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Short communication

MAXIMUM MODULUS, MAXIMUM TERM AND INTERPOLATION ERROR OF AN ENTIRE FUNCTION

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ABSTRACT: Let f be analytic on the compact set $E \subset C$, of positive transfinite diameter and let C_r denote the largest equipotential curve of E such that f is analytic within C_r. Generally, the growth of an entire function is measured in terms of its order and type. Here we have established the relations between maximum modulus, maximum term and interpolation error of best uniform approximation to a function $f \in C(E) = \{ f$ holomorphic on int (E) and continuous on E $\}$ by algebraic polynomials and Lagrange polynomials, in the form of direct estimates.

Key words/phrases: Equipotential curve, extremal polynomials, interpolation errors, Lagrange interpolation polynomial, transfinite diameter

INTRODUCTION

Let E be a compact set in complex plane and $x^{(n)} = \{x_{no}, x_{n1}, \ldots, x_{nn}\}$ be a system of $(n+1)$ points of the set E such that

$$V\left(x^{(n)}\right) = \prod_{o \le j \le k \le n} \left|x_{nj} - x_{nk}\right| \quad \text{and}$$
$$\Delta^j\left(x^{(n)}\right) = \prod_{k=o_{k\neq j}}^{n} \left|x_{nj} - x_{nk}\right|, \; j = 0,1,\ldots,n.$$

Again, let $h^{(n)} = \{h_{n0}, h_{n1}, \ldots, h_{nn}\}$ be the system of $(n+1)$ points in E such that

$$V_n \equiv V\left(h^{(n)}\right) = \sup_{x^{(n)} \subset E} V\left(x^{(n)}\right) \text{ and}$$
$$\Delta^o\left(h^{(n)}\right) \le \Delta^j\left(h^{(n)}\right) \text{ for } \; j = 1,2,\ldots,n.$$

Such a system always exists and is called the nth extremal system of E. The polynomials

$$L^{(j)}\left(z, h^{(n)}\right) = \prod_{k=o_{k\neq j}}^{n} \left(\frac{z - h_{nk}}{h_{nj} - h_{nk}}\right), \; j = 0,1,\ldots,n,$$

are called Lagrange extremal polynomials and the limit $d \equiv d(E) = \lim_{n\to\infty} V_n^{2/n(n+1)}$ is called the transfinite diameter of E.

Let us define the best uniform approximation to $f \in C(E) = \{ f$ holomorphic on int (E) and continuous on E $\}$ as follows :

$$m_{n,1}(f, E) \equiv m_{n,1}(f) = \inf_{g\in P_n} \| f - g\| = \| f - P_n\|,$$

where $\|\cdot\|$ is the sup norm and $P_n(z)$ denotes the set of all polynomials of degree at most n.

We also define

$$m_{n,2}(f, E) \equiv m_{n,2}(f) = \| L_n - L_{n-1}\|, \; n \ge 2,$$
$$m_{n,3}(f, E) \equiv m_{n,3}(f) = \| L_n - f\|,$$

where
$n \in N$ and

$$L_n(z) = \sum_{j=o}^{n} L^{(j)}(z, h^{(n)}) f(h_{nj})$$

is the Lagrange interpolation polynomial of degree n.

Let

$f(z) = \sum_{n=o}^{\infty} a_n z^n$ be an entire function. We set $M(r, f) = \max_{|z|=r} | f(z)|$;

$m(r, f) = \max_{k\ge 1} \{|a_k| r^k\}$. Then $M(r, f), m(r, f)$ are called respectively the maximum modulus and maximum term, of $f(z)$ on the circle $|z| = r$. The order and type of an entire function are defined as

$$\limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} = r(f) \equiv r, 0 \le r \le +\infty,$$

and, for functions having order $r\ (0 < r < \infty)$,

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{r^r} = T(f) \equiv T, 0 \le T \le \infty;$$

if $r(f) = 0$ or ∞, then $T(f)$ is underlined.

One way of characterizing the growth of an entire function in terms of interpolation error is to relate $m_{n,j}(f), j = 1,2,3,$ with order r and type T. Various authors (Readdy, 1970; Winiarski 1970; Rice, 1971; Juneja, 1974; Massa, 1981; Kasana and Kumar, 1994) established a relation between the growth parameters and interpolation error of an entire function, but as compared to the direct estimates of $m_{n,j}(f)$ or $M(r,f)$ these are still rather crude.

Further, let

$$g(z) = \log\left|g z + g_o + \frac{g_1}{z} + \ldots\right|, |g| = 1/d,$$

denote the Green's function for E with pole at ∞ and $y(z) = g z + g_o + \frac{g_1}{z} + \ldots$. Let C_r be the curve defined by

$C_r = \{z \in C : |y(z)| d = r\}$, where $w = y(z)$ is holomorphic and maps the unbounded components of the complement of E on $|w| > 1$ such that $y(\infty) = \infty$ and $y'(\infty) > 0$ and $C \setminus \hat{E}$ is simply connected. Also,

$\overline{M}(r,f) = \sup_{z\in C_r} |f(z)|$ for $r > 1$. It is clear that for $r = d, C_r = E$.

The aim of this paper is to set up more precise interrelation between $\overline{M}(r,f)$, $m(r,f)$ and $m_{n,j}(f)$ for entire functions of relatively slow growth, in terms of direct estimates for these quantities.

We say an entire function slowly increasing if $\overline{M}(r,f)$ increases essentially, not faster than

$$\exp(c(b-1))\left[\frac{\log(r/de^e)}{2b}\right]^{b/b-1} \quad \text{.....................(1)}$$

for b =2 and arbitrary $c > 0$ (critical value b =2 has been found to be significant). For rapidly increasing functions there are direct estimates of $\overline{M}(r,f)$ and $m_{n,j}(f)$. If f increases like Equation (1) with $1 < b < 2$, for example Proposition 2 and Corollary 1 apply, and for still more rapidly increasing entire functions of classical order $r > 0$, Proposition 1 and Theorem 2 apply. Though in latter case a necessary and sufficient characterization of the growth of f in terms of $m_{n,j}(f)$ is possible, the results are still sharper than the limit relations.

RESULTS

Preliminary results

Now we mention some preliminary results which have been utilized in the sequel;

Lemma 1. (Winiarski, 1970). If $f(z)$ is an entire function of order r and type T, then

$$\limsup_{r\to\infty} \frac{\log\log \overline{M}(r,f)}{\log r} = r$$

and for $0 < r < \infty$,

$$\limsup_{r\to\infty} \frac{\log \overline{M}(r,f)}{r^r} = Td^r.$$

Lemma 2. If a function f is defined and bounded on a compact set E, then

$$m_{n,1}(f) \le \| f - L_n \| \le (n+2) m_{n,1}(f)$$

and

$$\| L_n - L_{n-1} \| \le 2(n+2) m_{n-1,1}(f), \text{ for } n = 2,3,\ldots,$$

where L_n is the Lagrange interpolation polynomial with nodes at extremal points h_{nj}.

Proof. The proof of this lemma is available as Lemma 2 in Winiarski (1970).

Lemma 3. (Kasana and Kumar, 1994). For every $f \in C(E)$ and $m_{n,j}(f), j = 1,2,3,$ there exists an entire function $h_j(z) = \sum_{n=o}^{\infty} m_{n,j}(f) z^{n+1}$ such that

$$\overline{M}(r,f) \le a_o + 2h_j(r/d)$$

or

$$\overline{M}(r,f) \le a_o + K_o \overline{M}(r,h_j). \quad \text{......................(2)}$$

Lemma 4. Let $f \in C(E)$ be entire. Then for sufficiently large values of n, and for $r > 2de^e$, we have

$$m_{n,j}(f) \le K\overline{M}(r,f)\left(\frac{de^e}{r}\right)^{n+1},$$

where K is a constant.

Proof. Winiarski (1970:266) has proved that for any $e > 0$,

$$m_{n,3}(f) \le k\overline{M}(r,f)\left(\frac{de^e}{r}\right)^{n}, \quad \text{......................(3)}$$

where k is a constant and $d > 0$ is the transfinite diameter of E. Using (3) with Lemma 2, the proof follows.

Lemma 5. Let $f \in C(E)$. Then f can be extended to an entire function if and only if

$$m^{1/n}_{n,j}(f) \to 0 \text{ as } n \to \infty,\ j = 1,2,3.$$

This lemma is a direct consequence of Lemma 1, Winiarski (1970, eq. 4.5) and inequality due to Walsh (1969:77).

Main results

Maximum modulus and interpolation error

In this section we first prove two propositions for a class of entire functions of order $r > 0$. Then, we restrict ourselves to entire function of slow growth in order to obtain a characterization theorem of desired precision.

We denote by $C^2[x,\infty)$ the class of twice continuously differentiable functions on $[x,\infty)$ and for any $a \in C^2[x,\infty)$ with $a'' > 0, set.$

$$A(r) = \exp\left\{\log r\,(a')^{-1}(\log r) - a\left((a')^{-1}\right)(\log r)\right\}. \quad \text{......................(4)}$$

Proposition 1. Let $f \in C(E)$, has an analytic continuation as an entire function and $a \in C^2[x,\infty)$ be such that

$$a'(x) \to \infty \text{ and } a''(x) > 0$$

for each $x \ge x_1$. If

$$\overline{M}(r,f) = O(A(r/de^e)), r \to \infty,$$

then we have

$$m_{n,j}(f) = O\left(\frac{1}{y\,(n+1)}\right), n \to \infty, \quad \text{................(5)}$$

where $y\,(x) = \exp a\,(x), x \ge x_1$.

Proof. By Lemma 4, we have for $r > 2de^e$,

$$m_{n,j}(f) \le K\overline{M}(r,f)\left[\frac{de^e}{r}\right]^{n+1}.$$

For $r/de^e = \exp a'$, above inequality corresponds to Equation (5).

We define G to be the class of functions $a \in C^2[x,\infty)$ for some $x_1 \ge 0$ for which there exists a function w such that $x - w(x) \ge x_1, x \ge x_1$.

$\lim_{x\to\infty} a'(x) = \infty, \lim_{x\to\infty} w^2(x)a''(x) = \infty, \lim_{x\to\infty} a''(x) = 0,$
and
$a''(x) + \partial w(x) \approx a''(x)$ as $x \to \infty$ on ∂ such that $|\partial| < 1$.

Proposition 2. Let $f \in C(E)$ be an analytic continuation as an entire function satisfying Equation (5) for some y such that $a\,(x) = \log y\,(x) \in \Gamma$. Then,

$$\overline{M}\left(\frac{r}{de^e}\right) = O\left(\left[a''\left((a')^{-1}\left(\log\frac{r}{de^e}\right)\right)\right]^{-1/2} A\left(\frac{r}{de^e}\right)\right),$$
$r \to \infty.$ ....(6)

Proof. Using Lemma 4, we have

$$\overline{M}(r,f) \le a_o + 2\sum_{n=o}^{\infty} m_{n,j}(f)(r/d)^{n+1}.$$

In view of Equation (5), it gives

$$\overline{M}(r,f) \le O\left(a_o + 2\sum_{n=o}^{\infty}(r/d)^{n+1}\frac{1}{y\,(n+1)}\right). \quad \text{......................(7)}$$

Define $h(x,t) = xt - a\,(t)$. The right side of Equation (7) is estimated as

$$\sum_{n=o}^{\infty}\frac{1}{y\,(n+1)}\left(\frac{r}{d}\right)^{n+1}$$
$$= \sum_{n=o}^{\infty}\exp[h(\log(r/d), n+1)], r \to \infty.$$

Since $a \in \Gamma$, the asymptotic relation

$$\sum_{n=o}^{\infty}\exp[h(\log(r/d),n+1)]$$
$$\approx\sqrt{2p}\left[a''\left((a')^{-1}\left(\log\left(\frac{r}{de^{e}}\right)\right)\right)\right]^{-1/2}A\left(\frac{r}{de^{e}}\right),r\to\infty \quad \text{......................} (8)$$

follows easily by Berg (1968, Theorem 28.3). Hence the proof is completed.

Remark. The conclusions of Proposition 1 and 2 are best possible in the sense that 0 can not be replaced by o in Equations (5) and (6), respectively. In case y grows at least as rapidly as $\exp(x^{t}), t\ge 2,$ this will be a consequence of Theorem 1 below. For Proposition 2 and general y with $a\in\Gamma$ this is also clear from Eq. (7) by choosing f with $m_{n,j}(f)=1/y\ (n+1)$.

Now we consider the case of slowly increasing entire function expressed in terms of a particular $y\ (x)$ of the form $y\ (x)=\exp(cx^{b}), c>0,$ it means that the Theorem 1 will cover the case $b\ge 2$ whereas Proposition 1 and 2 cover the case $b>1$ and $1<b<2$, respectively. Let G denote the class of functions $a\in C^{2}[x,\infty)$, for some $x_1\ge 0$ with $\lim_{x\to\infty}a''(x)=\infty, a''(x)$ exit for $x\ge x_1$ and $\lim_{x\to\infty}a'''(x)(a''(x))^{-3/2}=0$.

Theorem 1. Let $f\in C(E)$ has an analytic continuation as an entire function for some $a\in\Gamma$ and let $A(r)$ be defined by Equation (4). The following statements are equivalent:

(i) $\overline{M}(r,f)=O(A(r/de^{e})),\ r\to\infty$

(ii) $m_{n,j}(f)=O(1/y\ (n+1)),\ n\to\infty.$

Proof. The (i) $\Rightarrow$ (ii) follows by Proposition 1. For converse, consider first the case when $\lim_{x\to\infty}a''(x)=c>0$ with $c\to\infty$. then the proof follows by Proposition 2. If $\lim_{x\to\infty}a''(x)=\infty$, then it gives again

$$\overline{M}(r,f)=O\left[a_o+2\sum_{n=o}^{\infty}(r/d)^{n+1}\frac{1}{y\ (n+1)}\right],r\to\infty \quad \text{......................} (9)$$

Now we have to show that the right hand side is $O(A(r/d\,e^{e}))$ as $r\to\infty$. We set $h(x,x)=xx-a(x)$, by a result of Sirovich (1971:96–98) and Evgrafov (1979:18), we have

$$\int_o^{\infty}e^{h(x,x)}dx=e^{h(x,x_o(x))}\left\{\frac{2p}{-h_{xx}(x,x_o(x))}\right\}^{1/2},\ x\to\infty.$$

Here h_{xx} denotes the second derivative with respect to x and $x_o(x)=(a')^{-1}(x)$. The hypothesis of Sirovich (1971:98 case 2) are satisfied since, for each $x>x_1$, $h(x,x)$, a function of x has a global maximum at $x_o=x_o(x)=(a')^{-1}(x)$, it means that $h_x(x,x_o(x))=0$.

Moreover, $h_{xx}(x,x_o(x))=-a''\left((a')^{-1}(x)\right)\ne 0$ and we have $(a')^{-1}(x\to\infty)$, as $x\to\infty$, it follows by the definition of G that

$$-\lim_{x\to\infty}h_{xx}(x,x_o(x))=\infty,$$

as well as

$$-\lim_{x\to\infty}h_{xx}(x,x_o(x))h(x,x_o(x))^{3/2}=\infty.$$

Now we set $x_r=x_o\left(\log\left(\frac{r}{de^{e}}\right)\right)$ and $K_r=[X_r]$, where $[X_r]$ denotes the integral part of X_r. By Eq. (9), we have

$$\sum_{n=o}^{\infty}\left(\frac{r}{d}\right)^{n+1}\frac{1}{y\ (n+1)}=\sum_{n=o}^{\infty}\exp\left\{h\left(\log\left(\frac{r}{d}\right),n+1\right)\right\}$$
$$\approx\sum_{n=o}^{\infty}\exp\left\{h\left(\log\left(\frac{r}{R^{*}e^{e}}\right),n+1\right)\right\},r\to\infty,R^{*}<d.$$
$$\le\int_o^{K_r}\exp\left\{h\left(\log\left(\frac{r}{de^{e}}\right),x\right)\right\}dx$$
$$+\exp\left\{h\left(\log\left(\frac{r}{de^{e}}\right),K_r\right)\right\}$$
$$+\exp\left\{h\left(\log\left(\frac{r}{de^{e}}\right),K_r+1\right)\right\}$$
$$+\int_{K_r+1}^{K_r}\exp\left\{h\left(\log\left(\frac{r}{de^{e}}\right),x\right)\right\}dx$$

$$\leq 2\exp\left\{h\left(\log\left(\frac{r}{de^e}\right), x_r\right)\right\}$$
$$+\int_o^{K_r}\exp\left\{h\left(\log\left(\frac{r}{de^e}\right), x\right)\right\}dx.$$

In the consequence of definition of h, Eq. (9) and Eq. (4), we get

$$\overline{M}(r,f)=O\left(a_o+K_oA\left(\frac{r}{de^e}\right)\left[2+\sqrt{2p}\,a''\left((a')^{-1}\left(\log\left(\frac{r}{de^e}\right)\right)\right)^{-1/2}\right]\right),$$

holds if and only if

$$m_{n,j}(f)=O\left(\frac{1}{\exp(c(n+1))^b}\right),\ n\to\infty.$$

The asymptotic relation (Eq. 7) is given by this for y, in the case $b\in N$ and $b>2$, respectively.

Maximum term and interpolation error

Since $h_j(z)$ in an entire function and it is clear that $h_j(z)$ and $f(z)$ have the same maximum term and is denoted by $m(r,f)$. A satisfactory characterization of $m(r,f)$ in terms of interpolation error holds for large dass of entire functions, including those of order $r>0$ and type $T\geq 0$.

Theorem 2. Let $f\in C(E)$ has an analytic continuation as an entire function with maximum term $m(r,f)$ and let a, y and $A(r)$ be. Then condition (Eq. 5) is equivalent to

$$m(r,f)=O(A(r/de^e)), r\to\infty.$$

Proof. Let Eq. (5) be satisfied, so that for each $r>0$, $n\geq n_o$,

$$m_{n,j}(f)\left(\frac{r}{de^e}\right)^{n+1}\leq M_o\left(\frac{r}{de^e}\right)^{n+1}\frac{1}{y\ (n+1)}. \quad\text{.................... (10)}$$

For fixed r, the maximum over x of the function

$(r/de^e)^{n+1}/y\ (n+1)$

is attained at $x=(a')^{-1}(\log(r/de^e))$ and has value $A(r/de^e)$,

provided $(a')^{-1}(\log(r/de^e))>x_1$. Therefore,

$$\max_{n\geq n_o} m_{n,j}(f)\left(\frac{r}{de^e}\right)^{n+1}$$
$$\leq M_o\max_{n\geq n_o}\left(\frac{r}{de^e}\right)^{n+1}\frac{1}{y\ (n+1)}=M_oA\left(\frac{r}{de^e}\right)$$

for $r>r_o$. Let $r_1>r_o$ large enough, so that

$$m_{n_o,j}(f)\left(\frac{r}{de^e}\right)^{n+1}\geq\max\left\{m_{n,j}(f)\left(\frac{r}{de^e}\right)^{n+1}; 0\leq n<n_o\right\}.$$

For each $r>r_1$, it also gives that $m(r,f)\leq M_oA(r/de^e)$. Conversely, (Eq. 10) gives that

$$m_{n,j}(f)\leq\frac{M_oA(r/de^e)}{(r/de^e)^{n+1}},\ r>r_o, \quad\text{................. (11)}$$

where r_o and M_o are constant. Taking $[r/de^e]^n=\exp a'$ for some $n\in N$ in (Eq. 11), we have for n large enough, since $a'(x)\to\infty$ as $x\to\infty$, this implies (Eq. 5).

Maximum modulus and maximum term

Corollary 1. Let $f\in C(E)$ has an analytic continuation as an entire function satisfying $m(r,f)=O(A(r/de^e)),\ r\to\infty$ with $A(r)$ defined by Equation (4), for $a\in\Gamma$. Then

$$\overline{M}(r,f)=O\left(\left[a''\left((a')^{-1}\left(\log\frac{r}{de^e}\right)\right)\right]^{-1/2}A\left(\frac{r}{de^e}\right)\right),\ r\to\infty.$$

This is a direct consequence of Theorem 2 and Proposition 2.

Corollary 2. Let $f\in C(E)$ be given as in Theorem 1. The following statements are equivalent:

(a) $m(r,f)=O(A(r/de^e)),\ r\to\infty.$

(b) $\overline{M}(r,f)=O(A(r/de^e)),\ r\to\infty.$

Combining Theorems 1 and 2 we can easily establish the above equivalent and it is concerned with functions of slow growth only.

REFERENCES

1. Berg, L. (1968). *Asympotische Darstellungin and Entwicklungen.* VEB Deuscher Vertag der Wissen schaften, Berlin, Germany
2. Evgrafov, M.A. (1979), *Asymptotic Estimates and Entire Functions*, 3rd edn. (Russian) Nauka, Moscow.
3. Juneja, O.P. (1974), Approximation of an entire function. J. *Approx. Theory* **4**(11):343–349.

4. Kasana, H.S and Kumar, D (1994). On approximation and interpolation of entire functions with index-pair (p, q), *Publications Mathematiques* **38:**255–267.

5. Massa, S. (1981). Remarks on the growth of an entire function and the degree of approximation. *Riv. Mat. Univ. Parma* **4(**7):1–6.

6. Readdy, A.R. (1970). Approximation of an entire function. *J. Approx. Theory* **3**:128–137.

7. Rice, J.R. (1971). The degree of convergence for entire functions, *Duke Math. J.* **38**:429–440.

8. Sirovich, L. (1971). *Techniques of Asymptotic Analysis*. Springer-Verlag New York.

9. Walsh, J.L. (1969). Interpolation and approximation by rational functions in the complex domain. Colloq. Publications, Vol. 20, Amer.Math. Soc. Providence R.I.

10. Winiarski, T.N. (1970). Approximation and interpolation of entire functions. *Ann. Polon. Math.* **29**:259–273.