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NUMERICAL SOLUTION OF TWO-DIMENSIONAL NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS USING HYBRID METHODS

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BSTRACT

This paper is concerned with the numerical solution of two- dimensional non-linear partial differential equations using a hybrid method. The solution technique involves discritizing the non-linear system of partial differential equations (PDEs) to obtain a corresponding non-linear system of algebraic difference equations to be solved at each time level. To linearize the resulting system of difference equations, Newton Ralphson root finding algorithm is applied at each time level with a discrete-node-update scheme. Numerical experiments are computationally more efficient than the existing methods such as Srivastava et al and Duffy, D.J.

INTRODUCTION

Here we consider the two-dimensional coupled non-linear partial differential equations which are a special form of incompressible Navier-Stokes equation without having pressure term and continuity equation Srivastava et al, (2011). It has enormous applications in fluid dynamics. Earlier studies by both researchers have been carried out using other numerical techniques. But this paper uses the proposed scheme in solving it, as an application of the scheme to solving Initial-Boundary Value Problems of non-linear systems of partial differential equations. We consider the equations

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0\\
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0\\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}-\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=0
\end{aligned}\right\} \qquad (1.1)$$

subject of the initial conditions

$$\left.\begin{aligned} u(x,y,0) &= \Gamma_1(x,y);\ (x,y)\in\Omega \\ v(x,y,0) &= \Gamma_2(x,y);\ (x,y)\in\Omega \end{aligned}\right\} \qquad (1.2)$$

and boundary conditions

$$u(x,y,t) = \varphi_1(x,y);\ (x,y)\in\partial\Omega,\ t>0$$
$$v(x,y,t) = \varphi_2(x,y);\ (x,y)\in\partial\Omega,\ t>0 \qquad (1.3)$$

Where $\Omega=\{(x,y);a_1\le x\le b_1,\ a_2\le y\le b_2\}$ and $\partial\Omega$ is its boundary, $u(x,y,t)$ *and* $v(x,y,t)$ are the velocity components in x *and* y directions, respectively, to be determined, Γ *and* φ are known functions. Re is the Reynolds number.

$$-\frac{1}{2\mathrm{Re}}\left(\frac{v^{l+1}_{i+1,j}-2v^{l+1}_{i,j}+v^{l+1}_{i-1,j}}{(\Delta x)^2}+\frac{v^{l}_{i+1,j}-2v^{l}_{i,j}+v^{l}_{i-1,j}}{(\Delta x)^2}\right)-$$

The interior nodes constitute 9 subsets, $S_1,S_2,\cdots,S_9\subset\Omega$. Nodes in $S_1,S_2,\cdots,S_8$ capture the boundary conditions, while the solution space covered by S_9 is discretized without directly applying the boundary conditions-they capture these conditions in proxy from the other nodes binding them.

MATERIALS AND METHODS

We proceed to discretize the system (1.1) using the Crank-Nicolson scheme as follows;

The first partial differential equation:

$$\frac{u^{l-1}_{i,j}-u^{t}_{i,j}}{\Delta t}+\frac{1}{2}\left(u^{l+1}_{i,j}\,\frac{u^{l+1}_{i+1,j}-u^{l+1}_{i+1,j}}{2\Delta x}+u^{l}_{i,j}\,\frac{u^{l}_{i+1,j}-u^{l}_{i,1,j}}{2\Delta x}\right)+$$

$$\frac{1}{2}\left(v^{l+1}_{i,j}\,\frac{u^{l+1}_{i,j+1}-u^{l+1}_{i,j-1}}{2\Delta y}+v^{l}_{i,j}\,\frac{u^{l}_{i,j+1}-u^{l}_{i,j-1}}{2\Delta y}\right)$$

$$-\frac{1}{2\mathrm{Re}}\left(\frac{u^{l+1}_{i+1,j}-2u^{l+1}_{i,j}+u^{l+1}_{i-1}}{(\Delta x)^2}+\frac{u^{l}_{i+1,j}-2u^{l}_{i,j}+u^{l}_{i-1,j}}{2\Delta y}\right)-$$

$$\frac{1}{2\mathrm{Re}}\left(\frac{u^{l+1}_{i+1,j}-2u^{l+1}_{i,j}+u^{l+1}_{i-1}}{(\Delta x)^2}+\frac{u^{l}_{i+1,j}-2u^{l}_{i,j}+u^{l}_{i-1,j}}{(\Delta y)^2}\right)=0 \quad (3.1)$$

The second partial differential equation:

$$\frac{v^{l-1}_{i,j}-v^{t}_{i,j}}{\Delta t}+\frac{1}{2}\left(u^{l+1}_{i,j}\,\frac{v^{l+1}_{i+1,j}-v^{l+1}_{i+1,j}}{2\Delta x}+u^{l}_{i,j}\,\frac{v^{l}_{i+1,j}-v^{l}_{i,1,j}}{2\Delta x}\right)+$$

$$\frac{1}{2}\left(v^{l+1}_{i,j}\,\frac{v^{l+1}_{i,j+1}-v^{l+1}_{i,j-1}}{2\Delta y}+v^{l}_{i,j}\,\frac{u^{l}_{i,j+1}-v^{l}_{i,j-1}}{2\Delta y}\right)$$

$$\frac{1}{2\mathrm{Re}}\left(\frac{v^{l+1}_{i+1,j}-2v^{l+1}_{i,j}+v^{l+1}_{i,j-1}}{(\Delta x)^2}+\frac{v^{l}_{i+1,j}-2v^{l}_{i,j}+v^{l}_{i-1,j}}{(\Delta y)^2}\right)=0 \quad (3.2)$$

To simplify the resulting system, we multiply through by Δt and do the following substitutions.

Let

$$\lambda_1=\frac{\Delta t}{4\Delta x};\ \ \lambda_2=\frac{\Delta t}{4\Delta y};\ \phi_1=-\frac{\Delta t}{2(\Delta t)^2\,\mathrm{Re}};\ \phi_2=-\frac{\Delta t}{2(\Delta y)^2\,\mathrm{Re}}$$

To obtain

$$u^{l-1}_{i,j}-u^{l}_{i,j}+\lambda_1\left\{u^{l+1}_{i,j}\left(u^{l+1}_{i+j}-u^{l+1}_{i-1}\right)+u^{l}_{i,j}\left(u^{l}_{i+1,j}-u^{l}_{i-1,j}\right)\right\}+\lambda_2\left\{v^{l+1}_{i,j}\left(u^{l+1}_{i,j+1}-u^{l+1}_{i,j-1}\right)+\right.$$

$$\left.v^{l}_{i,j}\left(u^{l}_{i,j+1}-u^{l}_{i,j-1}\right)\right\}+\phi_1\left\{u^{l+1}_{i+1,j}-2u^{l+1}_{i,j}+u^{l+1}_{i-1,j}+u^{l}_{i+1,j}-2u^{l}_{i,j}+u^{j}_{i-1,j}\right\}+$$

$$\phi_2\left\{u^{l+1}_{i,j+1}-2u^{l+1}_{i,j}+u^{l+1}_{i,j-1}+u^{l}_{i,j+1}-2u^{l}_{i,j}+u^{j}_{i,j-1}\right\}=0 \quad (3.3)$$

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and

$$v_{i,j}^{l-1}-v_{i,j}^{l}+\lambda_1\left\{u_{i,j}^{l+1}\left(v_{i+1,j}^{l+1}-v_{i-1,j}^{l+1}\right)+u_{i,j}^{l}\left(v_{i+1,j}^{l}-u_{i-1,j}^{l}\right)\right\}+\lambda_2\left\{v_{i,j}^{l+1}\left(v_{i,j-1}^{l+1}-v_{i,j-1}^{l+1}\right)+\right.$$

$$\left.v_{i,j}^{l}\left(v_{i,j+1}^{l}-v_{i,j-1}^{l}\right)\right\}+\phi_1\left\{v_{i+1,j}^{l+1}-2v_{i,j}^{l+1}+v_{i-1,j}^{l+1}+v_{i+1,j}^{l}-2v_{i,j}^{l}+v_{i-1,j}^{j}\right\}+$$

$$\phi_2\left\{v_{i,j+1}^{l+1}-2v_{i,j}^{l+1}+v_{i,j-1}^{l+1}+v_{i,j+1}^{l}-2v_{i,j}^{l}+v_{i,j-1}^{j}\right\}=0\text{, } (3.4)$$

Equate the L.H.S of the equations to F_1 *and* F_2 respectively. That is,

$$F_1=u_{i,j}^{l-1}-u_{i,j}^{l}+\lambda_1\left\{u_{i,j}^{l+1}\left(u_{i+1,j}^{l+1}-u_{i-1,j}^{l+1}\right)+u_{i,j}^{l}\left(u_{i+1,j}^{l}-u_{i-1,j}^{l}\right)\right\}+$$

$$\lambda_2\left\{v_{i,j}^{l+1}\left(u_{i,j+1}^{l+1}-u_{i,j-1}^{l+1}\right)+v_{i,j}^{l}\left(u_{i,j+1}^{l}-u_{i,j-1}^{l}\right)\right\}+\phi_1\left\{u_{i+1,j}^{l+1}-2u_{i,j}^{l+1}+u_{i-1,j}^{l+1}+\right.$$

$$\left.u_{i+1,j}^{l}-2u_{i,j}^{l}+u_{i-1,j}^{l}\right\}+\phi_2\left\{u_{i,j+1}^{l+1}-2u_{i,j}^{l+1}+u_{i,j-1}^{l+1}+u_{i,j+1}^{l}-2u_{i,j}^{l}+u_{i,j-1}^{l}\right\}$$

$$F_2=v_{i,j}^{l-1}-v_{i,j}^{l}+\lambda_1\left\{u_{i,j}^{l+1}\left(v_{i,j}^{l+1}-v_{i-1,j}^{l+1}\right)+u_{i,j}^{l}\left(v_{i+1,j}^{l}-v_{i-1,j}^{l}\right)\right\}+$$

$$\lambda_2\left\{v_{i,j}^{l+1}\left(v_{i,j+1}^{l+1}-v_{i,j-1}^{l+1}\right)+v_{i,j}^{l}\left(v_{i,j+1}^{l}-v_{i,j-1}^{l}\right)\right\}+\phi_1\left\{v_{i+1,j}^{l+1}-2v_{i,j}^{l+1}+v_{i-1,j}^{l+1}+\right.$$

$$\left.v_{i+1,j}^{l}-2v_{i,j}^{l}+v_{i-1,j}^{l}\right\}+\phi_2\left\{v_{i,j+1}^{l+1}-2v_{i,j}^{l+1}+v_{i,j-1}^{l+1}+v_{i,j+1}^{l}-2v_{i,j}^{l}+v_{i,j-1}^{l}\right\}$$

(3,5)

These equations, (1.1a), apply to all nods, (i, j), in the solution domain. In other to implement the Newton-Raphson iterative algorithm, we obtain the Jacobian matrix for the $(i, j)th$ node by finding the partial derivatives of $F_1\,and\,F_2$ with respect to $u_{i,j}^{l+1}$ *and* $v_{i,j}^{l+1}$ form a current time level l,

$$J_{11}(i,j)=\frac{\partial F_1}{\partial u_{i,j}^{l+1}}=1+\lambda_1\left(u_{i+1,j}^{l+1}-u_{i-1,j}^{l+1}\right)-2\left(\phi_1+\phi_2\right)$$

$$J_{12}(i,j)=\frac{\partial F_1}{\partial v_{i,j}^{l+1}}=\lambda_2\left(u_{i,j+1}^{l+1}-u_{i,j-1}^{l+1}\right) \qquad (3.6)$$

$$J_{21}(i,j)=\frac{\partial F_2}{\partial v_{i,j}^{l+1}}=\lambda_1\left(v_{i+1,j}^{l+1}-v_{i-1,j}^{l+1}\right)$$

$$J_{22}(i,j)=\frac{\partial F_2}{\partial v_{i,j}^{l+1}}=1+\lambda_2\left(v_{i,j+1}^{l+1}-v_{i,j-1}^{l+1}\right)-2\left(\phi_1+\phi_2\right)$$

Thus, $$J(i,j)=\begin{bmatrix} J_{11}(i.j) & J_{12}(i,j) \\ J_{22}(i,j) & J_{22}(i.l) \end{bmatrix}\begin{bmatrix} \dfrac{\partial F_1}{\partial u_{i,j}^{l+1}} & \dfrac{\partial F_1}{\partial v_{i,j}^{l+1}} \\ \dfrac{\partial F_2}{\partial u_{i,j}^{l+1}} & \dfrac{\partial F_2}{\partial v_{i,j}^{l+1}} \end{bmatrix} \qquad (3.7)$$

MATHEMATICAL FORMULATION

Applying boundary conditions:

Equation (1.1) are used directly for nodes in S_9. However for nodes in $S_1, S_2, \cdots, S_g$ the boundary conditions $\varphi_q u$ *and* $\varphi_q v(q=1,\cdots,4)$ are applied as follows

$S_1=\{(1,1)\}$; **Subject to boundary conditions** φ_1 *and* φ_4

$$F_1=u_{1,1}^{l-1}-u_{1,1}^{l}+\lambda_1\left\{u_{1,1}^{l+1}\left(u_{2,1}^{l+1}-\varphi_4^{l+1}u(y)\right)+u_{1,1}^{l}\left(u_{2,1}^{l}-\varphi_4^{l}u(y)\right)\right\}+$$

$$\lambda_2\left\{v_{1,1}^{l+1}\left(u_{1,2}^{l+1}-\varphi_1^{l+1}u(x)\right)+v_{1,1}^{l}\left(u_{1,2}^{l}-\varphi_1^{l}u(x)\right)\right\}+\phi_1\left\{u_{2,1}^{l+1}-2u_{1,1}^{l+1}+\right.$$

$$\left.\varphi_4^{l+1}u(y)+u_{2,1}^{l}-2u_{1,1}^{l}+\varphi_4^{l}(y)\right\}+\varphi_2\left\{u_{1,2}^{l+1}-2u_{1,1}^{l+1}+\varphi_1^{l+1}u(x)+u_{1,2}^{l}-\right.$$

$$\left.2u_{1,1}^{l}+\varphi_1^{l}u(x)\right\} \qquad (4.1)$$

$$F_2=v_{1,1}^{l-1}-v_{1,1}^{l}+\lambda_1\left\{u_{1,1}^{l+1}\left(v_{2,1}^{l+1}-\varphi_4^{l+1}v(y)\right)+u_{1,1}^{l}\left(v_{2,1}^{l}-\varphi_4^{l}v(y)\right)\right\}+$$

$$\lambda_2\left\{v_{1,1}^{l+1}\left(v_{1,2}^{l+1}-\varphi_1^{l+1}v(x)\right)+v_{1,1}^{l}\left(v_{1,2}^{l}-\varphi_1^{l}v(x)\right)\right\}+\phi_1\left\{v_{2,1}^{l+1}-2v_{1,1}^{l+1}+\right.$$

$$\left.\varphi_4^{l+1}v(y)+v_{2,1}^{l}-2v_{1,1}^{l}+\varphi_4^{l}v(y)\right\}+\varphi_2\left\{v_{1,2}^{l+1}-2v_{1,1}^{l+1}+\varphi_1^{l+1}v(x)+v_{1,2}^{l}-\right.$$

$$\left.2v_{1,1}^{l}+\varphi_1^{l}v(x)\right\} \qquad (4.2)$$

Jacobian matrix

$$J_{11}(1,1)=\frac{\partial F_1}{\partial u_{1,1}^{l+1}}=1+\lambda_1\left(u_{2,1}^{l+1}-\varphi_4^{l+1}u(y)\right)-2(\phi_1+\phi_2)$$

$$J_{12}(1,1)=\frac{\partial F_1}{\partial v_{1,1}^{l+1}}=1+\lambda_2\left(u_{1,2}^{l+1}-\varphi_1^{l+1}u(x)\right)$$

$$J_{21}(1,1)=\frac{\partial F_2}{\partial u_{1,1}^{l+1}}=\lambda_1\left(v_{2,1}^{l+1}-\varphi_4^{l+1}v(y)\right)$$

$$J_{22}(1,1)=\frac{\partial F_2}{\partial v_{1,1}^{l+1}}=1+\lambda_2\left(v_{1,2}^{l+1}-\varphi_4^{l+1}v(x)\right)-2(\phi_1+\phi_2)$$

Thus, $J(1,1)=\begin{bmatrix}J_{11}(1,1) & J_{12}(1,1)\\ J_{21}(1,1) & J_{22}(1,1)\end{bmatrix}=\begin{bmatrix}\frac{\partial F_1}{\partial u_{1,1}^{l+1}} & \frac{\partial F_1}{\partial v_{1,1}^{l+1}}\\ \frac{\partial F_2}{\partial u_{1,1}^{l+1}} & \frac{\partial F_2}{\partial v_{1,1}^{l+1}}\end{bmatrix}$ (4.3)

$S_2=\{(m,1)\}$; **Subject to boundary conditions** φ_1 *and* φ_2

$$F_1=u_{m,1}^{l+1}-u_{m-1}^{l}+\lambda_1\left\{u_{m,1}^{l+1}\left(\varphi_2^{l+1}u(y)-u_{m-1,1}^{l+1}\right)+u_{m,1}^{l}\left(\varphi_2^{1}u(y)-u_{m-1,1}^{l}\right)\right\}+$$
$$\lambda_2\left\{v_{m,1}^{l+1}\left(u_{m,2}^{l+1}-\varphi_1^{l+1}u(x)\right)+v_{m,1}^{l}\left(u_{m,2}^{l}-\varphi_1^{l}u(x)\right)\right\}+\phi_1\left\{\varphi_2^{l+1}u(y)-2u_{m,1}^{l+1}+\right.$$
$$\left.u_{m-1,1}^{l+1}+\varphi_2^{l}u(y)-2u_{m,1}^{l}+u_{m-1,1}^{l}\right\}+\phi_2\left\{u_{m,2}^{l+1}-2u_{m,2}^{l+1}+\varphi_1^{l+1}u(x)+u_{m,2}^{l}-\right.$$
$$\left.2u_{m,1}^{l}+\phi_1^{l}u(x)\right\} \quad (4.4)$$

$$F_2=v_{m,1}^{l+1}-v_{m-1}^{l}+\lambda_1\left\{u_{m,1}^{l+1}\left(\varphi_2^{l+1}v(y)-v_{m-1,1}^{l+1}\right)+u_{m,1}^{l}\left(\varphi_2^{1}v(y)-v_{m-1,1}^{l}\right)\right\}+$$
$$\lambda_2\left\{v_{m,1}^{l+1}\left(v_{m,2}^{l+1}-\varphi_1^{l+1}v(x)\right)+v_{m,1}^{l}\left(v_{m,2}^{l}-\varphi_1^{l}v(x)\right)\right\}+\phi_1\left\{\varphi_2^{l+1}v(y)-2v_{m,1}^{l+1}+\right.$$
$$\left.v_{m-1,1}^{l+1}+\varphi_2^{l}v(y)-2v_{m,1}^{l}+v_{m-1,1}^{l}\right\}+\phi_2\left\{v_{m,2}^{l+1}-2v_{m,2}^{l+1}+\varphi_1^{l+1}v(x)+v_{m,2}^{l}-\right.$$
$$\left.2v_{m,1}^{l}+\phi_1^{l}v(x)\right\} \quad (4.5)$$

Jacobian matrix

$$J_{11}(m,1)=\frac{\partial F_1}{\partial u_{m,1}^{l+1}}=1+\lambda_1\left(\varphi_2^{l+1}-u(y)-u_{m-1,1}^{l+1}\right)-2(\phi_1+\phi_2)$$

$$J_{12}(m,1)=\frac{\partial F_1}{\partial v_{m,1}^{l+1}}=\lambda_2\left(u_{m,2}^{l+1}-\varphi_1^{l+1}u(x)\right)$$

$$J_{21}(m,1)=\frac{\partial F_2}{\partial u_{m,1}^{l+1}}=\lambda_1\left(\varphi_2^{l+1}v(y)-v_{m-1,1}^{l+1}\right)$$

$$J_{22}(m,1)=\frac{\partial F_2}{\partial v_{m,1}^{l+1}}=1+\lambda_2\left(v_{m,2}^{l+1}-\varphi_4^{l+1}v(x)\right)-2(\phi_1+\phi_2)$$

Thus, $J(m,1)=\begin{bmatrix}J_{11}(m,1) & J_{12}(m,1)\\ J_{21}(m,1) & J_{22}(m,1)\end{bmatrix}=\begin{bmatrix}\frac{\partial F_1}{\partial u_{m,1}^{l+1}} & \frac{\partial F_1}{\partial v_{m,1}^{l+1}}\\ \frac{\partial F_2}{\partial v_{m,1}^{l+1}} & \frac{\partial F_2}{\partial v_{m,1}^{l+1}}\end{bmatrix}$ (4.6)

$S_3=\{(m,n)\}$; **Subject to boundary conditions** φ_2 *and* φ_3

$$F_1=u_{m,n}^{l+1}-u_{m,n}^{l}+\lambda_1\left\{u_{m,n}^{l+1}\left(\varphi_2^{l+1}u(y)-u_{m-1,n}^{l+1}\right)+u_{m,n}^{l}\left(\varphi_2^{1}u(y)-u_{m-1,n}^{l}\right)\right\}+$$
$$\lambda_2\left\{v_{m,n}^{l+1}\left(\varphi_3^{l+1}u(x)-u_{m,n-1}^{l+1}\right)+v_{m,n}^{l}\left(\varphi_3^{l}u(x)-u_{m,n-1}^{l}\right)\right\}+\phi_1\left\{\varphi_2^{l+1}u(y)-2u_{m,n}^{l+1}+\right.$$
$$\left.u_{m-1,n}^{l+1}+\varphi_2^{l}u(y)-2u_{m,n}^{l}+u_{m-1,n}^{l}\right\}+\phi_2\left\{u_{m,2}^{l+1}u(x)-2u_{m,n}^{l+1}+u_{m,n-1}^{l+1}\varphi_1^{l+1}u(x)-\right.$$
$$\left.2u_{m,n}^{l}+u_{m,n-1}^{l}\right\} \quad (4.7)$$

$$F_2=v_{m,n}^{l+1}-v_{m,n}^{l}+\lambda_1\left\{u_{m,n}^{l+1}\left(\varphi_2^{l+1}v(y)-v_{m-1,n}^{l+1}\right)+u_{m,n}^{l}\left(\varphi_2^{1}v(y)-v_{m-1,n}^{l}\right)\right\}+$$
$$\lambda_2\left\{v_{m,n}^{l+1}\left(\varphi_3^{l+1}v(x)-v_{m,n-1}^{l+1}\right)+v_{m,n}^{l}\left(\varphi_3^{l}v(x)-v_{m,n-1}^{l}\right)\right\}+\phi_1\left\{\varphi_2^{l+1}v(y)-2v_{m,1}^{l+1}+\right.$$
$$\left.v_{m-1,n}^{l+1}+\varphi_2^{l}v(y)-2v_{m,n}^{l}+v_{m-1,n}^{l}\right\}+\phi_2\left\{\varphi_3^{l+1}v(x)-2v_{m,n}^{l+1}+v_{m,n-1}^{l+1}+\varphi_3^{l+1}v(x)-\right.$$
$$\left.2v_{m,n}^{l}+v_{m,n-1}^{l}\right\} \quad (4.8)$$

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Jacobian matrix

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$J_{12}(m,n)=\frac{\partial F_1}{\partial v_{m,n}^{l+1}}=\lambda_2\left(\varphi_3^{l+1}u(x)-u_{m,n,1}^{l+1}\right)$$

$$J_{21}(m,n)=\frac{\partial F_2}{\partial u_{m,n}^{l+1}}=\lambda_1\left(\varphi_2^{l+1}v(y)-v_{m-1,n}^{l+1}\right)$$

$$J_{22}(m,n)=\frac{\partial F_2}{\partial v_{m,n}^{l+1}}=1+\lambda_2\left(\varphi_3^{l+1}v(x)-v_{m,n-1}^{l+1}\right)-2(\phi_1+\phi_2)$$

Thus,

$$J(m,n)=\begin{bmatrix} J_{11}(m,n) & J_{12}(m,n) \\ J_{21}(m,n) & J_{22}(m,n)\end{bmatrix}=\begin{bmatrix} \frac{\partial F_1}{\partial u_{m,n}^{l+1}} & \frac{\partial F_1}{\partial v_{m,n}^{l+1}} \\ \frac{\partial F_2}{\partial v_{m,n}^{l+1}} & \frac{\partial F_2}{\partial v_{m,n}^{l+1}}\end{bmatrix}$$

$S_4=\{(1,n)\}$; **Subject to boundary conditions** φ_3 *and* φ_4

$$F_1=u_{1,n}^{l+1}-u_{1,n}^{l}+\lambda_1\left\{u_{1,n}^{l+1}\left(u_{2,n}^{l+1}+\varphi_4^{l+1}u(y)-u_{1,n}^{l+1}\right)+u_{2,n}^{l}\left(u_{2,n}^{l}+\varphi_4^{l}u(y)\right)\right\}+$$

$$\lambda_2\left\{v_{1,n}^{l+1}\left(\varphi_3^{l+1}u(x)-u_{1,n-1}^{l+1}\right)+v_{1,n}^{l}\left(\varphi_3^{l}u(x)-u_{1,n-1}^{l}\right)\right\}+\phi_1\left\{u_{2,n}^{l+1}-2u_{1,n}^{l+1}+\right.$$

$$\varphi_4^{l+1}u(y)+u_{2,n}^{l}-2u_{1,n}^{l}+\varphi_4^{l}u(y)\Big\}+\phi_2\left\{\varphi_3^{l+1}u(x)-2u_{1,n}^{l+1}+u_{1,n-1}^{l+1}+\varphi_3^{l+1}u(x)-\right.$$

$$\left.2u_{1,n}^{l}+u_{1,n-1}^{l}\right\} \qquad (4.9)$$

$$F_2=v_{1,n}^{l+1}-v_{1,n}^{l}+\lambda_1\left\{u_{1,n}^{l+1}\left(v_{2,n}^{l+1}+\varphi_4^{l+1}v(y)\right)+u_{1,n}^{l}\left(v_{2,n}^{l}+\varphi_4^{l}v(y)\right)\right\}+$$

$$\lambda_2\left\{v_{1,n}^{l+1}\left(\varphi_3^{l+1}v(x)-v_{1,n-1}^{l+1}\right)+v_{1,n}^{l}\left(\varphi_3^{l}v(x)-v_{1,n-1}^{l}\right)\right\}+\phi_1\left\{v_{2,n}^{l+1}-2v_{1,n}^{l+1}+\right.$$

$$\varphi_4^{l+1}v(y)+v_{2,n}^{l}-2v_{1,n}^{l}+\varphi_4^{l}v(y)\Big\}+\phi_2\left\{\varphi_3^{l+1}v(x)-2v_{1,n}^{l+1}+v_{1,n-1}^{l+1}+\varphi_3^{l+1}v(x)-\right.$$

$$\left.2v_{1,n}^{l}+v_{1,n-1}^{l}\right\} \qquad (4.10)$$

Jacobian matrix

$$J_{11}(1,n)=\frac{\partial F_1}{\partial u_{1,n}^{l+1}}=1+\lambda_1\left(u_{2,n}^{l+1}-\varphi_4^{l+1}u(y)\right)-2(\phi_1+\phi_2)$$

$$J_{12}(1,n)=\frac{\partial F_1}{\partial v_{1,n}^{l+1}}=\lambda_2\left(\varphi_3^{l+1}u(x)-u_{1,n,1}^{l+1}\right)$$

$$J_{21}(1,n)=\frac{\partial F_2}{\partial u_{1,n}^{l+1}}=\lambda_1\left(v_{2,n}^{l+1}-\varphi_4^{l+1}v(y)\right)$$

$$J_{22}(1,n)=\frac{\partial F_2}{\partial v_{1,n}^{l+1}}=1+\lambda_2\left(\varphi_3^{l+1}v(x)-v_{1,n-1}^{l+1}\right)-2(\phi_1+\phi_2)$$

Thus, $J(1,n)=\begin{bmatrix} J_{11}(1,n) & J_{12}(1,n) \\ J_{21}(1,n) & J_{22}(1,n)\end{bmatrix}=\begin{bmatrix} \frac{\partial F_1}{\partial u_{1,n}^{l+1}} & \frac{\partial F_1}{\partial v_{1,n}^{l+1}} \\ \frac{\partial F_2}{\partial v_{1,n}^{l+1}} & \frac{\partial F_2}{\partial v_{1,n}^{l+1}}\end{bmatrix}$

$S_5=\{(i,1)\}$; **Subject to boundary conditions** φ_1

$$F_1=u_{i,1}^{l+1}-u_{i,1}^{l}+\lambda_1\left\{u_{i,1}^{l+1}\left(u_{i+1,1}^{l+1}-u_{1-1,1}^{l+1}\right)+u_{i,1}^{l}\left(u_{i+1,1}^{l}-u_{i-1,1}^{l}\right)\right\}+$$

$$\lambda_2=\left\{v_{i,1}^{l+1}\left(u_{i,2}^{l+1}-\varphi_1^{l+1}u(x)\right)+v_{i,1}^{l}\left(u_{i,2}^{l}-\varphi_1^{l}u(x)\right)\right\}+\phi_1\left\{u_{i+1}^{l+1}-2u_{i,1}^{l+1}+u_{i-1,1}^{l+1}+\right.$$

$$\left.u_{i+1}^{l}+2u_{i,1}^{l}+u_{i-1,1}^{l}\right\}+\phi_2\left\{u_{i,2}^{l+1}-2u_{i,1}^{l+1}+\varphi_1^{l+1}u(x)+u_{i,2}^{l+1}-2u_{i,1}^{l}+\varphi_1^{l}u(x)\right\}$$

(4.11)

$$F_2=v_{i,1}^{l+1}-v_{i,1}^{l}+\lambda_1\left\{u_{i,1}^{l+1}\left(v_{i+1,1}^{l+1}-v_{1-1,1}^{l+1}\right)+u_{i,1}^{l}\left(v_{i+1,1}^{l}-v_{i-1,1}^{l}\right)\right\}$$

$$\lambda_2=\left\{v_{i,1}^{l+1}\left(v_{i,2}^{l+1}-\varphi_1^{l+1}v(x)\right)+v_{i,1}^{l}\left(v_{i,2}^{l}-\varphi_1^{l}v(x)\right)\right\}+\phi_1\left\{v_{i+1}^{l+1}-2v_{i,1}^{l+1}+v_{i-1,1}^{l+1}+\right.$$

$$v_{i+1}^{l}+2v_{i,1}^{l}+v_{i-1,1}^{l}\Big\}+\phi_2\Big\{v_{i,2}^{l+1}-2v_{i,1}^{l+1}+\varphi_1^{l+1}v(x)+v_{i,2}^{l+1}-2v_{i,1}^{l}+\varphi_1^{l}v(x)\Big\}$$
(4.12)

Jacobian matrix

$$J_{11}(i,1)=\frac{\partial F_1}{\partial u_{i,1}^{l+1}}=1+\lambda_1\left(u_{i,+1,1}^{l+1}-u_{i-1,1}^{l+1}\right)-2(\phi_1+\phi_2)$$

$$J_{12}(i,1)=\frac{\partial F_1}{\partial v_{i,1}^{l+1}}=\lambda_2\left(u_{1,2}^{l+1}\varphi_1^{l+1}u(x)\right)$$

$$J_{21}(i,1)=\frac{\partial F_2}{\partial u_{i,1}^{l+1}}=\lambda_1\left(v_{1,+1,1}^{l+1}-v_{i-1,1}^{l+1}\right)$$

$$J_{22}(i,1)=\frac{\partial F_2}{\partial v_{i,1}^{l+1}}=1+\lambda_2\left(v_{1,2}^{l+1}-\varphi_1^{l+1}v(x)\right)-2(\phi_1+\phi_2)$$

Thus, $J(i,1)=\begin{bmatrix}J_{11}(i,1) & J_{12}(i,1)\\ J_{21}(i,1) & J_{22}(i,1)\end{bmatrix}=\begin{bmatrix}\frac{\partial F_1}{\partial u_{i,1}^{l+1}} & \frac{\partial F_1}{\partial v_{i,1}^{l+1}}\\ \frac{\partial F_2}{\partial v_{i,1}^{l+1}} & \frac{\partial F_2}{\partial v_{i,1}^{l+1}}\end{bmatrix}$

$S_6=\{(m,j)\}$; **Subject to boundary conditions** φ_2

$$F_1=u_{m,j}^{l+1}-u_{m,j}^{l}+\lambda_1\Big\{u_{m,j}^{l+1}\left(\varphi_2^{l+1}u(y)-u_{m-1,j}^{l+1}\right)+u_{m,j}^{l}\left(\varphi_2^{l}u(y)-u_{m-1,j}^{l}\right)\Big\}+$$
$$\lambda_2=\Big\{v_{m,j}^{l+1}\left(u_{m,j+1}^{l+1}+u_{m,j-1}^{l+1}\right)+v_{m,j}^{l}\left(u_{m,j+1}^{l}+u_{m,j-1}^{l}\right)\Big\}+\phi_1\Big\{\varphi_2^{l+1}u(y)-2u_{m,j}^{l+1}+$$
$$u_{m-1,j}^{l+1}+\varphi_2^{l}u(y)-2u_{m,j}^{l}+u_{m-1,j}^{l}\Big\}+\phi_2\Big\{u_{m,j+1}^{l+1}-2u_{m,j}^{l+1}+u_{m,j-1}^{l+1}+u_{m,j+1}^{l}-$$
$$2u_{m,j}^{l}+u_{m,j-1}^{l}\Big\} \qquad (4.12)$$

$$F_1=v_{m,j}^{l+1}-v_{m,j}^{l}+\lambda_1\Big\{u_{m,j}^{l+1}\left(\varphi_2^{l+1}v(y)-v_{m-1,j}^{l+1}\right)+u_{m,j}^{l}\left(\varphi_2^{l}v(y)-v_{m-1,j}^{l}\right)\Big\}+$$
$$\lambda_2=\Big\{v_{m,j}^{l+1}\left(v_{m,j+1}^{l+1}+v_{m,j-1}^{l+1}\right)+v_{m,j}^{l}\left(v_{m,j+1}^{l}+v_{m,j-1}^{l}\right)\Big\}+\phi_1\Big\{\varphi_2^{l+1}v(y)-2v_{m,j}^{l+1}+$$
$$v_{m-1,j}^{l+1}+\varphi_2^{l}v(y)-2v_{m,j}^{l}+v_{m-1,j}^{l}\Big\}+\phi_2\Big\{v_{m,j+1}^{l+1}-2v_{m,j}^{l+1}+v_{m,j-1}^{l+1}+v_{m,j+1}^{l}-$$
$$2v_{m,j}^{l}+v_{m,j-1}^{l}\Big\} \qquad (4.13)$$

Jacobian matrix

$$J_{11}(m,j)=\frac{\partial F_1}{\partial u_{m,j}^{l+1}}=1+\lambda_1\left(\varphi_2^{l+1}u(y)-u_{m-1,j}^{l+1}\right)-2(\phi_1+\phi_2)$$

$$J_{12}(m,j)=\frac{\partial F_1}{\partial v_{m,j}^{l+1}}=\lambda_2\left(u_{m,j+1}^{l+1}+u_{m,j-1}^{l+1}\right)$$

$$J_{21}(m,j)=\frac{\partial F_2}{\partial u_{m,j}^{l+1}}=\lambda_1\left(\varphi_2^{l+1}v(y)-v_{m-1,j}^{l+1}\right)$$

$$J_{22}(m,j)=\frac{\partial F_2}{\partial v_{m,j}^{l+1}}=1+\lambda_2\left(v_{m,j+1}^{l+1}+v_{m,j-1}^{l+1}\right)-2(\phi_1+\phi_2)$$

Thus, $J(m,j)=\begin{bmatrix}J_{11}(m,j) & J_{12}(m,j)\\ J_{21}(m,j) & J_{22}(m,j)\end{bmatrix}=\begin{bmatrix}\frac{\partial F_1}{\partial u_{m,j}^{l+1}} & \frac{\partial F_1}{\partial v_{m,j}^{l+1}}\\ \frac{\partial F_2}{\partial v_{m,j}^{l+1}} & \frac{\partial F_2}{\partial v_{m,j}^{l+1}}\end{bmatrix}$

$S_7=\{(i,n)\}$; **Subject to boundary conditions** φ_3

$$F_1=u_{i,n}^{l+1}-u_{i,n}^{l}+\lambda_1\Big\{u_{i,n}^{l+1}\left(u_{i+,n}^{l+1}+u_{i-1,n}^{l+1}\right)+u_{i,n}^{l}\left(u_{i+1,n}^{l}+u_{i-1,n}^{l}\right)\Big\}+$$
$$\lambda_2\Big\{v_{i,n}^{l+1}\left(\varphi_3^{l+1}u(x)-u_{i,n-1}^{l+1}\right)+v_{i,n}^{l}\left(\varphi_3^{l}u(x)-u_{i,n-1}^{l}\right)\Big\}+\phi_1\Big\{u_{i+n,1}^{l+1}-2u_{i,n}^{l+1}+$$
$$u_{i-1,n}^{l+1}u_{i+1,n}^{l}-2u_{i,n}^{l}+u_{i-1,n}^{l}\Big\}+\phi_2\Big\{\varphi_3^{l+1}u(x)-2u_{1,n}^{l+1}+u_{i,n-1}^{l+1}+\varphi_3^{l}u(x)-$$
$$2u_{i,n}^{l}+u_{i,n-1}^{l}\Big\} \qquad (4.14)$$

$$F_2=v_{i,n}^{l+1}-v_{i,n}^{l}+\lambda_1\Big\{u_{i,n}^{l+1}\left(v_{i+1,n}^{l+1}+v_{i-1,n}^{l+1}\right)+u_{i,n}^{l}\left(v_{i+1,n}^{l}+v_{i-1,n}^{l}\right)\Big\}+$$
$$\lambda_2\Big\{v_{i,n}^{l+1}\left(\varphi_3^{l+1}v(x)-v_{i,n-1}^{l+1}\right)+v_{i,n}^{l}\left(\varphi_3^{l}v(x)-v_{i,n-1}^{l}\right)\Big\}+\phi_1\Big\{v_{i+n,1}^{l+1}-2v_{i,n}^{l+1}+$$
$$v_{i-1,n}^{l+1}+v_{i+1,n}^{l}-2v_{i,n}^{l}+v_{i-1,n}^{l}\Big\}+\phi_2\Big\{\varphi_3^{l+1}v(x)-2v_{1,n}^{l+1}+v_{i,n-1}^{l+1}+\varphi_3^{l}v(x)-$$
$$2v_{i,n}^{l}+v_{i,n-1}^{l}\Big\} \qquad (4.15)$$

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Jacobian matrix

$$J_{11}(i,n)=\frac{\partial F_1}{\partial u_{i,n}^{l+1}}=1+\lambda_1\left(u_{i+1,n}^{l+1}+u_{i-1,n}^{l+1}\right)-2(\phi_1+\phi_2)$$

$$J_{12}(i,n)=\frac{\partial F_1}{\partial v_{i,n}^{l+1}}=\lambda_2\left(\varphi_3^{l+1}u(x)-u_{i,n-1}^{l+1}\right)$$

$$J_{21}(i,n)=\frac{\partial F_2}{\partial u_{i,n}^{l+1}}=\lambda_1\left(v_{i+1,n}^{l+1}+v_{i-1,n}^{l+1}\right)$$

$$J_{22}(i,n)=\frac{\partial F_2}{\partial v_{i,n}^{l+1}}=1+\lambda_2\left(\varphi_3^{l+1}v(x)-v_{i,n-1}^{l+1}\right)-2(\phi_1+\phi_2)$$

Thus, $$J(i,n)=\begin{bmatrix}J_{11}(i,n) & J_{12}(i,n)\\ J_{21}(i,n) & J_{22}(i,n)\end{bmatrix}=\begin{bmatrix}\frac{\partial F_1}{\partial u_{i,n}^{l+1}} & \frac{\partial F_1}{\partial v_{i,n}^{l+1}}\\ \frac{\partial F_2}{\partial v_{i,n}^{l+1}} & \frac{\partial F_2}{\partial v_{i,n}^{l+1}}\end{bmatrix}$$

$S_8=\{(1,j)\}$; **Subject to boundary conditions** φ_4

$$F_1=u_{1,j}^{l+1}-u_{1,j}^{l}+\lambda_1\left\{u_{1,j}^{l+1}\left(u_{2,j}^{l+1}+\varphi_4^{l+1}u(y)-u_{1,j}^{l+1}\right)+u_{1,j}^{l}\left(u_{2,j}^{l}+\varphi_4^{l}u(y)\right)\right\}+\lambda_2\left\{v_{1,j}^{l+1}\left(u_{1,j+1}^{l+1}+u_{1,j-1}^{l+1}\right)+v_{1,j}^{l}\left(u_{1,j+1}^{l}+u_{1,j-1}^{l}\right)\right\}+\phi_1\left\{u_{2,j}^{l+1}-2u_{1,j}^{l+1}+\varphi_4^{l+1}u(y)+u_{2,j}^{l}-2u_{1,j}^{l}+\varphi_4^{l}u(y)\right\}+\phi_2\left\{u_{1,j+1}^{l+1}-2u_{1,j}^{l+1}+u_{1,j-1}^{l+1}+u_{1,j+1}^{l+1}-2u_{1,j}^{l}+u_{1,j-1}^{l}\right\}$$

(4.16)

$$F_2=v_{1,j}^{l+1}-v_{1,j}^{l}+\lambda_1\left\{u_{1,j}^{l+1}\left(v_{2,j}^{l+1}+\varphi_4^{l+1}v(y)\right)+u_{1,j}^{l}\left(v_{2,j}^{l}+\varphi_4^{l}v(y)\right)\right\}+\lambda_2\left\{v_{1,j}^{l+1}\left(v_{1,j+1}^{l+1}+v_{1,j-1}^{l+1}\right)+v_{1,j}^{l}\left(v_{1,j+1}^{l}+v_{1,j-1}^{l}\right)\right\}+\phi_1\left\{v_{2,j}^{l+1}-2v_{1,j}^{l+1}+\varphi_4^{l+1}v(y)+v_{2,j}^{l}-2v_{1,j}^{l}+\varphi_4^{l}v(y)\right\}+\phi_2\left\{v_{1,j+1}^{l+1}-2v_{1,j}^{l+1}+v_{1,j-1}^{l+1}+v_{1,j+1}^{l+1}-2v_{1,j}^{l}+v_{1,j-1}^{l}\right\}$$

(4.17)

Jacobian matrix

$$J_{11}(1,j)=\frac{\partial F_1}{\partial u_{1,j}^{l+1}}=1+\lambda_1\left(u_{2,j}^{l+1}-\varphi_4^{l+1}u(y)\right)-2(\phi_1+\phi_2)$$

$$J_{12}(1,j)=\frac{\partial F_1}{\partial v_{1,j}^{l+1}}=\lambda_2\left(u_{1,j+1}^{l+1}+u_{1,j-1}^{l+1}\right)$$

$$J_{21}(1,j)=\frac{\partial F_2}{\partial u_{1,j}^{l+1}}=\lambda_1\left(v_{2,j}^{l+1}-\varphi_4^{l+1}v(y)\right)$$

$$J_{22}(1,j)=\frac{\partial F_2}{\partial v_{1,j}^{l+1}}=1+\lambda_2\left(v_{1,j+1}^{l+1}+v_{1,j-1}^{l+1}\right)-2(\phi_1+\phi_2)$$

Thus, $$J(i,n)=\begin{bmatrix}J_{11}(1,j) & J_{12}(1,j)\\ J_{21}(1,n) & J_{22}(1,j)\end{bmatrix}=\begin{bmatrix}\frac{\partial F_1}{\partial u_{1,j}^{l+1}} & \frac{\partial F_1}{\partial v_{1,j}^{l+1}}\\ \frac{\partial F_2}{\partial v_{1,j}^{l+1}} & \frac{\partial F_2}{\partial v_{1,j}^{l+1}}\end{bmatrix} \quad (4.18)$$

equates (3.11) are solved for each node (i,j) in the solution domain, Ω, one node at a time, and the new values of $u_{i,j}^{l+1}$ *and* $v_{i,j}^{l+1}$ are used to test for convergence repeatedly until the stop criterion $\|F\|_\infty \le \varepsilon$ satisfied for a time level, l, using the Newton-Raphson algorithm as developed above. The process is re-initiated at the next time level $l+1$, and iterated with the Newton-Raphson algorithm with same stop criterion. This repeats until the solutions at desired time values or steady-state solutions are achieved.

Numerical Experiments

Consider the two-dimensional coupled non-linear Burger's equations,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$
$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

with a solution domain

$\Omega=\{(x,y);0\le x\le 1,0\le y\le 1\}$

Subject to initial conditions

$$u(x,y,0)=\tfrac{3}{4}-\frac{1}{4\left(1+e^{\mathrm{Re}(-4x+4y)/32}\right)};\ v(x,y,0)=\tfrac{3}{4}+\frac{1}{4\left(1+e^{\mathrm{Re}(-4x+4y)/32}\right)}$$

and boundary conditions

$$u(0,y,t)=\tfrac{3}{4}-\frac{1}{4\left(1+e^{\mathrm{Re}(4y-t)/32}\right)};\ u(1,y,t)=\tfrac{3}{4}-\frac{1}{4\left(1+e^{\mathrm{Re}(-4+4y-t)/32}\right)}$$

$$v(0,y,t)=\tfrac{3}{4}+\frac{1}{4\left(1+e^{\mathrm{Re}(4y-t)/32}\right)};\ v(1,y,t)=\tfrac{3}{4}+\frac{1}{4\left(1+e^{\mathrm{Re}(-4+4y-t)/32}\right)}$$

$$u(x,0,t)=\tfrac{3}{4}-\frac{1}{4\left(1+e^{\mathrm{Re}(4x-t)/32}\right)};\ u(x,1,t)=\tfrac{3}{4}-\frac{1}{4\left(1+e^{\mathrm{Re}(-4x+4-t)/32}\right)}$$

$$v(x,0,t)=\tfrac{3}{4}+\frac{1}{4\left(1+e^{-\mathrm{Re}(4x-t)/32}\right)};\ v(x,1,t)=\tfrac{3}{4}+\frac{1}{4\left(1+e^{\mathrm{Re}(-4x+4-t)/32}\right)}$$

where Re = Reynolds number, is taken as 500. We are to obtain the values of the solutions at time, t = 0.5 and 2. We compute the errors associated with the scheme at selected nodes.

Numerical Solution Using the Proposed Scheme

The solution domain $\Omega=\{(x,y);0\le x\le 1,0\le y\le 1\}$ is uniformly discretized into 20 × 20 grids. That is suppose $m\,and\,n$ and n are the internal nodes in the $x-$ and $y-$ directions respectively.

Since the computational domain $\Omega=\{(x,y);0\le x\le 1,0\le y\le 1\}$ defines a square geometry such that $L_x=L_y=1$, the uniform mesh size $\Delta x\times\Delta y$ is defined by

$$\Delta x=\Delta y=\frac{L_x}{m+1}$$

$$=\frac{1}{20}=0.05$$

A time step, $\Delta t=0.0001$ is used. This implies that the number of time levels (or step increments) required to obtain solutions to the partial differential equations $t=0.5\ and\ 2$ at times are respectively;

$$\ell_{0.5}=\frac{0.5}{\Delta t}=\frac{0.5}{0.0001}=5000$$ time levels

$$\ell_{2.0}=\frac{0.2}{\Delta t}=\frac{0.2}{0.0001}=20000$$ time levels

Number of internal nodes to be solved for $u(x,y,t)$ and $v(x,y,t)$ each is $m\times n$

$=19\times 19=361$

However, including boundary nodes, total number of nodes in and on boundary of solution domain is $=(m+1)\times(n+2)=21\times 21$

= 441 nodes

But boundary node values are updated at each time level, $l\Delta t,(where\,l=1,2,3,\cdots,5000$ for time, $t=0.5,\ and\ 1,2,3,\cdots,20000$ for time $t=2.0$.

For the particular problem, the boundary conditions are Cauchy.

Because of the number of non-linear finite difference equations involved at each time, level, calculated as $(361\times 2)\times(361\times 2)=722\times 722$

$=722\times 722$ finite difference equations,

MATLAB® program is written to implement the discretization as well as solve

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the resulting system of non-linear equations using the proposed scheme.

Since the scheme incorporates the Newton-Raphson root-finding algorithm, at each time level, $l\Delta t$, an auto initialization process is designed to automate the initial guesses for $f_{i,j}^{l+1}$ as follows

$$f_{i,j(0)}^{l+1} = f_{i,j}^{l} + \frac{f_{i,j}^{l}}{\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f_{i,j}^{l}}$$

where f defines $u\, and\; v$

This ensures that the initial guess $f_{i,j(0)}^{l+1}$ for each node (i,j) is determined in direct proportion to the previous time value $f_{i,j}^{l}\,\forall(i,j)\in\Omega$.

The solutions to the initial Boundary value problems are given in tables (1.1a) and (1.1b) at specific nodes, and the corresponding perspective views are shown in Figs (1.1c) and (1.1d).

Table (1.1a) Numerical solutions; $\mathrm{Re}=500, \Delta x=\Delta y=0.05, \Delta t=0.0001\, at\; t=0.5$

$u(x,y,0.5)$

(x,y)	**Exact Value**	**Srivastava et al**	**Error**	**Research Result**	**Error**
(0.1,0.1)	0.50010	0.48732	0.012780	0.48738	0.01272
(0.5,0.1)	0.50000	0.50002	0.000020	0.50001	0.00001
(0.9,0.1)	0.50000	0.50000	0.000000	0.50000	0.00000
(0.3,0.3)	0.50010	0.49531	0.004790	0.49537	0.00473
(0.7,0.3)	0.50000	0.50001	0.000010	0.50001	0.00001
(0.1,0.5)	0.75000	0.74990	0.000100	0.74990	0.00010
(0.5,0.5)	0.50010	0.49438	0.005720	0.49445	0.00565
(0.9,0.5)	0.50000	0.49977	0.000230	0.50001	0.00001
(0.3,0.7)	0.75000	0.75001	0.000010	0.75001	0.00001
(0.7,0.7)	0.50010	0.49323	0.006870	0.49450	0.00560
(0.1,0.9)	0.75000	0.75000	0.000000	0.75000	0.00000
(0.5,0.9)	0.75000	0.75001	0.000010	0.75000	0.00000
(0.9,0.9)	0.50010	0.47390	0.026200	0.46962	0.03048

$v(x,y,0.5)$

(x,y)	**Exact Value**	**Srivastava et al**	**Error**	**Research Result**	**Error**
(0.1,0.1)	0.99990	1.01268	0.01278	1.03040	0.03050
(0.5,0.1)	1.00000	0.99990	0.00001	0.99999	0.00001
(0.9,0.1)	1.00000	1.00000	0.00000	1.00000	0.00000
(0.3,0.3)	0.99990	1.00469	0.00479	0.99999	0.00009
(0.7,0.3)	1.00000	1.99990	0.00001	0,99999	0.00001
(0.1,0.5)	0.75000	0.75010	0.00010	0.75010	0.00010
(0.5,0.5)	0.99990	1.00562	0.00572	1.00558	0.00568
(0.9,0.5)	1.00000	1.00023	0.00023	0.99999	0.00001
(0.3,0.7)	0.75000	0.74999	0.00001	0.74999	0.00001
(0.7,0.7)	0.99990	1.00677	0.00687	1.00552	0.00562
(0.1,0.9)	0.75000	0.75000	0.00000	0.75000	0.00000
(0.5,0.9)	0.75000	0.74999	0.00001	0.75000	0.00000
(0.9,0.9)	0.99990	1.02610	0.02620	1.03040	0.03050

Table (1.1b) Numerical solutions; $\text{Re} = 500, \Delta x = \Delta y = 0.05, \Delta t = 0.0001\, at\ t = 2.0$

$u(x, y, 2.0)$

(x, y)	**Exact Value**	**Srivastava et al**	**Error**	**Research Result**	**Error**
(0.1,0.1)	0.50000	0.49725	0.00275	0.49716	0.00284
(0.5,0.1)	0.50000	0.50024	0.00024	0.50023	0.00023
(0.9,0.1)	0.50000	0.49932	0.00687	0.50037	0.00037
(0.3,0.3)	0.50000	0.50687	0.00687	0.50687	0.00687
(0.7,0.3)	0.50000	0.49929	0.00071	0.49954	0.00046
(0.1,0.5)	0.50048	0.43945	0.06103	0.43954	0.06094
(0.5,0.5)	0.50000	0.49958	0.00042	0.9963	0.00037
(0.9,0.5)	0.50000	0.51378	0.01378	0.50255	0.00255
(0.3,0.7)	0.50048	0.41654	0.08394	0.41703	0.08345
(0.7,0.7)	0.50000	0.51054	0.01054	0.49107	0.00893
(0.1,0.9)	0.75000	0.75003	0.00003	0.75005	0.00005
(0.5,0.9)	0.50048	0.42895	0.07153	0.43666	0.06382
(0.9,0.9)	0.50000	0.56293	0.06293	0.50641	0.00641
Mean Error			**0.02427**		**0.01825**

$v(x, y, 2.0)$

(x, y)	**Exact Value**	**Srivastava et al**	**Error**	**Research Result**	**Error**
(0.1,0.1)	1.00000	1.00276	0.00276	1.00282	0.00282
(0.5,0.1)	1.00000	0.99975	0.00025	0.99977	0.00023
(0.9,0.1)	1.00000	1.00068	0.00068	0.99963	0.00037
(0.3,0.3)	1.00000	0.99313	0.00687	0.99311	0.00689
(0.7,0.3)	1.00000	1.00072	0.00072	1.00046	0.00046
(0.1,0.5)	0.99952	1.06055	0.06103	1.06051	0.06099
(0.5,0.5)	1.00000	1.00041	0.00041	1.00035	0.00035
(0.9,0.5)	1.00000	0.98622	0.01378	0.99745	0.00255
(0.3,0.7)	0.99952	1.08346	0.08394	1.08303	0.08351
(0.7,0.7)	1.00000	0.98946	0.01054	1.00895	0.00895
(0.1,0.9)	0.75000	0.74997	0.00003	1.74995	0.00005
(0.5,0.9)	0.99952	1.07105	0.07153	1.06337	0.06385
(0.9,0.9)	1.00000	0.93707	0.06293	0.99345	0.00655
Mean Error			**0.02427**		**0.01827**

CONCLUSION

This paper has investigated the convergence of the hybrid methods for solving partial differential equation using the Crank-Nicolson implicit finite difference scheme, which is a second order accurate in time and space, can be fused with the Newton Ralphson root search algorithm in providing high- accuracy.

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