Scientia Africana, Vol. 13 (No.2), December 2014. Pp219-224
 ISSN 1118 - 1931

CONSTRUCTION OF EXTENDED EXPONENTIAL GENERAL LINEAR METHODS 524 FOR SOLVING SEMI-LINEAR PROBLEMS

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Received: 23-07-14
Accepted: 11-11-14

ABSTRACT

*This paper introduces a new approach for constructing higher order of **EEGLM** which have become very popular and novel due to its enviable stability properties. This paper also shows that methods **524** is stable with its characteristics root lies in a unit circle.*
Numerical experiments indicate that Extended Exponential General Linear Methods perform better than existing Methods.

Key words: General Linear Methods, Exponential methods.

INTRODUCTION

The inspiration of this paper is drawn from the paper of Osisiogu and Bazuaye (2014) which gave a general framework for the construction of extended exponential general linear methods. We are interested in the problem

$$y'(t) = My(t) + G(y(t)),\ 0 \le t \le T,$$
Given $y(0)$. (1.1)

have attracted a lot of interest.

For given starting values $y_0,\ y_1, \cdots y_{q-1}$, the theoretical approximation y_{n+1} at time t_{n+1}, $n \le q-1$, is given by the recurrence relation or formula

$$y_{n+1} = e^{hL}y_n + h\sum_{i=1}^{s} B_i(hL)N(Y_{ni}) + h\sum_{k=1}^{q-1} V_k(hL)N(y_{n-k})$$
(1.2)

The internal stages Y_{ni}, $1 \le i \le s$, are defined through

$$Y_{ni} = e^{c_i hL}y_n + h\sum_{j=1}^{i-1} A_{ij}^{(1)}(hL)N(Y_{nj}) + h\sum_{k=1}^{q-1} U_{ik}^{(1)}(hL)N'(y_{n-k})$$

$$+ h^2\sum_{j=1}^{i-1} A_{ij}^{(2)}(hL)N'(Y_{nj}) + h^2\sum_{k=1}^{q-1} U_{ik}^{(2)}(hL)N'(y_{n-k})$$

$$Y_{n1} = y_{n1} = y_n$$
(1.2)

Construction of Extended Exponential General Linear Methods Order Five Step Two Stage Order Four.

The extended exponential general linear methods order five step two stage order four (known as methods 524) is given as

$$y_{n+1} = e^{hL}y_n + hB_1N(Y_{n1}) + hB_2N(Y_{n2}) + hV_1N(y_{n-1}) + hV_2N(y_{n-2}) + hV_3N(y_{n-3})$$

$$Y_{n2} = e^{c_2hL}y_n + hA_{21}N(Y_n) + hU_{21}^{(1)}N(y_{n-1}) + hU_{22}^{(1)}N(y_{n-2}) + hU_{23}^{(1)}N(y_{n-3}) + h^2A_{ij}^{(2)}N'(y_n)$$

$$+ h^2U_{21}^{(2)}N'(y_{n-1}) + h^2U_{22}^{(2)}N'(y_{n-2}) + h^2U_{23}^{(2)}N'(y_{n-3}) \qquad (2.1)$$

$$c_1^1A_{21}^{(1)} + (-1)^1U_{21}^{(1)} + (-2)^1U_{22}^{(1)} + (-3)^1U_{23}^{(1)}$$

$$+ (-1)^0U_{21}^{(2)} + (-2)^0U_{22}^{(2)} + (-3)^0U_{23}^{(2)} = \psi_2$$

$$-U_{21}^{(1)} - 2U_{22}^{(1)} - 3U_{23}^{(1)} + U_{21}^{(2)} + U_{22}^{(2)} + U_{23}^{(2)} = \psi_2$$

(2.2)

$$\frac{c_1^2}{2!}A_{21}^{(1)} + \frac{(-1)^2}{2!}U_{21}^{(1)} + \frac{(-2)^2}{2!}U_{22}^{(1)} + \frac{(-3)^2}{2!}U_{23}^{(1)}$$

$$+(-1)^1U_{21}^{(2)} + (-2)^1U_{22}^{(2)} + (-3)^1U_{23}^{(2)} = c_2^3\psi_3(c_2hL)$$

$$\frac{1}{2}U_{21}^{(1)} + 2U_{22}^{(1)} + \frac{9}{2}U_{23}^{(1)}$$

$$-U_{21}^{(1)} + -2U_{22}^{(2)} - 3U_{23}^{(2)} = \psi_3 \qquad (2.3)$$

$$\frac{c_1^3}{3!}A_{21}^{(1)} + \frac{(-1)^3}{3!}U_{21}^{(1)} + \frac{(-2)^3}{3!}U_{22}^{(1)} + \frac{(-3)^3}{3!}U_{23}^{(1)}$$

$$+\frac{(-1)^2}{2!}U_{21}^{(2)} + \frac{(-2)^2}{3!}U_{22}^{(2)} + \frac{(-3)^2}{2!}U_{23}^{(2)} = C_2^4\psi_4(C_2hL)$$

$$\frac{-1}{3!}U_{21}^{(1)} - \frac{8}{3!}U_{22}^{(1)} - \frac{27}{3!}U_{23}^{(1)}$$

$$+\frac{1}{2!}U_{21}^{(2)} + \frac{4}{2!}U_{22}^{(2)} + \frac{9}{2!}U_{23}^{(2)} = \psi_4 \qquad (2.4)$$

$$\frac{c_1^4}{4!}A_{21}^{(1)} + \frac{(-1)^4}{4!}U_{21}^{(1)} + \frac{(-2)^4}{4!}U_{22}^{(1)} + \frac{(-3)^4}{4!}U_{23}^{(1)}$$

$$\frac{c_1^3}{3!}A_{21}^{(2)} + \frac{(-1)^3}{3!}U_{21}^{(2)} + \frac{(-2)^3}{3!}U_{22}^{(2)} + \frac{(-3)^3}{3!}U_{23}^{(2)} = C_2^5\psi_5(C_2hL)$$

$$\frac{1}{4!}U_{21}^{(1)} + \frac{16}{4!}U_{22}^{(1)} + \frac{81}{4!}U_{23}^{(1)}$$

$$-\frac{1}{3!}U_{21}^{(1)} - \frac{8}{3!}U_{22}^{(1)} - \frac{27}{3!}U_{23}^{(1)} = \psi_5 \qquad (2.5)$$

$$\frac{c_1^5}{5!}A_{21}^{(1)} + \frac{(-1)^5}{5!}U_{21}^{(1)} + \frac{(-2)^5}{5!}U_{22}^{(1)} + \frac{(-3)^5}{5!}U_{23}^{(1)}$$

$$+\frac{(-1)^4}{4!}U_{21}^{(2)} + \frac{(-2)^4}{4!}U_{22}^{(2)} + \frac{(-3)^4}{4!}U_{23}^{(2)} = C_2^6\psi_6(C_2hL)$$

$$-\frac{U_{21}^{(1)}}{5!} - \frac{32}{5!}U_{22}^{(1)} - \frac{243}{5!}U_{23}^{(1)}$$

$$+\frac{1}{4!}U_{21}^{(2)} + \frac{16}{4!}U_{22}^{(2)} + \frac{81}{4!}U_{22}^{(2)} = \psi_6 \qquad (2.6)$$

$$\frac{c_1^6}{6!}A_{21}^{(1)} + \frac{(-1)^6}{6!}U_{21}^{(1)} + \frac{(-2)^6}{6!}U_{22}^{(1)} + \frac{(-3)^6}{6!}U_{23}^{(1)}$$

$$+\frac{(-1)^5}{5!}U_{21}^{(2)} + \frac{(-2)^5}{5!}U_{22}^{(2)} + \frac{(-3)^5}{5!}U_{23}^{(2)} = C_2^7\psi_7(C_2hL)$$

$$\frac{1}{6!}U_{21}^{(1)} + \frac{64}{6!}U_{22}^{(1)} + \frac{729}{6!}U_{23}^{(1)}$$

$$=\frac{1}{5!}U_{21}^{(2)} - \frac{32}{5!}U_{22}^{(2)} - \frac{243}{5!}U_{23}^{(2)} = \psi_7 \qquad (2.7)$$

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Similarly,

$$c_1^0 B_1 + c_2^0 B_2 + (-1)^0 V_1 + (-2)^0 V_2 + (-3)^0 V_3 = \psi_2(hL)$$

$$B_1 + B_2 + V_1 + V_2 + V_3 = \psi_1 \qquad (2.8)$$

$$c_1^1 B_1 + c_2^1 B_2 + (-1)^1 V_1 + (-2)^1 V_2 + (-3)^1 V_3 = \psi_2(hL)$$

$$B_2 - V_1 - 2V_2 - 3V_3 = \psi_2 \qquad (2.9)$$

$$\frac{c_1^2 B_1}{2!} + \frac{c_2^2 B_2}{2!} + \frac{(-1)^2 V_1}{2!} + \frac{(-2)^2 V_2}{2!} + \frac{(-3)^2 V_3}{2!} = \psi_3$$

$$\frac{1}{2!} B_2 + \frac{1}{2} V_1 + \frac{4V_2}{2} + \frac{9}{2} V_3 = \psi_3 \qquad (2.10)$$

$$\frac{c_1^3 B_1}{3!} + \frac{c_2^3 B_2}{3!} + \frac{(-1)^3 V_1}{3!} + \frac{(-2)^3 V_2}{3!} + \frac{(-3)^3 V_3}{3!} = \psi_4$$

$$\frac{B_2}{6} - \frac{V_1}{6} - \frac{8V_2}{6} - \frac{27}{6} V_3 = \psi_4 \qquad (2.11)$$

$$\frac{c_1^4 B_1}{4!} + \frac{c_2^4 B_2}{4!} + \frac{(-1)^4 V_1}{4!} + \frac{(-2)^4 V_2}{4!} + \frac{(-3)^4 V_3}{4!} = \psi_5$$

$$\frac{B_2}{24} + \frac{V_1}{24} + \frac{16V_2}{24} + \frac{81V_3}{24} = \psi_5 \qquad (2.12)$$

Simplifying, (2.2) to (2.12), we have

$$\begin{aligned}
B_1 &= \frac{1}{6}(6\varphi_1 + 5\varphi_2 - 10\varphi_3 - 30\varphi_4 - 24\varphi_5) \\
B_2 &= \frac{1}{12}(3\varphi_2 + 11\varphi_3 + 18\varphi_4 + 12\varphi_5) \\
V_1 &= \frac{1}{2}(-3\varphi_2 + \varphi_3 + 12\varphi_4 + 12\varphi_5) \qquad (2.13) \\
V_2 &= \frac{1}{6}(3\varphi_2 + 2\varphi_3 - 18\varphi_4 - 25\varphi_5) \\
V_3 &= \frac{1}{12}(-\varphi_2 - \varphi_3 + 6\varphi_4 + 12\varphi_5)
\end{aligned}$$

Also,

$$\begin{aligned}
U_{21}^{(1)} &= \frac{3}{23}(6\varphi_2 + 44\varphi_3 + 157\varphi_4 + 336\varphi_5 + 420\varphi_6 + 240\varphi_7) \\
U_{22}^{(1)} &= \frac{3}{46}(66\varphi_2 + 277\varphi_3 + 692\varphi_4 + 936\varphi_5 + 480\varphi_6 - 120\varphi_7) \\
U_{23}^{(1)} &= \frac{1}{207}(78\varphi_2 + 1124\varphi_3 + 2961\varphi_4 + 1608\varphi_5 - 5580\varphi_6 - 7920\varphi_7) \\
U_{21}^{(2)} &= \frac{6}{23}(18\varphi_2 + 63\varphi_3 + 126\varphi_4 + 157\varphi_5 + 110\varphi_6 + 30\varphi_7) \qquad (2.14) \\
U_{22}^{(2)} &= \frac{3}{23}(9\varphi_2 - 3\varphi_3 - 6\varphi_4 + 136\varphi_5 + 400\varphi_6 + 360\varphi_7) \\
U_{23}^{(2)} &= \frac{2}{69}(6\varphi_2 + 67\varphi_3 + 180\varphi_4 + 129\varphi_5 - 270\varphi_6 - 450\varphi_7)
\end{aligned}$$

Stability Analysis of Methods 524

We recall the Scheme given by

$$y_{n+1} = e^{hl} y_n + hB_1(hL)\, N(Y_{n1}) + hB_2(hL)\, N(Y_{n2}) + hV_1 N(y_{n-1}). \qquad (3.1)$$

$$Y_{n1} = y_n$$

$$Y_{n2} \;=\; e^{c_2 hL} y_n \;+\; hA_{21}(hL)\,N(y_n)\;+hU_{21}^{(1)}\,N(N(y_{n-1})+h^2 A_{21}^{(2)}(hL)N'(y_n)+ h^2 A_{21}^{(2)} N(N(y_n)+h^2 U_{21}^{(2)} N(N(y_{n-1}) \tag{3.2}$$

Again the first characteristic polynomial is given by

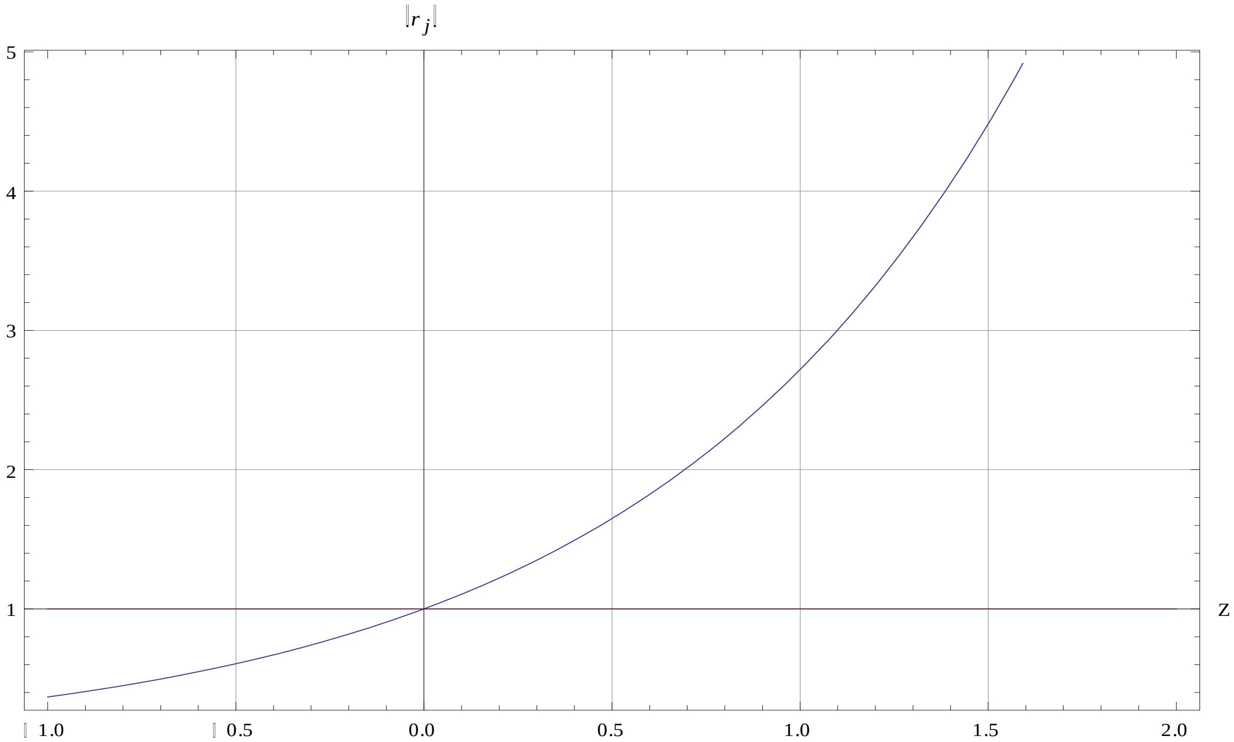
$$y_{n+1} = e^z y_n \qquad (hl = z)$$

$$y_{n+1} - e^z y_n = 0$$

dividing through by y_n

$$\frac{y_{n+1}}{y_n} - \frac{e^z y_n}{y_n} = 0$$

$$r - e^z = 0 \tag{3.3}$$



The stability graph of order five step two and stage order four scheme using MATHEMATICA

The graph shows that the method is zero stable since the first characteristic polynomial lies in a unit circle.

The stability matrix is

$$M(z) = e^Z + ZB_1 + ZB_2 e^Z + Z^2 B_2 A^{(1)}{}_{21} + Z^3 B_2 a_{21}{}^{(2)} + Z^3 B_2 U_{21}{}^{(2)} + Z^2 B_2 U^{(1)}{}_{21} + ZV_1$$

Scientia Africana, Vol. 13 (No.2), December 2014. Pp219-224
 ISSN 1118 - 1931

$$Det(\lambda I - M(Z)) = [Det\begin{pmatrix} \lambda - 0 & \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ z^3 b_2 a_{21}^{(2)} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ z^3 b_2 U_{21}^{(2)} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ z^2 b_2 a_{21}^{(1)} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ z^2 b_2 U_{21}^{(1)} & 0 \end{pmatrix} - zb_1$$

$$-ZV_1 - \frac{99}{100} zb_2 - \frac{99}{100}]$$

$$= -\frac{99\lambda}{50} + \lambda^2 - 2\lambda zb_1 - \frac{99}{50}\lambda zb_2 - \frac{99}{100} z^2 b_2 a_{21}^{(1)} - \frac{99}{100} z^3 b_2 a_{21}^{(2)} - z^3 b_2 a_{21}^{(1)} b_1 - z^4 b_2 a_{21}^{(2)} b_1 - \frac{99}{100} z^3 b^2{}_2 a_{21}^{(1)}$$

$$-\frac{99}{100} z^4 b^2{}_2 a_{21}^{(2)} . - \frac{99}{100} z^2 b^2{}_2 U_{21}^{(1)} - Z^3 b_1 b_2 U_{21}^{(1)} - \frac{99}{100} z^3 b^2{}_2 U_{21}^{(1)} - \frac{99}{100} z^3 b_2 U_{21}^{(2)} - z^4 b_2 U_{21}^{(2)} b_1 - \frac{99}{100} z^4 b^2{}_2 U_{21}^{(2)}$$

$$-2Z\lambda V_1 - Z^3 a_{21}^{(1)} b_2 V_1 - Z^4 a_{21}^{(2)} b_2 V_1 - Z^3 U_{21}^{(1)} b_2 V_1 - Z^4 b_2 U_{21}^{(2)} V \qquad (3.4)$$

inserting the values of the coefficients in we have

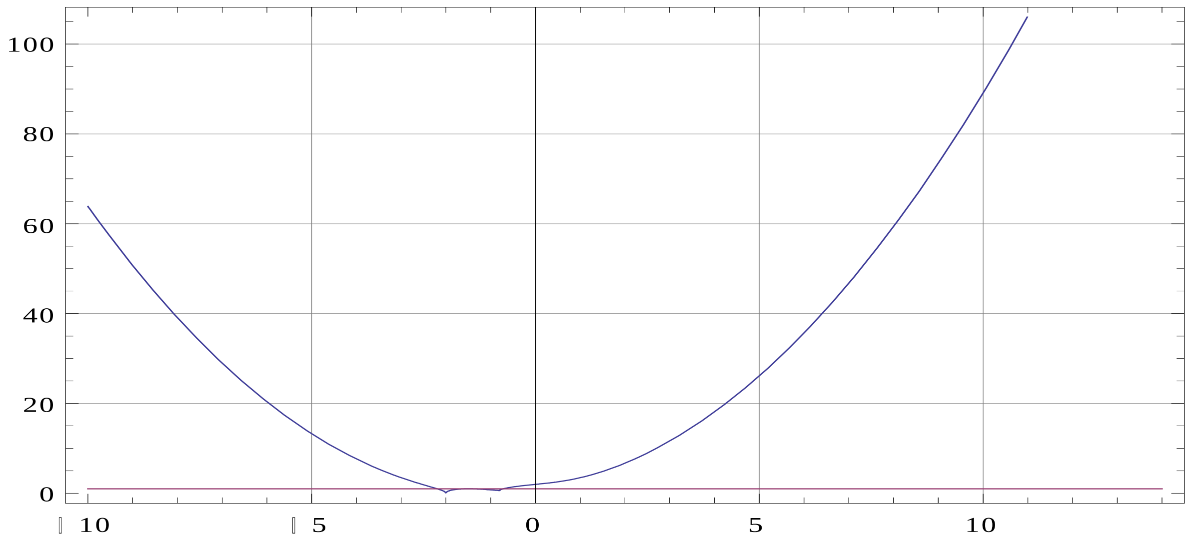
$$-\frac{4109586921z^2}{2500000000} - \frac{519907216550 67z^3}{25000000000000} - \frac{149288158442 79z^4}{50000000000000} - \frac{99\lambda}{50} - \frac{163473z\lambda}{500000} + \lambda^2 \qquad (3.5)$$

Solving equation (3.5) we have

$$\lambda = \frac{3}{10000000}\left(3300000 + 544910z - \sqrt{2}\sqrt{\begin{pmatrix} 5445000000000 + 1798203000000z + 9280878834050z^2 + \\ 11553493701126z^3 + 1658757316031z^4 \end{pmatrix}}\right)$$

$$\lambda = \frac{3}{10000000}\left(3300000 + 544910z + \sqrt{2}\sqrt{\begin{pmatrix} 5445000000000 + 1798203000000z + 9280878834050z^2 + \\ 11553493701126z^3 + 1658757316031z^4 \end{pmatrix}}\right) \qquad (3.6)$$

The graphs are shown below



The stability graph of for order five step two and stage order four (524) Scheme using MATHEMATICA. The interval between (-2.3,-0.5)

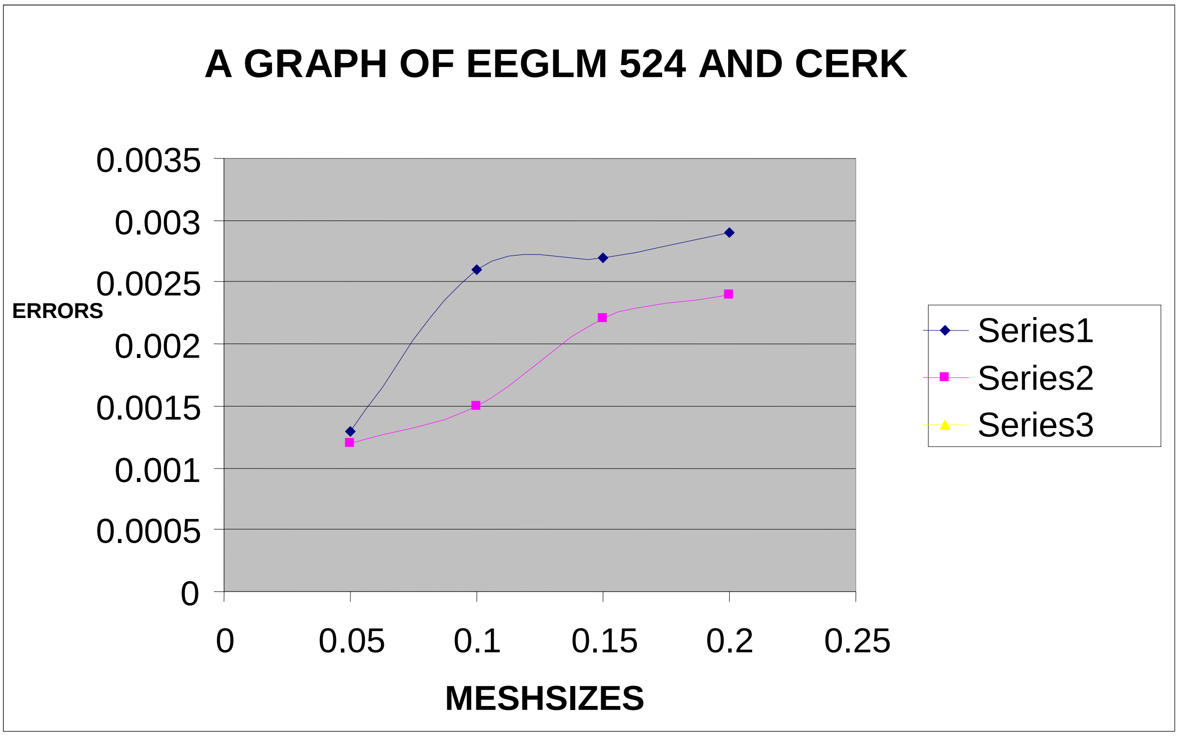
Numerical Experiments

In this section, we illustrate theoretical results given in section two, accuracies of 524 methods.

Problem

$y' = -10(y-1)^2, y(0) = 2 \ \ with \ t\varepsilon[0,1]$

Exact solution: $y = \dfrac{2+10t}{1+10t}$



The results obtained via our propose scheme 524 is very impressive.

This paper has investigated the construction of Extended Exponential General linear methods of 524 with its stabilities. Stability analysis show that the method is zero stable with its characteristics root lies in the unit disc.

Experimental experience reveals that our Methods perform better than the already existing ones.

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