical point $\left(\frac{K}{\alpha}, \frac{K}{\beta+\mu}, \frac{K}{\gamma+\mu}\right)$ is equivalent to solving the characteristic equation (3)

$$\det\left[J\left(\frac{K}{\alpha}, \frac{K}{\beta+\mu}, \frac{K}{\gamma+\mu}\right) - \xi I\right] = 0, \tag{3}$$

where ξ stands for eigenvalues and by expanding the matrix, we get

$$(\alpha + \xi)[(\beta + \mu) + \xi)(\gamma + \mu) + \xi) = 0 \tag{4}$$

Solving (4), we get $\xi_1 = -\alpha$; $\xi_2 = -(\beta + \mu)$; $\xi_3 = -(\gamma + \mu)$

Since all parameters are assumed to be positive, it implies that all eigenvalues are negative. Therefore, the system is stable around the equilibrium point $\left(\frac{K}{\alpha}, \frac{K}{\beta+\mu}, \frac{K}{\gamma+\mu}\right)$.

2.3 Positivity of the model solution and boundness

The positivity of the solution of a mathematical model depends on the nature of the model and the variables involved. In the model (2), it is necessary to restrict the solution to positive values to maintain physical relevance or mathematical consistency. In this case, it is necessary to restrict

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the concentration of methane gas to positive values to ensure that the extraction process is physically possible.

Theorem 2.1. *The solution of the model (1) with initial conditions* $Q_{RZ}(0) \geq 0, Q_{SZ}(0) \geq 0,$ $Q_{WZ}(0) \geq 0$ *and* $Q_{PZ}(0) \geq 0$ *is positive in* $R^4{}_{+} \forall\, t \geq 0$.

Proof. Starting with the equation Q_{RZ},

$\frac{dQ_{RZ}}{dt} = K - \alpha Q_{RZ}$, it can be expressed as an inequality

$$\frac{dQ_{RZ}}{dt} \geq -\alpha Q_{RZ}\,. \tag{5}$$

Solving (5), it is seen that

$$Q_{RZ}(t) \geq Q_{RZ}(0)\exp\left(-\alpha \int Q_{RZ}(t)dt\right) \geq 0 \text{ for } t \geq 0. \tag{6}$$

From (6), it can be concluded that $Q_{RZ}(t) \geq 0, \forall\, t \geq 0$.
In the same way, $Q_{SZ}(t), Q_{WZ}(t)$ and $Q_{PZ}(t)$ can be shown to be positive in in $R^4{}_{+}$for all $t \geq 0$.
Therefore, the solution of the model (2) is positive.
The total dynamics of the model (2) is obtained by adding all the equations to get $Q(t)$ (the total quantity of methane gas in the system) in the following way

$$\frac{d}{dt}Q(t) = K - \mu Q_{SZ} - \mu Q_{WZ}. \tag{7}$$

Since there a minimum loss of methane gas (small μ) in the water $Q_{SZ}(t) \cong Q_{Wz}(t) = Q(t)$ and Equation (7) becomes

$$\frac{d}{dt}Q(t) = K - \mu Q(t) \tag{8}$$

Solving (8),

$$Q(t) = \frac{K}{\mu} + \left(Q(0) - \frac{K}{\mu}\right)e^{-\mu t} \tag{9}$$

From Equation (9), if $t \to 0, Q(t) \to Q(0)$ and whenever $t \to \infty, Q(t) \leq \frac{K}{\mu}$. And finally, it is seen that $Q(t)$ is bounded as $0 \leq Q(t) \leq \frac{K}{\mu}$. Therefore, the feasible region is defined as

$$\Omega = \left\{(Q_{RZ}(t), Q_{SZ}(t), Q_{WZ}(t), Q_{PZ}(t)) \in R_+^4 : Q(t) \leq \frac{K}{\mu}\right\}.$$

This means that the quantity of the methane gas to be harvested from the Lake Kivu is bounded.

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2.4 Existence and uniqueness of the solution

Existence of a solution means that there is at least one solution that satisfies the system of ordinary differential equations (2). On the other hand, the uniqueness of a solution means that there is only one solution that satisfies the system (2).

Theorem 2. 2. *The solution of the model (1) with initial conditions* $Q_{RZ}(0) \geq 0, Q_{SZ}(0) \geq 0,$ $Q_{WZ}(0) \geq 0$ *and* $Q_{PZ}(0) \geq 0$ *exists and is unique in* $R_+^4, \forall\, t \geq 0.$

Proof. The right-hand sides of the system (2) can be expressed as follows:

$$f_1 = K - \alpha Q_{RZ}\,,$$
$$f_2 = \alpha Q_{RZ} - (\beta + \mu)Q_{SZ}\,,$$
$$f_3 = \beta Q_{SZ} - (\gamma + \mu)Q_{WZ}\,,$$
$$f_4 = \gamma Q_{WZ}\,.$$

It is obvious to obtain that $\frac{\partial f_i}{\partial x_i}$ are continuous and $\left|\frac{\partial f_i}{\partial x_i}\right| < \infty$ for $i = 1,2,3,4$ where $x_1 = Q_{PZ}, x_2 = Q_{SZ}, x_3 = Q_{WZ}$ and $x_4 = Q_{PZ}$. Therefore, by using the Picard-Lindelöf theorem (Braun, M. & Golubitsky, M., 1983), the system (2) has a unique solution.

2.5 Estimation of parameters

Parameter estimation is the process of determining the values of the parameters of a model that best fit the available data and those to be estimated are $(\alpha\,, \beta, \gamma, \mu)$. The goal is to develop a model that accurately captures the behavior of the real-world system being studied and that can be used to make predictions and guide decision-making. The difference between observed and fitted values produced by the model is calculated as shown by Equation (10)

$$y_i = f(x_i, \theta) + \epsilon_i; \tag{10}$$

where $y_i = (y_1, y_2, \dots, y_n)^T$, is the vector of response data, x_i stands for covariate variables, $\theta = (\alpha, \beta, \gamma, \mu)$ is the vector of regression parameters and $\epsilon_i = (\epsilon_1, \epsilon_2, \dots, \epsilon_n)^T$is the vector of error terms. The sum squared error SSE is given by Equation (11)

$$SSE = \sum_{i=1}^{n}\left(y_i - f(x_i, \theta)\right)^2. \tag{11}$$

Parameters will be estimated in order to be able to numerically simulate the optimized states and predict scenarios related to methane production.

After guessing initial parameters using earlier studies, the data are found by introducing noise to the solution of the system (2). The choice of parameter estimation technique depends on the specific problem and the type of data being analyzed. It is often necessary to try different methods and compare their performance to determine the most suitable approach. However, in this paper, the Least Squares approach is employed because it aids to minimize the residuals in a dataset. For further information on parameter estimation using Least Squares techniques refer to

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(Lakshmi, K. et al., 2021; Dismuke, C. & Lindrooth, R., 2006; Ndanguza et al., 2020) among others. As a result, estimated values for the parameters utilized in the model are shown in Table 1.

Table 1. Estimation of parameters by Least Square

Parameter	**Definition**	**Initial Value**	**Reference**	**Estimated**
K	The total quantity of methane Gas inflow into RZ	0.14 (km^3/year)	Wüest et al. 2012	Known constant
α	The rate of extraction of the methane Gas from the PRZ	0.8234	Guessed	0.8212
β	The proportion of methane gas transferred to the washing zone	0.99	Guessed	0.8364
γ	The proportion of methane gas transferred from washing zone to the production zone	0.99	Guessed	0.9553
μ	The proportion of methane gas discharged in the water during the separation and washing stages.	0.001	Calculated	0.0005

The selection and estimation of appropriate parameter values is a critical step in the development and validation of mathematical models. The values of these parameters are often estimated from experimental data, and the accuracy and precision of these estimates can have a significant impact on the reliability of the model predictions. Some of these parameters are found in literature or computed from the collected data and others are estimated using simulated data.

3. Numerical Simulations

Mathematical models are built using simulation tools, which allow researchers to simulate the behavior of the system being studied under different conditions. Simulation tools may include software programs, computer models, or physical models. This section comprises the numerical solution of the system of ordinary differential equations (2). The model numerical solutions have been simulated using MATLAB software and the results are displayed in Figure 1 where the x-axis is the time in year and y-axis the quantity of methane gas in km^3 pear year. The Runge-Kutta fourth order method has been used to compute the numerical solution. The initial guessed

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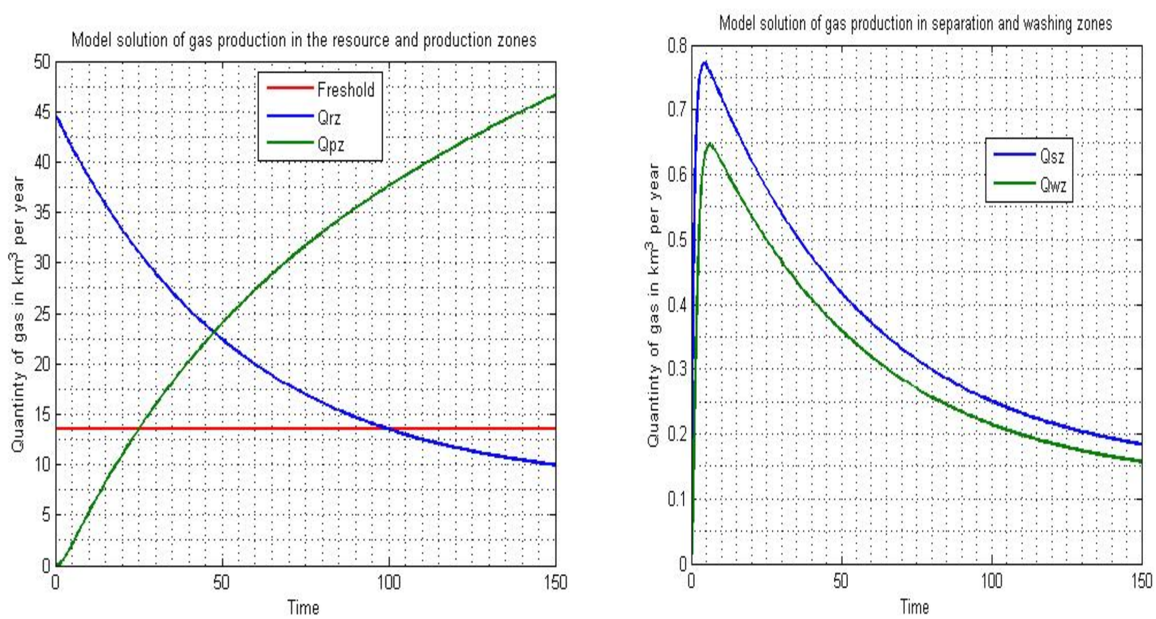
values of the variables of the model used in the simulations are from literature from previous researchers such as (Wüest, A. et al., 2009).



(a) (b)

Figure 2. Model solution representing the dynamics of gas in the Resource, Production zones (a) and Separation, Washing (b)

The reserve of the gas in the resource zone is $44.7\ km^3$ (Wüest, A. et al., 2009; Wüest et al., 2012). This is considered as the initial value of the quantity of methane gas in the resource zone. From the report of (Alfred W., 2019), the economically exploitable quantity is $5\ mol/m^3$ equivalent to $13.48\ km^3$; that is the minimum quantity of Methane Gas in the Resource zone (EEQ: Economic exploitable quantity) considered as the exploitable threshold gas. Figure 2(a) illustrates that the quantity of methane in the resource zone decreases over time, eventually reaching the threshold after 100 years (Wüest et al., 2012). In contrast, the methane quantity in the production zone increases exponentially. Similarly, Figure 2(b) shows that, initially, the methane quantities in both the separation and washing zones fluctuate, with an increase followed by a decrease, corresponding to the depletion of methane in the resource zone.

3.1 Validation and verification of the model

Since the mathematical model has been developed, it must be validated and verified to ensure that it accurately represents the real-world system being studied. This involves comparing the model's predictions to simulated data. Validation refers to the process of assessing whether the model accurately represents the real-world system it is intended to simulate or predict. Figure 3 represents the comparison between the model solution and simulated data. It is seen that the model fits the simulated data at 98%.

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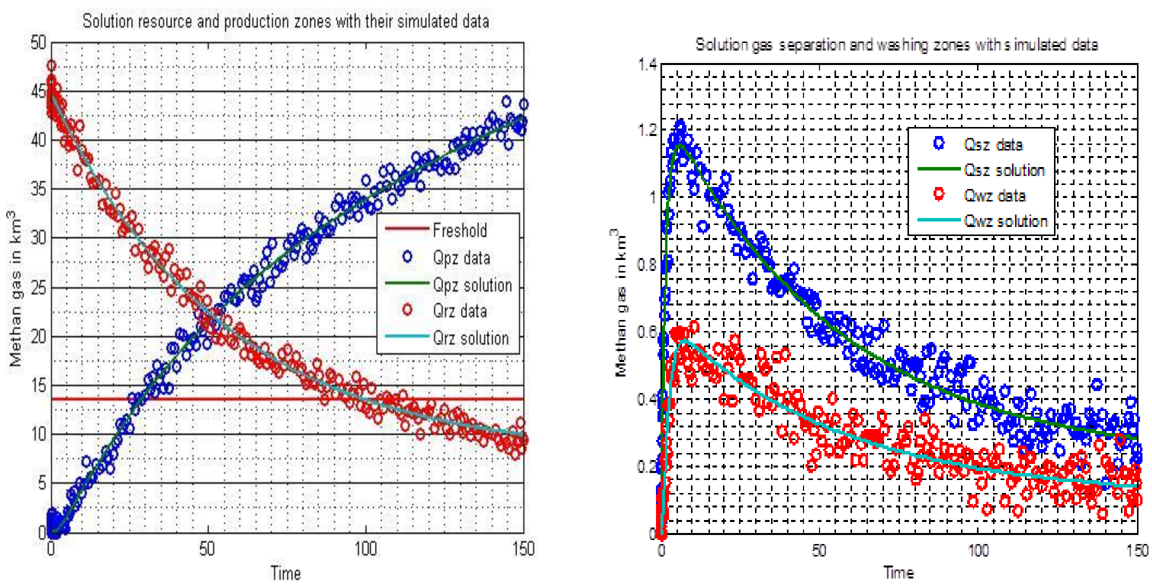

(a) (b)

Figure 3. Fitness of the model to the simulated data for the Resource, Production zones (a) and Separation, Washing (b)

Data were simulated and then compared to the predictions of the model. From Figure 3, it is seen that the predictions match the data, and then the model can be considered valid. However, if there are significant differences between the predictions and the data, then the model may need to be refined or updated to better reflect the behavior of the system.

3.2 Discussions

While discussing the results for the mathematical model (2), there are several scenarios regarding the extraction rate that should be considered and referred to in relation to the existing literature.

Scenario 1: No extraction of the gas from Lake Kivu

Starting with a scenario where there is no extraction, i.e. $\alpha = 0$. The quantity of the gas in the lake will grow following the function $Q_{RZ} = Kt + Q_{RZ}(0)$. This scenario is explained by Figure 4. In the event that no gas is extracted from Lake Kivu, the buildup of these gases could potentially result in a catastrophic occurrence. When the gases accumulate, the pressure would rise, causing the gas to bubble to the surface. If a substantial volume of gas were to suddenly leak from the lake, it might cause asphyxiation and death for anyone nearby. Alternatively, if a limnic eruption were to occur, it could release a significant amount of these gases into the atmosphere and potentially cause widespread harm to human and animal populations in the surrounding region. Even though there is no exact quantity of methane gas in Lake Kivu that could trigger a catastrophe, scientists have estimated that the lake contains around 60 billion cubic meters of methane and 300 billion cubic meters of CO_2 (Wüest et al., 2012). To prevent a limnic eruption,

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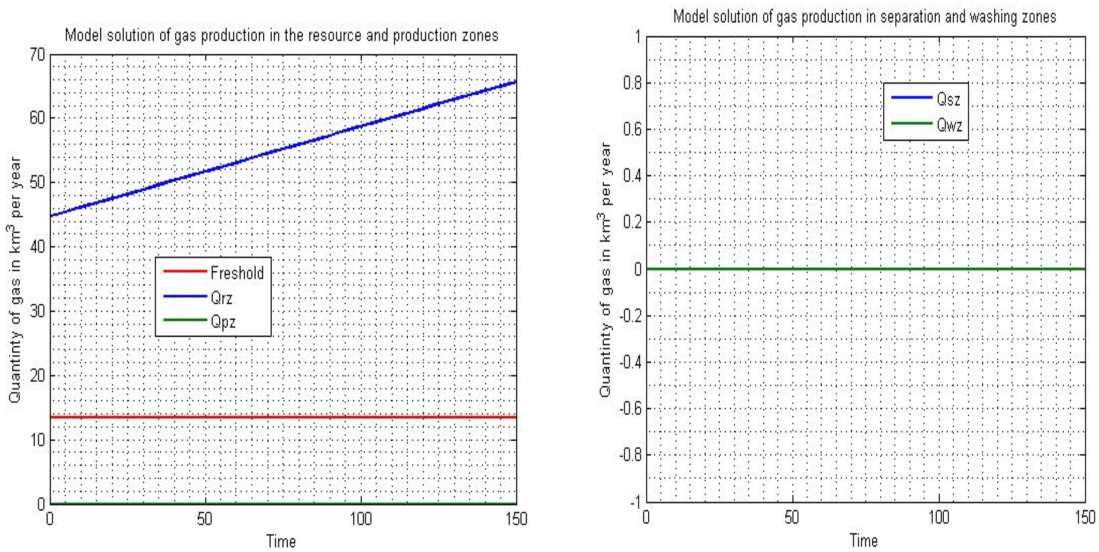
it is important to carefully manage the gas concentrations in Lake Kivu through a process known as degassing. This involves extracting the dissolved gas from the lake and converting it into usable energy, such as electricity. Proper management and monitoring of the gas levels in Lake Kivu are crucial to prevent a catastrophic event.



(a) (b)

Figure 4. Model solution representing the dynamics of gas in the Resource, production zones (a) separation, washing zones (b) when there is no extraction process

Without gas extraction, it would be required to regularly monitor the lake's gas levels to detect any symptoms of harmful buildup. Other options to lessen the risk of a gas release could be investigated, such as limiting human activity around the lake or deploying technologies to absorb and store the gases. Overall, failure to extract gas from Lake Kivu would represent a substantial risk to both human life and the ecosystem, necessitating careful management and monitoring (Schoell, M. et al., 1988).

Scenario 2: The extraction rate is increased at $\alpha = 0.308$ from $\alpha = 0$

The lake contains enormous volumes of dissolved methane and carbon dioxide, which might be collected and used to generate energy. However, the extraction of these gases must be done carefully to avoid harming the environment and the people who live nearby.

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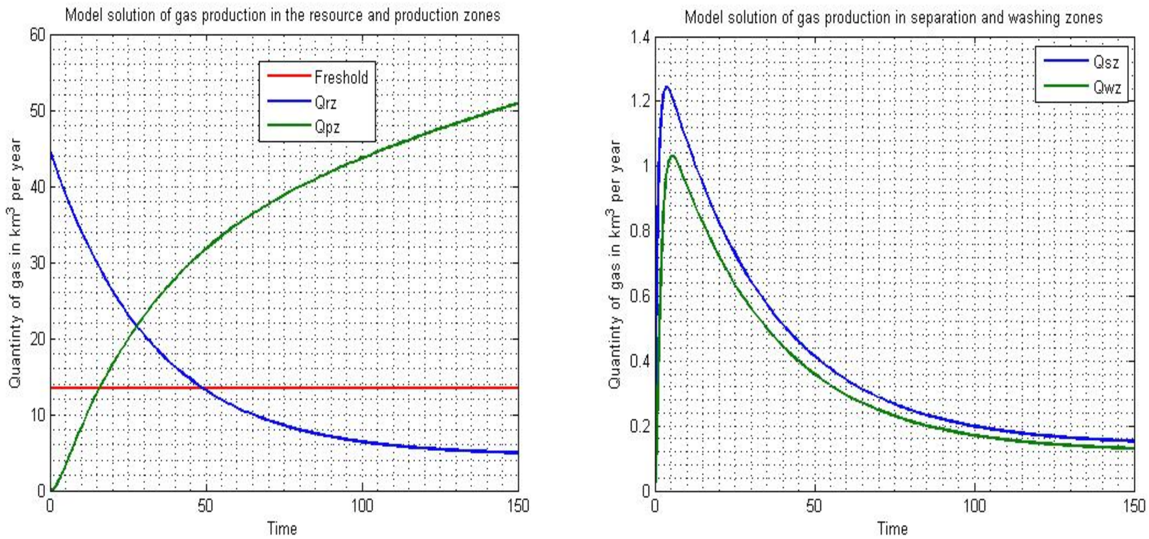

(a) (b)

Figure 5. Model solution representing the dynamics of gas in the Resource and production zones (a) and separation, washing zones (b) when there is no extraction process

From Figure 5, it is seen that increasing the extraction rate of methane gas requires a combination of investment in technology, resources, and expertise. However, if done correctly, it can lead to higher profits for the company and increased access to this important energy resource.

Scenario 3: When the proportion of methane gas used for production of electricity (λ_1) , the proportion of methane gas vented in the atmosphere (λ_2) and the proportion of the one returned in the water are considered

Methane gas can be used as a source of energy for electricity production or other industrial processes. However, if methane is not captured and utilized, it can be vented or leaked into the atmosphere, where it is a potent greenhouse gas that contributes to climate change. The proportion of methane gas used for energy production and the proportion vented into the atmosphere will depend on various factors, such as the efficiency of the production process, the availability of infrastructure for capturing and utilizing the gas, and regulations or incentives to reduce emissions. Once the gas is produced, it can generate electricity by rate (λ_1) or vented in atmosphere by rate (λ_2). Their values can be found in different literatures such as (Wüest, A. et al., 2009; Wüest, A. et al., 2012).

λ_1: The proportion of methane gas used for production of energy (electricity) estimated as 0.90 (90%).

λ_2: The proportion of methane gas vented in the atmosphere while burning the gas to get electricity or gas vented due to excess of it or returned to water, estimated as 0.10 (10%)

The quantity of electricity (QEL) found after transition from gas (QPZ) is explained by Figure 6.

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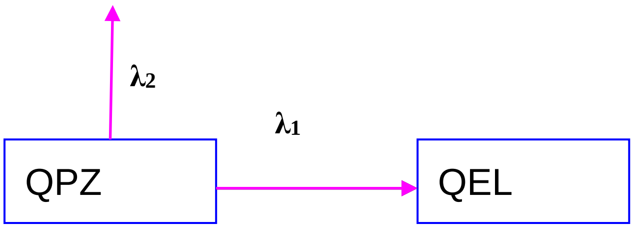

Figure 6. Transition from methane gas to electricity generation

From Figure 6, the ordinary differential equations can be formulated as in (12)

$$\frac{dQ_{PZ}}{dt} = -(\lambda_1 + \lambda_2)Q_{PZ} \tag{12a}$$
$$\frac{dQ_{EL}}{dt} = \lambda_1 Q_{PZ} \tag{12b}$$

where $Q_{PZ}(0)$ is the quantity of methane gas transferred from washing zone to the production zone Q_{PZ} (in $km^3/year$) and $Q_{EL}$is the quantity of generated electricity (in MW). From the Equation (12a), it is seen that the quantity of produced methane gas will be exponentially decreasing as shown in Figure 3a. The good news for Q_{PZ} is that γ from Figure 2 will be continuously filling the gap.
Regarding the quantity of electricity shared to electrical grids will be increasing as shown in Figure 7.
Solving (12a);

$$Q_{PZ}(t) = Q_{PZ}(0)e^{-(\lambda_1+\lambda_2)t} \tag{13}$$

Plugging (13) into (13b), we get

$$\frac{dQ_{EL}}{dt} = \lambda_1 Q_{PZ}(0)e^{-(\lambda_1+\lambda_2)t} \tag{14}$$

Solving (14),

$$Q_{EL}(t) = Q_{EL}(0) - \frac{\lambda_1 Q_{PZ}(0)}{\lambda_1+\lambda_2+\lambda_3}\exp[-(\lambda_1 + \lambda_2)t]$$

The washed gas flow with a high concentration of CH_4 is transported to the shore through tubes to be converted to electricity with rate λ_1. For unit harmonization, a conversion on what quantity of methane should generate what quantity of electricity is documented (Joselyne, N. M. et al., 2020).

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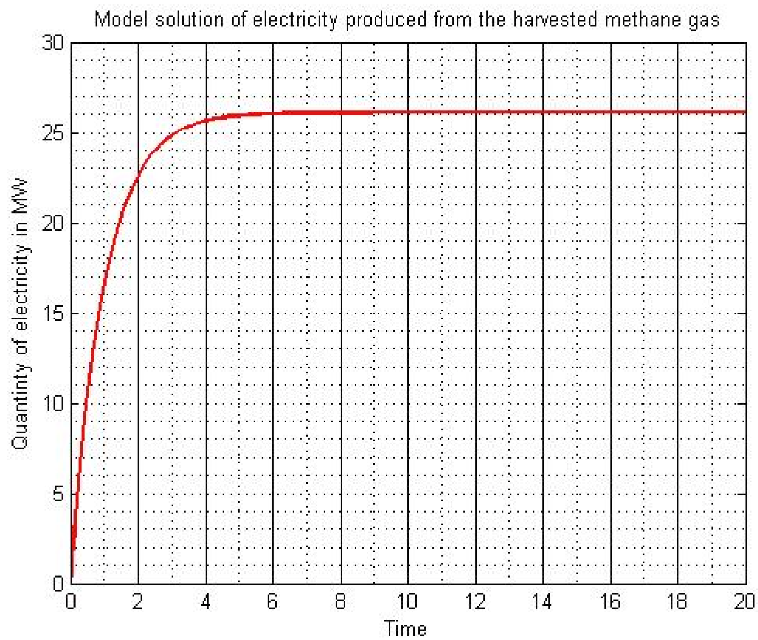

Figure 7. Model solution of the quantity of electricity equivalent to 26 MW

The first phase of the KivuWatt project utilized three generators to produce 26 MW of electricity for the local grid. This aligns with the data presented in Figure 7, which shows that electricity production remains constant from year 10 onward. In the second phase, nine additional 75 MW generators will be installed, effectively doubling Rwanda's power generation. Consequently, electricity production will no longer remain constant but will instead increase. Lake Kivu's methane is projected to have the capacity to create 700 MW of power over a 55-year period (Bolson, N. et al., 2021).

4. Conclusion

This study developed a mathematical model to analyze the process of methane gas extraction from Lake Kivu, focusing on the rates of methane generation, extraction, and redistribution within the lake's deep layers. The model provides valuable insights into the dynamics of methane recovery while considering key operational processes, including bubble-driven extraction and gas handling in separation and washing zones. The findings highlight the importance of carefully managing extraction rates to ensure the stability of the lake's stratified layers, thereby minimizing the risk of uncontrolled gas releases. Moreover, the system will be stable when the yearly average extraction of methane gas from Resource Zone is 0.1704 $km^3/year$, 0.1672 $km^3/year$ in the separation zone and 0.1464 $km^3/year$ in the washing zone.

The model identifies critical parameters and thresholds, making it a valuable tool for designing efficient and sustainable methane gas extraction strategies that maximize energy recovery while preserving the ecological balance of Lake Kivu. The proposed mathematical model provides

https://dx.doi.org/10.4314/rjeste.v7i1.10

essential insights for companies interested in extracting methane gas from the lake. By simulating the extraction process, the model predicts the volume of gas that can be extracted, the duration of the extraction process, and the amount of methane required to produce a specified quantity of electricity. Numerical simulations explored extraction rate scenarios, supported by a two-compartment model for gas-to-electricity conversion. The mathematical model provides valuable insights for optimizing methane extraction from Lake Kivu efficiently and sustainably.

Simulation results indicate that the methane concentration in the resource zone decreases over time, approaching a threshold after approximately 100 years. This aligns with the operational outcomes of the KivuWatt project, whose first phase utilized three gensets to generate 26 MW of electricity for the local grid. These findings validate the model's applicability in supporting sustainable and efficient energy production from Lake Kivu's methane reserves.

It is important to acknowledge that the accuracy of mathematical models depends heavily on the quality of the data and assumptions used in their development. In this study, simulated data were utilized, highlighting the need for future research to incorporate real-world data collected from operating companies. Additionally, several factors influencing the extraction process, such as the presence of impurities in the gas, the stability of stratified layers, and the environmental impact of extraction, were not included in the current model.

Future research should expand the scope by incorporating other critical constituents, including salinity, nutrients, gases, and temperature, alongside methane, to study the global stability of Lake Kivu comprehensively. By employing advanced tools and techniques, researchers can develop more accurate and reliable mathematical models that enhance our understanding of complex systems and support informed decision-making for sustainable resource management.

Acknowledgement

The authors acknowledge the support of the University of Rwanda (UR) through the research grant implemented by the University of Rwanda, College of Science and Technology (UR-CST), School of Science, Department of Mathematics.

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6. References

Bolson, N., Yutkin, M., & Patzek, T. (2021). Energy efficiency and sustainability assessment for methane harvesting from Lake Kivu. Energy, 225, 120215. https://doi.org/10.1016/j.energy.2021.120215.

Braun, M., & Golubitsky, M. (1983). *Differential equations and their applications* (Vol. 2). New York: Springer-Verlag.

Descy, Jean-Pierre, Darchambeau, François, Schmid, Martin (2012), *Lake Kivu: Limnology and biogeochemistry of a tropical great lake,* Aquatic Ecology Series.

Dismuke, C., & Lindrooth, R. (2006). Ordinary least squares. *Methods and designs for outcomes research*, *93*, 93-104.

Doevenspeck, M. (2007). Lake Kivu's methane gas: natural risk, or source of energy and political security? Afrika Spectrum, 42(1), 95-110. https://nbn-resolving.org/urn:nbn:de:0168-ssoar-105410.

Esiara, K., 2016. Symbion Power to produce 50 MW from methane. East African. https://www.esi-africa.com/top-stories/us-symbion-power-to-generate-50mw-in-rwanda/

Owkes, N., & Mason, D. (2019). Modelling the Surface Water Pollution in Lake Kivu. In *Proceedings of the Mathematics in Industry Study Group*. https://www.wits.ac.za/media/wits-university/conferences/misgsa/documents/2019/kivupollution19.pdf

Joselyne, N. M., Waweru, M., Gitau, A. N., Alphonse, U., & Patrick, M. (2020). Evaluating Anaerobic Digestion Technology in Reducing the Quantity of Solid Waste: Case of Kigali Dumpsite. *Journal of Geoscience and Environment Protection*, *8*(05), 204. 10.4236/gep.2020.85013

Lakshmi, K., Mahaboob, B., Rajaiah, M., & Narayana, C. (2021). Ordinary least squares estimation of parameters of linear model. *J. Math. Comput. Sci.*, *11*(2), 2015-2030. https://doi.org/10.28919/jmcs/5454.

Ndanguza, D., Tchuenche, J. M., & Haario, H. (2013). Statistical data analysis of the 1995 Ebola outbreak in the Democratic Republic of Congo. *Afrika Matematika*, *24*, 55-68. https://doi.org/10.1007/s13370-011-0039-5.

Ndanguza, D., Nyirahabinshuti, A., & Sibosiko, C. (2020). Modeling the effects of toxic wastes on population dynamics. *Alexandria Engineering Journal*, *59*(4), 2713-2723. https://doi.org/10.1016/j.aej.2020.05.013.

Osborn, T. R. (1980). Estimates of the local rate of vertical diffusion from dissipation measurements. *Journal of physical oceanography*, *10*(1), 83-89. https://doi.org/10.1175/1520-0485(1980)010<0083:EOTLRO>2.0.CO;2.

Schmid, M., Halbwachs, M., Wehrli, B., & Wüest, A. (2005). Weak mixing in Lake Kivu: new insights indicate increasing risk of uncontrolled gas eruption. Geochemistry, Geophysics, Geosystems, 6(7), https://doi.org/10.1029/2004GC000892.

Schmid, M., Lorke, A., Wüest, A., Halbwachs, M., & Tanyileke, G. (2003). Development and sensitivity analysis of a model for assessing stratification and safety of Lake Nyos during artificial degassing. *Ocean Dynamics*, *53*, 288-301. https://doi.org/10.1007/s10236-003-0032-0

Schmid, M., Busbridge, M., & Wü est, A. (2010). Double-diffusive convection in Lake Kivu.

Limnology and Oceanography, *55*(1), 225-238. https://doi.org/10.4319/lo.2010.55.1.0225

Schmid, Martin, et al. "*How hazardous is the gas accumulation in Lake Kivu? Arguments for a risk assessment in light of the Nyiragongo volcano eruption of 2002*." Actavulcanologica 14.1-2 (2002): 115-122.

Schoell, M., Tietze, K., & Schoberth, S. M. (1988). Origin of methane in Lake Kivu (East-Central Africa). *Chemical Geology*, *71*(1-3), 257-265. https://doi.org/10.1016/0009-2541(88)90119-2.

Stichting Deltares, (2021). The revision of management prescriptions for the development of Lake Kivu gas resources. Africa Energy Services Group Texas A&M University.

Tietze, K. (1978), Geophysikalische Untersuchung des Kivusees und seiner ungewöhnlichen Methangaslagerstätte—Schichtung, Dynamik und Gasgehalt des Seewassers, es Sciences thesis, 150 pp., Christian Albrechts Univ., Kiel, Germany.

Wüest, A., Jarc, L., & Schmid, M. (2009). Modelling the reinjection of deep-water after methane extraction in Lake Kivu. *EAWAG and BTC for the Governments of Rwanda and DR Congo*, *141*. http://www.wrq.eawag.ch/organisation/abteilungen/surf/kivu/methane_harvesting/index_EN

Wüest, A., Jarc, L., Bürgmann, H., Pasche, N., & Schmid, M. (2012). Methane formation and future extraction in Lake Kivu. *Lake Kivu: Limnology and biogeochemistry of a tropical great lake*, 165-180.

https://dx.doi.org/10.4314/rjeste.v7i1.10