Decomposition of BH-Lattices

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ABSTRACT: In this paper we study further properties of BH-lattices which is a subclass of BH-monoids. We furnish certain examples of BH-monoids which are not BH-lattice. We give a characterization of BH-lattices in terms of bounded BH-lattices and commutative l-groups. Also we prove that every BH-lattice is a direct product of Heyting algebra and commutative l-group under certain condition. Further we obtain the decomposition theorem in terms of Boolean algebra and a commutative l-group.

Key words/phrases: Brouwer-Heyting(BH) monoid, BH-lattice, Brouwerian algebra, Heyting algebra, l-group

INTRODUCTION

It is Ward and Dilworth [16] initiated the study of residuated lattices and as a result a study on lattice ordered Semigroups with residuation as an operation have been introduced by K.L.N Swamy [10] under the name DRI-Semigroups. Two algebras Brouwerian and Heyting are generalizations to Boolean algebra (lattice) and dual to each other. There is a mismatch in the literature regarding the nomenclature of these two algebras. Brouwerian algebra defined in [7] by Nordhaus, EA and Leolapidus is called as Heyting algebra by Birkhoff G [1]. To avoid this mismatch, we recall the two definitions in the preliminaries and also to clear the confusion between the two algebras Swamy [9] introduced the notion of Brouwer-Heyting monoids (for short BH-monoids) as a general class containing both Brouwerian algebra, its dual Heyting algebra. He also observed that DRI-semigroups and dual DRI-semigroups are examples of BH-monoids. Further he obtained decomposition theorems for both BH-Monoids as well as for BH-lattices.

This paper investigates further properties of BH-lattices and decomposition theorems on BH-lattices. It is divided into 3 sections. The first section is preliminaries in which we recall all the existing literature on BH-monoids and BH-lattices. In section two, we obtain further properties on BH-lattices which we use in the sequel. The last section is devoted for decomposition theorems. In this paper we prove every BH-lattice can be represented as a direct product of Heyting algebra and a commutative l-group with certain conditions.

The following abbreviations are used in this paper. Po-group means partially ordered group, l-group means lattice-ordered groups, BH means Brouwer-Heyting, DRI means Dually residuated lattice ordered.

Preliminaries

In this section, we recall certain definitions and results concerning Heyting algebra [3, 6] and Brouwer-Heyting lattices [9] which will be used in the sequel.

Definition 1.1. A bounded lattice \((L, \vee, \wedge)\) in which to each \(a, b\) there is a least \(x\) such that \(x \lor a \geq b\) is called a Brouwerian algebra. The least element is denoted by \(2^{-b}\).

Definition 1.2. A bounded lattice \((L, \vee, \wedge)\) is called a Heyting algebra if for any given elements \(a\) and \(b\) in \(L\), there is a greatest \(x\) such that \(x \land a \leq b\).

Remark 1.1. The greatest element \(x\) is denoted by \(a \rightarrow b\). Clearly \(a \rightarrow b\) is unique.

Lemma 1.1. Every Boolean algebra is a Heyting
algebra, with \( a \to b \) given by \( a' \lor b \).

**Theorem 1.1.** Let \( L \) be a Heyting algebra and \( x, y, z \in L \). Then the following hold:

1. \( y \leq x \to y \)
2. \( 1 = x \to 1 \)
3. \( x \leq y \iff x \to y = 1 \)
4. \( x \land (x \to y) = x \land y \)
5. \( y \land (x \to y) = y \) and \( ((x \land y) \to x) \land z = z \)
6. \( y = 1 \to y \)
7. If \( x \leq y \), then \( (z \to x) \leq (z \to y) \)

**Theorem 1.2.** Let \( L \) be a Heyting algebra and \( x, y, z \in L \). Then the following hold:

1. \( x \land y \to (y \to z) = (x \land y) \to (x \to z) \)
2. \( x \to (y \to z) = (x \land y) \to (z \to y) \)
3. \( (x \to y) \land (x \to z) = x \land (y \lor z) \)
4. \( (x \to y) \land z = ((z \land x) \to (z \land y)) \land z \)
5. \( L \) is a bounded distributive lattice.

**Note:** An equivalent definition of 1.2 is:

**Definition 1.3.** A non-empty set \( L \) with three binary operations \( \land, \lor \) and \( \to \) and two distinguished elements \( 0 \) and \( 1 \) is a Heyting Algebra if the following conditions hold:

\[
(\text{H}_1) \quad (L, \land, \lor, 0, 1) \text{ is a lattice with } 0, 1
\]
\[
(\text{H}_2) \quad x \land (x \to y) = x \land y
\]
\[
(\text{H}_3) \quad x \land (y \to z) = x \land [(x \land y) \to (x \land z)]
\]
\[
(\text{H}_4) \quad (x \land y) \to x = 1.
\]

**Definition 1.4.** A system \((G, \circ, e, \rho)\) is called a partially ordered group (po-group) if \((G, \circ, e)\) is a group, \((G, \rho)\) is a poset and for all \(a, b, x, y \in G\), \(xpy \Rightarrow (a \circ x \circ b)(a \circ y \circ b)\). And it is called a commutative po-group, if \(\circ\) is commutative.

**Definition 1.5.** An \(l\)-group is a po-group where \((G, \rho)\) is a lattice.

**Example 1.1.** The additive groups \(Z\) - integers, \(Q\) - rationals, \(R\) - reals are the simplest examples of \(l\)-groups.

**Definition 1.6.** A system \((G, \circ, e, \rho, \to)\) is a Brouwer-Heyting (for short BH) monoid if

1. \((G, \circ, e)\) is a commutative semi group with identity \(e\).
2. \((G, \rho)\) is a partially ordered set and \(\to\) is binary operation on \(G\) such that for all \(x, a, b \in G\), \((x \circ b)pa \Leftrightarrow xp(a \to b)\).

The following are examples of Brouwer-Heyting monoids.

**Example 1.2.** Let \((G, \circ, e, \rho)\) is a commutative po-group. Define \(a \to b = a \circ b^{-1}\)

**Example 1.3.** \((B, \lor, \land, 0, 1)\) is a Boolean algebra. Let \(\circ = \land, \rho \leq \) defined by \(a \circ b = a, e = 1, a \to b = a \lor b\).

**Example 1.4.** Let \((G, \lor, \land, 1)\) be a Heyting algebra. Let \(\circ = \land, \epsilon = 1, \rho \) is the lattice order, \(a \to b\) is the largest \(x\) such that \(b \land x \leq a\). That is, the new arrow operation is defined in terms of the arrow operation in the Heyting algebra by \(a \to b = a \lor b\).

**Example 1.5.** Let \((L, \lor, \land, 0)\) be Brouwerian algebra. If we take \(\rho = (\text{the dual of } \leq ) \geq, \rho \) is the lattice order, \(a \to b = a \land b\) is the smallest \(x\) such that \(x \lor b \geq a, \rho = \lor\) and the least 0 as identity element. Thus \((L, \lor, \land, 0)\) is a BH monoid where \(\rho\) is the dual ordering of the lattice \((L, \lor, \land)\).

**Example 1.6.** Let \((G, +, \leq, -)\) be a DRI-monoid. We have \(x + b \geq a \Leftrightarrow a - b \leq x\). (By the definition of DRI- monoid). Thus the dual of DRI-monoid is a BH monoid.

**Note:** Here after for the sake of convenience, we use \(\leq\) instead of \(\rho\).

**Theorem 1.3.** In BH monoid \((L, +, e, \leq, \to)\) the following hold for \(x, y, z \in L\)

1. \(y \leq z \Rightarrow x \circ y \leq x \circ z\)
2. \(x \leq (x \circ y) \to y\)
3. \((x \to y) \circ y \leq x\)
4. \(z \to (x \circ y) = (z \to y) \to x = (z \to x) \to y\)
5. \(e \to e = e\)
6. \(x \leq e \to x\)
7. \(e \leq x \to x\)
8. \(x \leq y \Rightarrow x \to z \leq y \to z\)
9. \(x \leq y \Rightarrow z \to y \leq z \to x\)
10. \(y \leq x \Rightarrow e \leq x \to y\)
11. \((x \to y) \circ (y \to z) \leq x \to z\)
12. \(I(x \circ y) \circ (z \circ y) \text{ exists for any } z \text{ and } z \circ (x \circ y) = (z \circ x) \circ (z \circ y)\)
13. \(I(x \circ y) \circ (z \to y) \text{ exists for any } z \text{ and } z \to (x \circ y) = (z \to x) \circ (z \to y)\)
14. \((x \land y) \to z = (x \to z) \land (y \to z)\).

**Definition 1.7.** A BH monoid \((L, +, e, \leq, \to)\) is a BH-lattice if

1. \((L, \leq)\) is a lattice with glb and lub denoted \(\land\)
and $\lor$ respectively

2. $x \circ (y \land z) = (x \circ y) \land (x \circ z), \forall x, y, z \in L$

3. $((y \rightarrow x) \land e) \circ x = x \land y, \forall x, y \in L.$

**Theorem 1.4.** A lattice $(L, \lor, \land, o, e, \rightarrow)$, where $(L, \lor, e)$ is a commutative monoid and $\rightarrow$ be a binary operation on $L$, is a BH-lattice if and only if

1. $(y \rightarrow x) \circ x \leq y$
2. $(x \land z) \rightarrow y \leq x \rightarrow y$
3. $x \leq (x \circ y), \forall x, y \in L$
4. $x \circ (y \land z) = (x \circ y) \land (x \circ z), \forall x, y, z \in L$
5. $((y \rightarrow x) \land e) \circ x = x \land y, \forall x, y \in L$

Since $\land$ is commutative, it follows that $(y \rightarrow x) \land e \circ x = ((x \rightarrow y) \land e) \circ y, \forall x, y \in L.$

**Remark 1.2.** Theorem 1.4 shows that BH-lattices can be defined by means of identities alone.

**Theorem 1.5.** Let $(L, \lor, \land, o, e, \rightarrow)$ be a BH-lattice and $a, b, x \in L$, then the following hold.

1. $x \rightarrow x = e$.
2. $L$ is distributive.
3. $(x \lor y) \circ (x \land y) = x \circ y$
4. $x = x \circ e = (x \lor e) \circ (x \land e)$.
5. If $e \leq x$, then $x$ is invertible.
6. $e \lor x$ is invertible.
7. If $x$ is invertible, then $e \rightarrow x$ is inverse of $x$.
8. If $y$ is invertible, then $x \rightarrow y = x \circ (e \rightarrow y)$
9. If $x$ and $y$ are invertible, then $xoy$ is invertible and $(e \rightarrow x) \circ (e \rightarrow y)$ is the inverse of $xoy$.
10. $e \rightarrow x$ is invertible.
11. If $G$ is the set of all invertible elements, then $G$ is a $1$-group.

**Theorem 1.6.** (Decomposition Theorem for BH monoids and BH-lattices)

1. BH monoid $L$ is direct product of po-group and a BH monoid with greatest element if and only if $e \rightarrow x$ is invertible and $x \rightarrow x = e, \forall x \in L.$
2. BH-lattice $L$ is direct product of a commutative $1$-group and a BH-lattice with greatest element.

**Further Properties of BH-lattices**

We begin with the following

**Example 2.1.** Boolean algebra, Heyting algebra given in example 1.3 and 1.4 above and $1$-groups are BH-lattices. Heyting algebra is an example of bounded BH-lattice while the unbounded $1$-groups are examples of unbounded BH-lattices.

Now we have the following examples which are BH-monoids but not BH-lattices.

**Example 2.2.** The commutative po-group $(G, \circ, e, \rho)$ given in example 1.2 above is not a BH-lattice.

**Example 2.3.** Consider the lattice given in Fig 1. Clearly it is a Brouwerian algebra and hence a BH-monoid. Since $[(c \rightarrow b) \land e] \circ b = (c \land b)$, it is not a BH-lattice. Thus

![Figure 1. Example of Brouwerian algebra.](image)

Brouwerian algebra is not a BH-lattice.

**Example 2.4.** Consider the set $A$ - the multiplicative semigroup of the set of non-negative integers ordered by the divisibility relation. Then $A$ is a DRI-semigroup with least element $1$ and greatest element $0$, for $x, y \in A, x \rightarrow y = \lfloor x/y \rfloor$, where $\lfloor x \rfloor$ is the floor function, $x \land y = \text{GCF}(x, y)$ and $x \lor y = \text{LCM}(x, y)$. Then for the BH-monoid induced from the above DRI-monoid, $[2 \rightarrow 3] \land e \circ 2 = (1 \land 1) \circ 2 = 2 \circ 2 = 4$. Hence it is not a BH-lattice. Hence DRI-monoid is not a BH-lattice.

**Note:** Here after $L$ stands for a BH-lattice $(L, \lor, \land, o, e, \rightarrow)$.

**Theorem 2.1.** In BH lattice $L$, for $x, y \in L$, if $x$ and $y$ are invertible, then $x \rightarrow y$ is invertible and $y \rightarrow x$ is the inverse of $x \rightarrow y$.

**Proof.** Let in BH lattice $L, x, y \in L$ are invertible. From theorem 2.9, $x \rightarrow y = xo(e \rightarrow y)$ and $y \rightarrow x = yo(e \rightarrow x)$. Hence $(x \rightarrow y) \circ (y \rightarrow x) = [xo(e \rightarrow y)] \circ [yo(e \rightarrow x)] = xo[[e \rightarrow y] \circ o(e \rightarrow x)]$ (associativity) $= xo[oe(e \rightarrow x)] = e$. Hence the result holds.

**Theorem 2.2.** For $x, y, z \in L$, the following properties hold.

1. $x \leq y \rightarrow (y \rightarrow x)$
2. $z \leq (xoz) \rightarrow (x \land y)$ or equivalently $(x \land y) \leq (xoz) \rightarrow z$
3. $x \land y \leq [(xoz) \rightarrow y] \land [(xoy) \rightarrow x]$
Proof. Proof is a consequence of the definition BH-monoid and 1 of theorem 1.3.

Remark 2.1. For a BH-lattice L and a, b ∈ L, b → a = \max\{x ∈ L : ax ≤ b\}

Theorem 2.3. Let x, y, z ∈ L. Then the following properties hold.
1. x ≤ ((x ∨ y)oz) → z, x ≤ ((x ∨ y)ox) → y
2. x → y ≤ (z → y) → (z → x)
3. x → y ≤ (x → z) → (y → z)

Proof.
1. By 1 of theorem 1.3, xoz ≤ (x ∨ y)oz, xoy ≤ (x ∨ y)ox ⇒ x ≤ (x ∨ y)ox → y.
2. By 4 of theorem 1.3, we have z → (xoy) = (z → x) → y = (z → y) → x. And by 1 of theorem 2.2 above we have x ≤ z → (z → x) ⇒ x → y ≤ [z → (z → x)] → y (by 8 of theorem 1.3). Thus we have (z → y) → (z → x) = z → (yo(z → x)) = (z → (z → x)) → y. Hence x → y ≤ (z → y) → (z → x).
3. By 4 and 9 of theorem 1.3 we have (y → z)oz ≤ y ⇒ x → y ≤ x → [(y → z)oz] = (x → z) → (y → z).

Corollary 2.1. Let x, y ∈ L. Then x → y ≤ e → (y → x)

Proof. Replace z by x in 3 of theorem 2.3

Notation: For x ∈ L, we shall denote e → x by the symbol x^r.

Theorem 2.4. For all x, y ∈ L the following properties hold.
1. (xoy)^r = x^r → y = y^r → x
2. x ≤ y ⇒ y ≤ x
3. x ≤ x^r and x^r = x^rr
4. (x ∨ y)^r = x^r ∧ y^r and x^r ∨ y^r ≤ (x ∧ y)^r
5. x → y ≤ y^r → x^r
6. x^r → y^r = y^r → x^r

Proof. (1) and (2) are clear from theorem 1.3. By 2 of theorem 2.2 it follows that x ≤ e → (e → x) = x^rr. Hence x ≤ x^r. Now replacing x^r in the place of x in the above inequality we have x^r ≤ x^rr. But by 2 in the inequality x ≤ x^r, we obtain that x^rr = e → (x^rr) ≤ (e → x) = x^r. So x^r = x^rr. By 9 and 13 of theorem 1.3, e → (x ∨ y) = (e → x) ∧ (e → y) = x^r ∧ y^r. Since x ∧ y ≤ x, y, it follows that x^r ≤ e → (x ∧ y) = (x ∧ y)^r and similarly y^r ≤ (x ∧ y)^r. Hence x^r ∨ y^r ≤ (x ∧ y)^r. (5) follows from 2 of theorem 2.3. By 7, 8 and 10 of theorem 1.5, x^rr → y^rr = x^rr → (o(e → y^rr)) = x^rr → (o(y^rr → y^rr)) = y^rr → x^rr.

Theorem 2.5. Let x, y ∈ L, then the following properties hold.
1. x ∧ y → x ≤ e
2. e ≤ x → (x ∧ y)

Proof. Follows from 8 and 9 of theorem 1.3 and 1 of theorem 1.5.

Theorem 2.6. For x, y ∈ L, x → (y → x) = x ⇒ x → (x → y) = y → (y → x)

Proof. Let x, y ∈ L. Then x → (x → y) = (x → (y → x)) → (x → y) = (x → (y → x)) → (y → x) (by 4 of theorem 1.3). Since from theorem 2.2, y ≤ x → (y → x) it follows that y → (y → x) ≤ (x → (x → y)) → (y → x) = x → (x → y). Hence y → (y → x) ≤ x → (y → x). Analogously x → (x → y) ≤ y → (y → x).

Theorem 2.7. Let a, b, x, y ∈ L such that a ≤ b and x ≤ y, then the following properties hold.
1. aox ≤ boy
2. (a → x) ∨ (b → y) ≤ b → x
3. a → y ≤ (a → x) ∧ (b → y)
4. a → y ≤ b → x

Proof. Suppose a ≤ b and x ≤ y. By 1 in theorem 1.3 a ≤ b implies aox ≤ box and x ≤ y implies box ≤ boy. Hence aox ≤ boy. By 8 and 9 in theorem 1.3, a ≤ b ⇒ a → x ≤ b → x and x ≤ y ⇒ b → y ≤ b → x. Hence (a → x) ∨ (b → y) ≤ b → x. And a ≤ b ⇒ a → y ≤ b → y and x ≤ y ⇒ a → y ≤ a → x. Hence a → y ≤ (a → x) ∧ (b → y). (4) follows from 2 and 3.

Theorem 2.8. Let x, y, z, t ∈ L. Then the following properties hold.
1. x → y ≤ xoz → yoz
2. (x → y)a(s → t) ≤ (xos) → (yot)

Proof. From 2, 4 and 8 of theorem 1.3, x → y ≤ [(xoz) → z] → y = xoz → yoz. From (1) we have x → y ≤ xos → yos and s → t ≤ yos → toy. Now by theorem 2.7 and 11 of theorem 1.3, we have (x → y)a(s → t) ≤ (xos) → (yos)a(yos → yot) ≤ xos → yot. Hence (2) holds.
Theorem 2.9. Let $x, y \in L$. $(x \land y) \rightarrow (x \lor y) = (x \rightarrow y) \land (y \rightarrow x)$

Proof. By 13 and 14 of theorem 1.3, $x \land y \rightarrow x \lor y = [\{(x \land y) \rightarrow x\} \land \{(x \land y) \rightarrow y\}] = [\{(x \rightarrow x) \land (y \rightarrow x)\}] = (x \land y) \land (y \rightarrow x)$

And also by 1, 3 and 12 of theorem 1.3, $(x \lor y) = (x \rightarrow y) \lor (y \rightarrow x)$

Hence we have $x \land y \rightarrow x \lor y = (x \rightarrow y) \land (y \rightarrow x)$.

Theorem 2.10. Let $x, y \in L$. $y \leq x \Rightarrow (y \rightarrow x)x = y$

Proof. By 9 of theorem 1.3 and 3 in the definition of BH-lattices, $y \leq x \Rightarrow (y \rightarrow x) \leq e$

Hence we have $y \leq x \Rightarrow (y \rightarrow x)x = y$.

Theorem 2.11. Let $x, y \in L$. $(x \rightarrow y) \land e)x = x$

Proof. By 13 of theorem 1.3 and theorem 2.10, $x = (x \rightarrow (x \lor y)) \land e)(x \lor y) = ((x \rightarrow y) \land e)x \lor y = (y \rightarrow x)x = x$

Theorem 2.12. Let $x, y \in L$. $[(x \rightarrow y) \land (y \rightarrow x)]x = x \land y$

Proof. Follows from theorem 2.9 and theorem 2.10.

Theorem 2.13. Let $x, y, z \in L$. $x \leq y \leq z \Rightarrow (x \rightarrow y) \land e)(y \rightarrow z) = x \rightarrow z$

Proof. Let $x \leq y \leq z$. By 4 of theorem 1.3 and theorem 2.10 we have $(y \rightarrow z)x = y$

Hence $(x \rightarrow y \rightarrow z) = (x \rightarrow y) \land (y \rightarrow z) \land e) = (x \rightarrow y) \land (y \rightarrow z) \land e)$

And by 8 of theorem 1.3 we have $x \rightarrow y \leq z \rightarrow z$. Hence by theorem 2.10 $(x \rightarrow y) \land (y \rightarrow z) = ((x \rightarrow z) \land (y \rightarrow z)) \land e) = x \rightarrow z$

Theorem 2.14. For $x, y, z \in L$ the following hold:

1. $(x \rightarrow y) \land e)(y \rightarrow z) = x \rightarrow z$
2. $(x \rightarrow y) \land e)(y \rightarrow z) = x \rightarrow z$
3. $(e \rightarrow x) \land e)(y \rightarrow z) = e \rightarrow (y \rightarrow z)$
5. \(x \ast y \leq (x \to y) \ast e\)
6. \(x \leq e \Rightarrow x \ast e = x\)

**Proof.** Using 8 of theorem 1.3 and theorem 2.9, \(x \ast y = (x \to y) \land (y \to x) = (x \land y) \rightarrow (x \lor y) \leq e\). If \(x = y\), then by 1 of theorem 1.5, \(x \ast y = e\). Conversely if \(x \ast y = e\), then \(x \ast y = (x \land y) \rightarrow (x \lor y) = e \Rightarrow e \leq (x \land y) \rightarrow (x \lor y) \Rightarrow (x \lor y) \leq (x \land y) \Rightarrow x \lor y = x \land y\) and consequently it follows that \(x = y\). Hence (1) holds. Evidently (2) is trivial. Since by theorem 2.9, \((x \lor y) \ast (x \lor y) = [(x \lor y) \land (x \lor y)] \rightarrow [(x \lor y) \lor (x \lor y)] = (x \lor y) \land (x \lor y)\). Hence (3) holds. Again \((x \to y) \land (y \to x) \leq x \to y\) and \((y \to z) \land (z \to y) \leq y \to z\). So \((x \ast y) \odot (y \ast z) = [(x \to y) \land (y \to x)] \odot (y \to z) \land (z \to y) \leq (x \to y) \odot (y \to z)\) by theorem 2.7 \(x \to z\). So (by 11 of theorem 1.3).

That is \((x \ast y) \odot (y \ast z) \leq x \to z\). By a similar argument \((x \ast y) \odot (y \ast z) \leq z \to x\). So it follows that \((x \ast y) \odot (y \ast z) \leq (x \to z) \land (z \to x) = x \ast z\). Hence (4) holds. By theorem 2.3 we have \(y \leq x \to (x \to x) \rightarrow (x \to y) = e \rightarrow (x \to y) \Rightarrow x \ast y = (x \to y) \land (y \to x) \leq (x \to y) \land (e \to (x \to y)) = (x \to y) \ast e\). So \(x \ast y \leq (x \to y) \ast e\). Finally let \(x \leq e \Rightarrow e \leq e \Rightarrow e \leq x\) (by 9 of theorem 1.3).

\(\Rightarrow x = e \land x \leq x \land (e \to x) = x \ast e \leq x\). Hence \(x \ast e = x\)

**Definition 2.1.** For \(n \in N\) and \(x \in L\), define \(x^n = x_{x \odot x \odot \ldots \odot x}\) (\(n\) times).

**Theorem 2.18.** If there exists an element \(x \in L\) such that \(e < x\), then the set \(L\) is an infinite and not bounded above.

**Proof.** Let \(e < x\) and \(x \neq e\). Consider the sequence \(\{x^n\}_{n \in N}\) where \(N\) is the set of non-negative integers. With 1 of theorem 1.3, \(x \leq x^2\).

If \(x^2 = x\), then by 2 of theorem 1.3, \(x \leq x^2 \Rightarrow x < x^2\). i.e. \(x \leq e\). Since \(e < x\), it follows that \(x = e\) which is a contradiction. So \(x^2 \neq x\). If \(x^2 = e\), then \(x = x \rightarrow e = x \rightarrow x^2 = (x \to x) \rightarrow e = x \rightarrow e\) since \(e < x\), by 9 of theorem 1.3, \(e \to x = e \rightarrow e\). This implies that \(x = e \to x \leq e\). Hence \(x = e\). In both cases there is a contradiction. So \(e < x < x^2\). This implies that \(x^2 \leq x^4\).

If \(x^2 = x^4\), then by 2 of theorem 1.3, \(x^2 \leq \{x^2 \odot x^2\} \rightarrow x^2 = e\). i.e. \(x^2 \leq e\). Hence \(x^2 < e\) which is a contradiction. Hence \(e < x < x^2 < x^3\) Suppose that \(e < x < x^2 < x^3 < \ldots < x^n\). The latter implies that \(x^n \leq x \odot x^2 \odot x^2 \odot \ldots \odot x^2 \leq x\). i.e. \(x^2 \leq e\). Hence \(x^2 = e\) which is a contradiction. Hence by the principle of mathematical induction, the chain \(e < x < x^2 < x^3 < \ldots < x^n \ldots\) does not terminate at some point. So, the sequence of elements \(x^n\) are all distinct. Hence the set \(L\) is infinite and unbounded above.

**Corollary 2.2.** If \(L\) is bounded above by the element \(t\), then \(t = e\).

**Theorem 2.19.** For \(x, y \in L\) the following properties hold.

1. \(x \leq e\) and \(y \leq e \Rightarrow x \odot y \leq x \lor y\)
2. \(e \leq x \) and \(e \leq y \Rightarrow x \land y \leq x \lor y\)
3. \(e \leq x\) and \(y \leq e \Rightarrow y \odot x \leq x\)

**Proof.** Follows from 1 of theorem 1.3.

**Theorem 2.20.** For \(x \in L\) and any \(n \in N\), \(x^n \leq e \Rightarrow x = e\).

**Proof.** Let \(x^n \leq e\). Repeatedly applying the associative and distributive properties of \(o\) over the operation \(v\), we have \((x^n v e) = x^n v (x^{n-1} v \ldots v x v e) = (x^n v e) v (x^{n-1} v \ldots v x v e) = x^n v \ldots v x v e\) (as \(x^n \leq e\) = \((x^n v e) v (x^{n-1} v \ldots v x v e) = (x^n v e) v (x^{n-1} v \ldots v x v e) = e \rightarrow x v e = e \Rightarrow x \leq e\). Hence \(x = e\) (by 2 of theorem 1.3). The converse follows from theorem 2.7 by induction.

**Corollary 2.3.** For \(x \in L\), \(x \ast e = e \Rightarrow x \leq e\).

**Proof.** Follows from theorem 2.17 and theorem 2.20.

**Corollary 2.4.** Let \(x, y \in L\) and \(y\) is invertible element. Then for any \(n \in N\), \(x^n \leq y^n \Rightarrow x \leq y\).

**Proof.** Let \(x^n \leq y^n\). Then \((x^n o (e \rightarrow y^n)) = x^n o (e \rightarrow y^n) = y^n o (e \rightarrow y^n) = (y^n o (e \rightarrow y^n)) = e\). Thus by theorem 2.20, \(x o (e \rightarrow y^n) \leq e\) and hence \(x \leq y\). The converse follows from 1 of theorem 2.7.

**Theorem 2.21.** For \(x \in L\) and any \(n \in N\), \(x^n \ast e = e \Rightarrow x = e\).

**Proof.** From theorem 2.20, \(x \leq e\). Observe that \(x = x \rightarrow e = x \rightarrow x^n = x \rightarrow (x o x^{n-1}) = (x \to x) \rightarrow x^n = e \rightarrow x^{n-1} = x^n \rightarrow x^{n-1}\) (by 4 of 1.3). Since \(x \leq e\), \(x \land e = x\). So by the distributive property of \(o\) over \(\land\) repeatedly, we have \(x^n = (x \land e)^n = x^n \land x^{n-1} \land \ldots \land x \land e = x^{n-1} \land \ldots x \land e = (x \land e)^{n-1} = x^{n-1}\). Hence \(x = x^n \rightarrow x^{n-1} = e\). The converse is trivial.
Corollary 2.5. In an $l$-group, every element other than the identity element has an infinite order.

Definition 2.2. An element $l \in L$ is called unity if and only if $xo(l \to x) = lol, \forall x \in L$.

Lemma 2.1. In $L$ with unity $l$, the following properties hold.
1. $lol = l$
2. $l \leq e$
3. $e = e \to l = (e \to l) \to l$

Proof. Let $x = e$ in definition 2.2. Then it is immediate that $lol = l$. Thus by 2 and 4 of theorem 1.3 it follows that $l \leq (lol) \to l = e$. Let $t = e \to l$. Then $t = e \to (lol) = (e \to l) \to l = t \to l$ i.e. $t = t \to l$. By 4 of theorem 1.3, $e = t \to l$ and 8 of theorem 1.3, $x = x \to x = x \to l \Rightarrow l \to x$. Thus $e \to l = e$.

Theorem 2.22. Unity element is unique, if it exists in $L$.

Proof. Let $l$ and $l'$ be unities. This implies $e \to l = e = e \to l'$ and $l \leq e \Rightarrow l \to l' \leq e \to l' = e$. Now by 3 in the definition of BH-lattices and definition of unity, $l \land l' = [(l \to l') \land e] \lor l' = (l \to l') \lor l' = l$. By a similar argument $l' \land l = l'$. Hence $l = l'$.

Corollary 2.6. If BH-lattice $L$ has unity, then $L$ is an $l$-group if and only if $l = e$.

Theorem 2.23. If $L$ with unity contains a least element $a$, then $a = l$

Proof. Since $a$ is least element of $L$, $a \leq e \Rightarrow (aoa) \leq eoa = a$ (by 1 of theorem 1.3). This implies $aoa = a$ (as $a$ is least). Thus $l = (l \to a)oa = (l \to a)oa = loa \leq eoa = a$ (since $l \leq e$ and by 1 of theorem 1.3). Since $a$ is least, $a = l$.

Theorem 2.24. If $L$ with unity $l$, contains an element $x$ such that $x \prec l$, then the set $L$ is an infinite set and unbounded below.

Proof. Let $x \prec l$ and $x \neq l$. Since by lemma 2.1 $l \leq e$, it follows that $x \prec l \leq e \Rightarrow x^2 \leq lox \leq x$ (by 1 of theorem 1.3). If $x^2 = x$, then $x = x^2 \leq lox \leq x \Rightarrow x = lox$. Then the above argument together with 1 and 4 of theorem 1.3, $e = e \to l = (x \to x) \to l = x \to (xol) = x \to [xo(l \to x)]ox = x \to xo(l \to x) = x \to l$. $\Rightarrow e \leq x \to l \Rightarrow l \to x$ which is a contradiction. Thus $x^2 \neq x$

If $x^2 = l$, then $x \to l = x \to x^2 = (x \to x) \to x = e \to x$. Since $x \leq e$, by 9 of theorem 1.3, $e \leq e \to x$. So $e \leq e \to x = x \to l \Rightarrow l \to x$ which is a contradiction. Hence $x^2 \neq l$. In both cases there is a contradiction, hence $x^2 < l \Rightarrow x^2o\Rightarrow x^2 < x^2$. If $x^2 = x^4$, then $l = x^2o(l \to l) = x^2o(x^2 \to l) = xol$. Hence by the same argument as above $l < x$, which is a contradiction. Hence $x^4 < x^2 < l \leq e$. In the similar fashion, the sequence of elements $x^2n$ are all distinct. Hence the set $L$ is infinite and unbounded below.

Corollary 2.7. If $L$ with unity is bounded below by the element $t$, then $t = l$.

Theorem 2.25. Let $L$ be a BH-lattice with unity $l$. Then $L_1 = \{x \in L : x \to l = e\}$ is a BH-lattice with least and greatest element. And $L_1$ is subset of the BH-lattice $L^z = \{x \in L : e \to x = e\}$.

Proof. By lemma 2.1, $lol = l$ and $e \to l = e$. Hence both $l$ and $e$ belong to $L_1$. For $x \in L_1$, $e \to x = (x \to l) \to x = (x \to x) \to l = e \to l = e$. Hence $e$ is the greatest element and $l$ is the least element in $L_1$.

Let $x, y \in L_1$. Then $x \leq e$ and $y \leq e$. This implies that $xy \leq e$. Hence $(xyo) \to l \leq e \to l = e$. Furthermore by theorem 2.14 and 8 of theorem 1.3, $(x \to l)ayo \leq (xyo) \to l$. This implies $y \leq (xyo) \to l$. Hence $e = y \to l \leq (xyo) \to l = (xyo) \to l$. Thus $(xyo) \to l = e$. By 14 of theorem 1.3, $(x \land y) \to l = (x \to l) \land (y \to l)$. Hence $L_1$ is closed under both $\land$ and $\lor$. Since $x, y \leq x \lor y = e$, by 8 of theorem 1.3, $e = (x \to l) \land (y \to l) \leq (x \lor y) \to l \leq e \to l = e$. Hence $x \lor y \in L_1$. Moreover $(x \to y) \to l = (x \to l) \to y = e \to y = (y \to l) \to y = (y \to y) \to l = e \to l = e$. So $x \to y \in L_1$. Hence $L_1$ is a BH-lattice with $l$ as least element and $e$ as greatest element. Clearly $L_1 \subseteq L^z$. Finally by the proof of theorem 4.2 of [9], $L^z$ is a BH-lattice with greatest element.

Theorem 2.26. $(L_1, o)$ is a group iff $\forall x, y, z \in L_1, x \to y = x \to z \Rightarrow y = z$.

Proof. Let $(L_1, o)$ be a group and let for any $x, y, z \in L_1$, $x \to y = x \to z$. Hence by 9 of theorem 1.5 it follows that $(x \to y)^{-1} = y \to x = z \to x = (x \to z)^{-1} \Rightarrow (y \to x) \to (e \to x) = (z \to x) \to (e \to x)$. Using 4 of theorem 1.3 and 8 of theorem 1.5, this
implies that \((y \to (x_0(e \to x)) = (z \to (x_0(e \to x)))\).
So \(y = y \to e = z \to e = z\).
Conversely let \(x, y, z \in L, x \to y = x \to z \Rightarrow y = z\).
Then \(e \to \{(y_0(e \to y)) = (e \to y) \to (e \to y) = e \to e \to e\). So that \(y_0(e \to y) = e\).
Hence \(y\) is invertible element. Hence \((L, o)\) is a group.

**Theorem 2.27.** \(L\) is an l-group if \((L, o)\) is a group and further \(a \to b\) is the solution of the equation \(box = a\).

**Proof.** If \((L, o)\) is a group, then by the definition of BH-lattice and 1 of theorem 1.3 \(L = (L, o, \leq, \to)\) is an l-group. Again by 7 and 8 theorem 1.5, \(bo(a \to b) = bo\{ao(e \to b)\} = a\). Hence \(a \to b\) is the solution of the equation \(box = a\).

**Theorem 2.28.** BH-lattice \(L\) bounded below is a Heyting Algebra if \(xoy = x\land y, \forall x, y \in L\).
Also, in \(L\), if \((L, \leq, \to)\) is a Heyting Algebra then, \(xoy = x \land y, \forall x, y \in L\).

**Proof.** The first part of the theorem is trivial. For the second part let \((L, \leq, \to)\) be a Heyting Algebra.
For \(a, b \in L, a \to b\) is the largest \(x\) such that \(x \land b \leq a\). \(a \land b \leq a \Rightarrow a \leq a \to b \Rightarrow a \land b \leq (a \to b) \land b = a \land b\) (by 4 of theorem 1.1, as \(L\) is Heyting Algebra). Hence \(a \land (a \to a) = a \land a \Rightarrow a \land e = a \Rightarrow a \leq e\). Hence \(e\) is the largest element of the lattice. Hence by 1 of theorem 1.3, \(aoa \leq a, b \Rightarrow \{aoa \land b\} = \{a \land b\} \land \{aoa \to a \land b\}\) (by 4 of theorem 1.1 in a Heyting Algebra \((L, \leq, \to)\) = \(a \land b\) (by 3 of theorem 1.1).

**Theorem 2.29.** Let \(L\) be with unity \(l\), if \((L, \land, \land)\) is a Boolean algebra, then \(o = \land\) and \(x' = l \to x\).

**Proof.** By theorem 2.23 and corollary 2.2, \(l\) is the least element and \(e\) is the greatest element of the Boolean algebra. Let \(x \in L\). Then there exists an element \(x' \in L\) such that \(x \land x' = e\) and \(x \land x' = l\).
Hence by theorem 2.28, \(o = \land\) and \(xox = x \land x' = l\). Hence \(x = l \to x\) = \(x_0(l \to x) = l\) and \(x_0 = l\) \(\Rightarrow x' \leq l \Rightarrow x. \Rightarrow e = x \land x' \leq x \land x' \Rightarrow x \land x' = e\).
So \(l \to x\) is the complement of \(x\) in the Boolean algebra. Hence by uniqueness of complement \(x' = l \to x\)

**Decomposition Theorems of BH-lattices**

**Lemma 3.1.** Let \(L\) be a BH-lattice and for \(x, y, z \in L, x \to (zox) \leq (x \to z)o(e \to z)\) holds. Then the following are equivalent.

1. \(H = \{x \in L : xo(z) \to x = e\}\)
2. \(H' = \{x \in L : xo(x) = x\}\)
3. \(B = \{x \in L : e \to x = e\}\)

**Proof.** Let \(x \to (zox) \leq (x \to z)o(e \to z)\). Since by theorem 2.8, \((x \to z)o(e \to z) \leq (xo(e \to z)), it follows that \((xo(e \to z)) = (x \to z)o(e \to z)\).
Let \(x \in H \Rightarrow xo(x) \to x = e \Rightarrow x \leq xo(x). As x \leq xo(x) \to x = e, it follows that xo(x) ≤ x(by 1 and 2 of theorem 1.3). So that xo(x) = x. Thus \(H\) is the set of all idempotent elements with respect to the operation \(o\). Also if xo(x) = x, then clearly xo(x) \to x = x \to x = e. Let \(x \in H \Rightarrow xo(x) \to x = e. Thus by 4 of theorem 1.3, e \to x = ((xo(x) \to x) \to x = (xo(x) \to xo(x) = e).
Hence \(x \in B\). Furthermore let \(y \in B \Rightarrow e \to y = e.\) Then by 1 and 3 of theorem 1.3, \(y = yo(e \to y) \leq e \Rightarrow y^2 ≤ y. And e = y^2 \Rightarrow y^2 = (y^2 \to y)o(e \to y) = y^2 \to y \Rightarrow y \leq y^2. Therefore \(e = y^2\).

**Theorem 3.1.** A BH-lattice \(L\) is direct product of Heyting algebra and a commutative l-group if \(\{x \to (zox) \leq (x \to z)o(e \to z)\)

2. there exists an idempotent element \(0 \in L\) such that \(0 ≤ x, for any idempotent element \(x \in L\).

**Furthermore if L is the direct product of a Heyting algebra and a commutative l-group, then condition (1) holds.**

**Proof.** Let the conditions (1) and (2) hold. Since by theorem 2.8, \((x \to z) \to (e \to z) \leq (xo(e \to z)), it follows that \((xo) \to (zox) = (x \to z) \to (e \to z)\).

BH-lattice with greatest element \(e\) is in similar line to the proof 2 of theorem 1.6 [9]. Further for \(x, y \in H\), as \(xoy ≤ x\) and \(xoy ≤ y\) it follows that \(xoy ≤ x \land y\). By theorem 2.7, \(x \land y ≤ x, y\) implies that \(x \land y = (x \land y)o(x \land y) ≤ xo(y). Hence \(x \land y\) = xo(y). Thus by theorem 2.28 \(H\) is a Heyting algebra. Moreover, as \(0, e \in H, H\) is non-trivial.

Now consider the set \(G = \{a \in L : (aoa) \to a = a\}. Since e \in G, G \neq \emptyset. Let a \in G. Then aoa(e \to a) = (aoa) \to (a) = (aoa) \to (a) = e\) (by (a) above). Hence \(a\) is the invertible element. Let \(a \in L\) be an invertible element. Then \(aoa(e \to a) = e = (aoaoe) \to (a) = (aoa) \to (aoa) = e\) (by (a) above and 7 of theorem 1.5) \(a = (aoa) \to a. Hence G\) is the set of all invertible elements of \(L\). By 11 of theorem 1.5, \(G\) is a commutative l-
group. As $e \in G$ and using 10 of theorem 1.5, $e \rightarrow x \in G, \forall x \in L, G$ is non-trivial.

Hence $L$ is the direct product of $G$ and $H$, by a proof in a comparable strip to the proof 2 of theorem 1.6 [9].

Furthermore if $L$ is the direct product of a Heyting algebra and a commutative l-group, then trivially condition (1) holds.

**Theorem 3.2.** BH-lattice $L$ is direct product of a BH-lattice with least and greatest elements and a commutative l-group if and only if
1. $(x \land y) \rightarrow (z \land y) \leq (x \rightarrow z)(y \rightarrow z)$
2. There exists an element $l$ such that $x \land (l \rightarrow x) = l$

**Proof.** Let the conditions given in (1) and (2) hold. Then by lemma 2.1, $lol = l$ and $e \rightarrow l = e$, and from (1) by 2 of theorem 2.8 it follows that $(x \land y) \rightarrow (z \land y) = (x \rightarrow z)(y \rightarrow z)$. Consider $G = \{x \in L : x \rightarrow l = x\}$ and $L_l = \{x \in L : x \rightarrow l = e\}$.

Then by theorem 2.25, $L_l$ is a BH-lattice with least element $l$ and greatest element $e$. Since both $e$ and $l$ are in $L_l$, it is non-trivial.

Fro $x, y \in G$, $(x \land y) \rightarrow l = (x \rightarrow l) \land (y \rightarrow l)$ and $(x \land y) \rightarrow l = (x \land y) \land l = (x \land l \land y \land l)$. Hence $G$ is closed under $\land$ and $\Rightarrow$. Furthermore $(x \land y) \rightarrow l = (x \land l \rightarrow y \land l) \land (x \land y \rightarrow l) = (x \land y \rightarrow l) \land l = x \land y \land l = e$. Hence $x$ is an invertible element. Thus $G$ is a $\land$ semi lattice and hence $G$ is a commutative l-group. By lemma 2.1, $e \in G$ and using 10 of theorem 1.5, $e \rightarrow x \in G, \forall x \in L$. So $G$ is non-trivial.

Now for $a \in L$, let $t = a \rightarrow l$ and $s = a \rightarrow t$. Then $l \rightarrow l = (a \rightarrow l) \rightarrow l = a \rightarrow (l \land l) = a \rightarrow l = t$ and $s \rightarrow l = (a \rightarrow (a \land l)) \rightarrow l = (a \rightarrow l) \rightarrow (a \rightarrow l) = e$. Thus $t \in G$ and $s \in L_l$. Since $l \leq e$ by 9 of theorem 1.3, $a = a \rightarrow e \leq a \rightarrow l$. Thus by theorem 2.10, $t = (a \rightarrow l) \land (a \rightarrow l) = a$.

Now let $t = t' s'$, where $t' \in G$ and $s' \in L_l$. Then $a \rightarrow l = (t' s') \rightarrow lol = (t' \rightarrow l) \land (s' \rightarrow l) = e \land (t' \rightarrow l) = t'$. Hence $t = t'$ and consequently $t = t' s' = t's' \Rightarrow (e \rightarrow t) = (e \rightarrow t) \land (s' \rightarrow l) = e$. Clearly $|e| = G \cap B$. Thus $L$ is the direct product of $B$ and $G$.

Conversely, if $L$ is the direct product of BH-lattice with least and greatest element $B$ and commutative l-group $G$, then trivially conditions (1) and (2) hold.

**Corollary 3.1.** For a BH-lattice $L$ with unity and bounded below the following are equivalent
1. $(x \land y) \rightarrow (z \land y) \leq (x \rightarrow z)(y \rightarrow z), \forall x, y, z \in L$.
2. $x \rightarrow (z \land y) \leq (x \rightarrow z)(e \rightarrow z), \forall x, z \in L$.
3. $L$ is the direct product of Heyting algebra and a commutative l-group.

**Theorem 3.3.** BH-lattice $L$ is direct product of a Boolean algebra and a commutative l-group if and only if
1. $x \rightarrow (y \land z) \leq (x \rightarrow y)(e \rightarrow z), \forall x, y, z \in L$.
2. there exists an element $l$ in $L$ such that $(l \rightarrow x) \land l \rightarrow (l \rightarrow x) = x$ for all $x \in L$.

**Proof.** Suppose that the condition in (1) and (2) hold. Let $G$ be the set of all invertible elements of $L$ and $H$ be the set of all idempotent elements of $L$. By lemma 3.1 and the same argument as in the proof of theorem 3.1, $H$ is a BH-lattice with greatest element $e$, $e = \land$ and $L$ is direct product of $G$ and $H$.

For any $x \in H$, by 9 of theorem 1.3, $x \land e \Rightarrow e \leq x \leq e \Rightarrow e = e \Rightarrow x$.

Hence as $l \in H$, it follows that $e = e \Rightarrow (l \rightarrow x)$. So by the condition given in 2 and 4 of theorem 1.3, $l \rightarrow (l \rightarrow x) = x \Rightarrow (l \rightarrow (l \rightarrow x)) = (l \rightarrow (l \rightarrow x) \rightarrow x) = (l \rightarrow (l \rightarrow x) \rightarrow e) = (l \rightarrow (l \rightarrow x) \rightarrow l = x \rightarrow l \rightarrow l = l \leq x$. Hence $H$ is bounded below by the element $l$. Thus by theorem 2.28, $H$ is a Heyting algebra.

Now for $x \in H$, $x \land (l \rightarrow x) = (l \rightarrow (l \rightarrow x)) \land (l \rightarrow x) = (l \rightarrow (l \rightarrow x))(l \rightarrow x) = l$. Moreover, $l = (l \rightarrow (x \lor (y \rightarrow x))) \land (l \rightarrow (x \lor (y \rightarrow x)))$ $= (l \rightarrow (x \lor (y \rightarrow x))) \land (x \lor (y \rightarrow x))$. This implies $l \rightarrow (x \rightarrow (y \rightarrow x)) = y$ and $l \rightarrow (x \rightarrow (y \rightarrow x)) = y$.

Hence $l \rightarrow (x \rightarrow (y \rightarrow x)) \leq y$ and $l \rightarrow (x \rightarrow (y \rightarrow x)) \leq y \leq l \rightarrow (l \rightarrow x) \leq y$. Hence $l \rightarrow (x \rightarrow (y \rightarrow x)) \leq y$ and $l \rightarrow (x \rightarrow (y \rightarrow x)) \leq y$. Hence $e = e \rightarrow l \rightarrow (l \rightarrow x) \rightarrow (x \lor (y \rightarrow x))) \land (x \lor (y \rightarrow x))$. Hence $H$ is a Boolean algebra. Thus $L$ is the direct product of Boolean algebra and a commutative l-group.

Conversely if $L$ is the direct product of a Boolean algebra $H$ and a commutative l-group $G$, then trivially condition (1) and (2) hold.

**Definition 3.1.** A BH-lattice $L$ is called idempotent if $x^2 = x, \forall x \in L$.

**Theorem 3.4.** An idempotent BH-lattice $L$ with unity $l$ is a direct product of Boolean algebra and commutative l-group if $l \rightarrow (l \rightarrow x) = x, \forall x \in L$. 


Proof. Suppose that \( l \rightarrow (l \rightarrow x) = x, \forall x \in L \). Let \( H = \{ x \in L : c \rightarrow x = e \} \) and \( G \) be the set of all invertible elements of \( L \). The proof of \( H \) is a BH-lattice with greatest element \( e \) is analogous to the proof \( 2 \) of theorem \( 1.6[9] \) and further \( L \) is the direct product of \( H \) and \( G \) can be obtained. Furthermore for \( x, y \in L, xoy \leq x \) and \( xoy \leq y \). Hence \( xoy \leq x \wedge y \). By theorem \( 2.7, x \wedge y = (x \wedge y) \circ (x \wedge y) \leq xoy \). Thus \( xoy = x \wedge y \). Finally by the same argument as in the proof of theorem \( 3.3, x = l \rightarrow x, \forall x \in H \). Thus \( H \) is a Boolean algebra. Conversely if \( L \) is the direct product of Boolean algebra and commutative \( l \)-group, then the condition \( l \rightarrow (l \rightarrow x) = x, \forall x \in L \) is trivial.

**Open problem**

1. Which group of BH-lattice can be decomposable in to irreducible non-trivial sub algebras of BH-lattices?

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