Analysis of two-operator boundary-domain integral equations for variable-coefficient Dirichlet and Neumann problems in 2D with general right-hand side

Markos F. Yimer and Tsegaye G. Ayele

1Department of Mathematics, College of Natural and Computational Sciences, Addis Ababa University, P. O. Box 1176, Addis Ababa Ethiopia, E-mail: markos.fisseha@aau.edu.et
2Department of Mathematics, College of Natural and Computational Sciences, Addis Ababa University, P. O. Box 1176, Addis Ababa Ethiopia, E-mail: tsegaye.ayele@aau.edu.et

ABSTRACT: The Dirichlet and Neumann boundary value problems for the linear second-order scalar elliptic differential equation with variable coefficients in a bounded two-dimensional domain are considered. The right-hand side the PDE belongs to $H^{-1}(\Omega)$ or $H^{-1}(\Omega)$, when neither classical nor canonical conormal derivatives of solutions are well defined. The two-operator approach and appropriate parametrix (Levi function) are used to reduce each of the problems to two different systems of two-operator boundary-domain integral equations (BDIEs). Although the theory of BDIEs in 3D is well developed, the BDIEs in 2D need a special consideration due to their different equivalence properties. As a result, we need to set conditions on the domain or on the associated Sobolev spaces to ensure the invertibility of corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs. The equivalence of the two-operator BDIE systems to the original problems, BDIE system solvability, solution uniqueness/nonuniqueness and invertibility BDIE system are analyzed in the appropriate Sobolev spaces. It is shown that the BDIE operators for the Neumann BVP are not invertible, and appropriate finite-dimensional perturbations are constructed leading to invertibility of the perturbed operators.

Keywords/phrases: Analysis, Boundary-Domain Integral Equations, Parametrix, Partial Differential Equations, Variable Coefficients.

INTRODUCTION

Partial differential equations (PDEs) with variable coefficients often arise in mathematical modelling of inhomogeneous media (e.g. functionally graded materials or materials with damage induced inhomogeneity) in solid mechanics, electromagnetics, thermal conductivity, fluid flows through porous media, and other areas of physics and engineering. Generally, explicit fundamental solutions are not available if the PDE coefficients are not constant, preventing reduction of boundary value problems (BVPs) for such PDEs to explicit boundary integral equations (BIEs), which could be effectively solved numerically. Nevertheless, for a rather wide class of variable-coefficient PDEs it is possible to use instead an explicit parametrix (Levi function) associated with the fundamental solution of the corresponding frozen-coefficient PDEs, and reduce BVPs for such PDEs to systems of boundary-domain integral equations (BDIEs) for further numerical solution of the latter, see for example Chkadua et al. (2009, 2011); Mikhailov (2002, 2005b, 2006). Still this (one-operator) approach does not work when the fundamental solution of the frozen-coefficient PDE is not known explicitly (as e.g. in the Lamé system of anisotropic elasticity). To overcome this difficulty, one can apply the so-called two-operator approach, formulated in Mikhailov (2005a) for a certain nonlinear problem, that employs a parametrix of another (second) PDE, not related with the PDE in question, for reducing the BVP to a BDIE system. Since the second PDE is rather arbitrary, one can always choose it in such a way, that its parametrix is known explicitly. The simplest choice for the second PDE is the one with an explicit fundamental solution.

For a function from the Sobolev space $H^1(\Omega)$, a classical conormal derivative in the sense of traces may not exist (Mikhailov, 2015, Appendix A). However, when this function satisfies...
a second order PDE with a right-hand side from $H^{-1}(\Omega)$, the generalized conormal derivative can be defined in the weak sense, associated with the first Green identity and an extension of the PDE right-hand side to $\tilde{H}^{-1}(\Omega)$, see (McLean, 2000, Lemma 4.3) and (Mikhailov, 2011, Definition 3.1). Since the extension is not unique, the conormal derivative appears to be an operator that is not unique, which is also nonlinear in $u$ unless a linear relation between $u$ and the PDE right-hand side extension is enforced. This creates some difficulties in formulating the BDIEs. These difficulties are addressed in Mikhailov (2015, 2018) presenting formulation and analysis of direct segregated BDIE systems equivalent to the Dirichlet and Neumann problems for the divergent type PDE with a variable scalar coefficient and a general right-hand side. This needed a nontrivial generalization of the third Green identity and its conormal derivative for such functions, which extends the approach implemented in Mikhailov (2006); Chkadua et al. (2009, 2010, 2011); Mikhailov (2013) for the PDE right-hand from $L_2(\Omega)$. In Ayele Tsegaye (2021b), using the two-operator approach in settings different from those in Ayele Tsegaye and Mikhailov (2010, 2011), a generalization of the two-operator third Green identity and its conormal derivative is derived and the two-operator BDIEs for variable-coefficient Dirichlet, Neumann and mixed BVPs are analyzed in 3D.

Nowadays, the theory of BDIEs in 3D is well developed (Mikhailov, 2002, 2005a; Chkadua et al., 2009, 2010, 2011), but the BDIEs in 2D need a special consideration due to their different equivalence properties. As a result, we need to set conditions on the domain or on the associated Sobolev spaces to ensure the invertibility of corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs (Dufera Tamrat and Mikhailov, 2015; Ayele Tsegaye et al., 2017, 2019; Ayele Tsegaye, 2021a; Ayele Tsegaye and Bekele Solomon, 2023; Ayele Tsegaye and Dagnaw Mulugeta, 2021a).

In this paper, we extend the results in (Ayele Tsegaye and Yimer Markos, 2023; Ayele Tsegaye, 2021a), and consider the Dirichlet and Neumann BVPs for the linear second-order scalar elliptic differential equation with variable coefficient in a two-dimensional bounded domain. The PDE right-hand side belongs to $H^{-1}(\Omega)$ or $\tilde{H}^{-1}(\Omega)$ when neither classical nor canonical conormal derivatives of solutions are well defined. The two-operator approach and appropriate parametrix (Levi function) are used to reduce each problem into two different systems of BDIEs. The properties of corresponding potential operators are investigated. The equivalence of the two-operator BDIE systems to the original problems, BDIE system solvability, solution uniqueness/nonuniqueness and invertibility BDIE system are analyzed in the appropriate Sobolev spaces. It is shown that the BDIE operators for the Neumann BVP are not invertible, and appropriate finite-dimensional perturbations are constructed leading to invertibility of the perturbed operators.

**Preliminaries**

**Conormal derivatives.** Let $\Omega$ be a domain in $\mathbb{R}^2$ bounded by a smooth curve $\partial \Omega$. Consider the scalar elliptic differential equation, which for sufficiently smooth function $u$ and $x \in \Omega$ has the following strong form,

$$Au(x) := A(x, \partial_x)u(x) = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left( a(x) \frac{\partial u(x)}{\partial x_i} \right) = f(x), \quad (1)$$

where $u$ is unknown function and $f$ is a given function in $\Omega$. We assume that $a \in C^s(\mathbb{R}^2)$ and $0 < a_{\min} \leq a(x) \leq a_{\max} < \infty, \quad \forall x \in \mathbb{R}^2$.

In what follows $D(\Omega) = C_0^\infty(\Omega)$, $H^s(\Omega) = H_s^2(\Omega)$, $H^s(\partial \Omega) = H_s^2(\partial \Omega)$ are the Bessel potential spaces, where $s \in \mathbb{R}$ is an arbitrary real number (McLean, 2000; Lions and Magenes, 1972). We recall that $H^s$ coincides with the Sobolev-Slobodetski spaces $W_s^2$ for any nonnegative $s$. We denote by $\tilde{H}^s(\Omega)$ the subspace of $H^s(\mathbb{R}^2)$,

$$\tilde{H}^s(\Omega) := \{ g : g \in H^s(\mathbb{R}^2), \; \text{supp}(g) \subset \Omega \}$$

while $H^s(\Omega)$ denotes the space of restriction on $\Omega$ of distributions from $H^s(\mathbb{R}^2)$,

$$H^s(\Omega) = \{ r_{\Omega} g : g \in H^s(\mathbb{R}^2) \}$$

where $r_{\Omega}$ denotes the restriction operator on $\Omega$. We will also use the notation $g|_{\partial \Omega} := r_{\partial \Omega} g$. We denote by $H^s_{\partial \Omega}$ the following subspace of $H^s(\mathbb{R}^2)$ (and $\tilde{H}^s(\Omega)$),

$$H^s_{\partial \Omega} := \{ g : g \in H^s(\mathbb{R}^2), \; \text{supp}(g) \subset \partial \Omega \}. \quad (2)$$

From the trace theorem (Lions and Magenes, 1972; Dautray and Lions, 1990; McLean, 2000) for $u \in H^1(\Omega)$, it follows that $\gamma^+ u \in H^2_\partial(\partial \Omega)$, where $\gamma^+ = \gamma^+_{\partial \Omega}$ is the trace operator on $\partial \Omega$ from $\Omega$. Let also $\gamma^{-1} : H^2_\partial(\partial \Omega) \to H^1(\Omega)$ denote a (non-unique) continuous right inverse to the trace operator $\gamma^+$, i.e., $\gamma^+ \gamma^{-1} w = w$ for $w \in H^2_\partial(\partial \Omega)$.
where the conormal derivative operator is generally, not i.e., $P$.

Let us consider also the operator $T_a^+$ the corresponding canonical (strong) conormal derivative operator on $\partial\Omega$ in the sense of traces,

$$T_a^+u := \sum_{i=1}^{2} a(x)n_i(x)\gamma^+\frac{\partial u(x)}{\partial x_i}$$

(3)

where $n(x)$ is the outward to $\Omega$ unit normal vector at the point $x \in \partial\Omega$. However, the classical conormal derivative operator is generally, not well defined if $u \in H^1(\Omega)$, see (Mikhailov, 2015, Appendix A). For $u \in H^1(\Omega)$, the PDE $Au$ in (1) is understood in the sense of distributions,

$$\langle Au, v \rangle_{\Omega} := -E_a(u,v), \quad \forall v \in D(\Omega),$$

(4)

where

$$E_a(u,v) := \int_{\Omega} a(x)\nabla u(x) \cdot \nabla v(x)\,dx$$

is a symmetric bilinear form and the duality brackets $(g, \cdot)_\Omega$ denote the value of a linear functional (distribution) $g$, extending the usual $L_2$ inner product. Since the set $D(\Omega)$ is dense in $\tilde{H}^1(\Omega)$, the above formula defines a continuous operator $A : H^1(\Omega) \to \tilde{H}^{-1}(\Omega) = [\tilde{H}^1(\Omega)]^*$,

$$\langle Au, v \rangle_{\Omega} := -E_a(u,v), \quad \forall u \in H^1(\Omega), \quad \forall v \in \tilde{H}^1(\Omega).$$

(5)

Let us consider also the operator

$$\tilde{A} : H^1(\Omega) \to \tilde{H}^{-1}(\Omega) = [H^1(\Omega)]^*$$

such that

$$\langle \tilde{A}u, v \rangle_{\Omega} := -E_a(u,v)$$

$$= -\int_{\Omega} a(x)\nabla u(x) \cdot \nabla v(x)\,dx$$

$$= -\int_{\Omega} E[a\nabla u](x) \cdot \nabla V(x)\,dx$$

$$= \langle \nabla \cdot E[a\nabla u], V \rangle_{\Omega},$$

(6)

for any linear operator $A$ such that $\tilde{A}u \in \tilde{H}^{-1}(\Omega)$ and is the extension of the functional $Au \in H^{-1}(\Omega)$, which domain is thus extended from $\tilde{H}^{-1}(\Omega)$ to the domain $H^1(\Omega)$ for $\tilde{A}u$.

Inspired by the first Green identity for smooth functions, we can define the generalized conormal derivative as in (McLean, 2000, Lemma 4.3), (Mikhailov, 2011, Definition 3.1) and (Kohr et al., 2015, Lemma 2.2).

**Definition 1.** Let $u \in H^1(\Omega)$ and $Au = r_\Omega \tilde{f}$ in $\Omega$ for some $\tilde{f} \in \tilde{H}^{-1}(\Omega)$. Then the generalized co-normal derivative $T_a^+(\tilde{f}, u) \in H^{-\frac{1}{2}}(\partial\Omega)$ is defined as

$$\langle T_a^+(\tilde{f}, u), w \rangle_{\partial\Omega} := \langle \tilde{f}, \gamma^{-1}w \rangle_{\partial\Omega} + E_a(u, \gamma^{-1}w)$$

$$= \langle \tilde{f} - \tilde{A}u, \gamma^{-1}w \rangle_{\partial\Omega}, \quad \forall w \in H^{\frac{1}{2}}(\partial\Omega),$$

(8)

that is

$$T_a^+(\tilde{f}, u) := (\gamma^{-1}u)(\tilde{f} - \tilde{A}u).$$

Due to (McLean, 2000, Lemma 4.3) and (Mikhailov, 2011, Theorem 3.2), we have the estimate

$$\|T_a^+(\tilde{f}, u)\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq \|u\|_{H^1(\Omega)}^{\alpha_1} + \|\tilde{f}\|_{\tilde{H}^{-\frac{1}{2}}(\Omega)}^{\alpha_2}$$

(9)

and for $u \in H^1(\Omega)$ such that $Au = r_\Omega \tilde{f}$ in $\Omega$ for some $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ the first Green identity holds in the following form:

$$\langle T_a^+(\tilde{f}, u), \gamma^{-1}v \rangle_{\partial\Omega} := \langle \tilde{f}, v \rangle_{\partial\Omega} + E_a(u, v)$$

$$= \langle \tilde{f} - \tilde{A}u, v \rangle_{\partial\Omega}, \quad \forall v \in H^1(\Omega).$$

(10)

As follows from Definition 1, the generalised conormal derivative is nonlinear with respect to $u$ for a fixed $\tilde{f}$, but linear with respect to the couple $(f, u)$, i.e.,

$$\alpha_1 T_a^+(\tilde{f}, u_1) + \alpha_2 T_a^+(\tilde{f}, u_2)$$

$$= T_a^+(\alpha_1 \tilde{f}_1, \alpha_1 u_1) + T_a^+(\alpha_2 \tilde{f}_2, \alpha_2 u_2)$$

$$= T_a^+(\alpha_1 \tilde{f}_1 + \alpha_2 \tilde{f}_2, \alpha_1 u_1 + \alpha_2 u_2)$$

(11)

for any real numbers $\alpha_1, \alpha_2$.


**Definition 2.** Let $s \in \mathbb{R}$ and $A_s : H^s(\Omega) \to D^*(\Omega)$ be a linear operator. For $t \geq -\frac{1}{2}$ we introduce the space

$$H^{s,t}(\Omega; A_s) := \{ g \in H^s(\Omega) : \exists \text{ there exists }$$

$$\tilde{f}_g \in \tilde{H}^t(\Omega) \text{ such that } A_s g|_\Omega = \tilde{f}_g|_\Omega \}$$

endowed with the norm

$$\| g \|_{H^{s,t}(\Omega; A_s)} := \left( \| g \|_{H^s(\Omega)}^2 + \| \tilde{f}_g \|_{\tilde{H}^t(\Omega)}^2 \right)^{\frac{1}{2}}.$$
and the inner product

\[(g, h)_{H^{s,t}(-\Omega; A\ast)} = (g, h)_{H^{s,t}(\Omega)} + (\tilde{f}_g, \tilde{f}_h)_{\tilde{H}^{s,t}(\Omega)}\]

The distribution \(\tilde{f}_g \in \tilde{H}^{s,t}(\Omega), t \geq -\frac{1}{2}\), in the above definition is an extension of the distribution \(A_u g|_\Omega \in H^s(\Omega)\), and the extension is unique (if it does exist) since any distribution from the space \(H^s(\mathbb{R}^2)\) with support in \(\partial \Omega\) is identically zero if \(t \geq -\frac{1}{2}\), see (McLean, 2000, Lemma 3.39) and (Mikhailov, 2011, Theorem 2.10). We denote this extension as an operator \(A_u\), i.e., \(A_u g = \tilde{f}_g\). The uniqueness implies that the norm \(\|g\|_{H^{s,t}(-\Omega; A\ast)}\) is well defined.

We will mostly use the operators \(A, B\) or \(\Delta\) as \(A_u\) in the above definition. Note that since \(Au - a\Delta u = \nabla a \cdot \nabla u \in L^2(\Omega)\), for \(u \in H^1(\Omega)\), we have \(H^{1,0}(\Omega; A) = H^{1,0}(\Omega; \Delta)\).

**Definition 3.** For \(u \in H^{1-\frac{1}{2}}(\Omega; A)\), we define the canonical conormal derivative \(T^+_a u \in H^{-1}(\partial \Omega)\) as

\[
\langle T^+_a u, w \rangle_{\partial \Omega} := \langle \tilde{A} u, \gamma^{-1} w \rangle_{\Omega} + \mathcal{E}_a(u, \gamma^{-1} w) = \langle \tilde{A} u - A^\ast u, \gamma^{-1} w \rangle_{\Omega}, \quad \forall w \in H^\frac{1}{2}(\partial \Omega), \quad i.e.,
\]

\[
T^+_a u := (\gamma^{-1})^\ast (\tilde{A} u - A^\ast u). \quad (12)
\]

The canonical conormal derivative \(T^+_a u\) is independent of (non-unique) choice of the operator \(\gamma^{-1}\), the operator \(T^+_a : H^{1-\frac{1}{2}}(\Omega; A) \to H^{-1}(\partial \Omega)\) is continuous, and the first Green identity holds in the following form,

\[
\langle T^+_a u, \gamma^+ v \rangle_{\partial \Omega} := \langle \tilde{A} u, v \rangle_{\Omega} + \mathcal{E}_a(u, v), \quad \forall v \in H^1(\Omega). \quad (13)
\]

The operator \(T^+_a : H^{1,0}(\Omega; A) \to H^{-\frac{1}{2}}(\partial \Omega)\) in Definition 3 is continuous for \(t \geq -\frac{1}{2}\). The canonical co-normal derivative is defined by the function \(u\) and the operator \(A\) and does not depend separately on the right-hand side \(\tilde{f}\) (i.e. its behavior on the boundary), unlike the generalised co-normal derivative defined in (8), and the operator \(T^+_a\) is linear. Note that the canonical co-normal derivative coincides with classical conormal derivative \(T^+_a u = a \frac{\partial u}{\partial n}\) if the latter does exist in the trace sense, see (Mikhailov, 2011, Corollary 3.14 and Theorem 3.16).

Let \(u \in H^{1-\frac{1}{2}}(\Omega; A)\). Then Definitions 1 and 3 imply that the generalised co-normal derivative for arbitrary extension \(\tilde{f} \in \tilde{H}^{-1}(\Omega)\) of the distribution \(Au\) can be expressed as

\[
\langle T^+_a (\tilde{f}, u), w \rangle_{\partial \Omega} := \langle T^+_a u, w \rangle_{\partial \Omega} + \langle \tilde{f} - A^\ast u, \gamma^{-1} w \rangle_{\Omega}, \quad \forall w \in H^\frac{1}{2}(\partial \Omega). \quad (14)
\]

Let us consider the auxiliary linear elliptic partial differential operator \(B\) defined by

\[
Bu(x) := (x, \partial_x)u(x) = \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left( b(x) \frac{\partial u(x)}{\partial x_i} \right), \quad (15)
\]

where \(b \in C^\infty(\Omega)\) and \(b(x) > 0\) for \(x \in \Omega\). Since for \(u \in H^{1,0}(\Omega, \Delta)\), \(Au - Bu = (a - b)\Delta u + \nabla (a - b) \cdot \nabla u \in L^2(\Omega)\), we have \(H^{1,0}(\Omega; A) = H^{1,0}(\Omega; B)\). Let \(u \in H^1(\Omega)\) and \(v \in H^{1,0}(\Omega; B)\). Then we write the first Green identity for operator \(B\) in the form

\[
\mathcal{E}_b(u, v) + \int_\Omega u(x) Bv(x) dx = \langle T^+_b v, \gamma^+ u \rangle_{\partial \Omega}, \quad (16)
\]

where

\[
\mathcal{E}_b(u, v) = \int_\Omega b(x) \nabla u(x) \cdot \nabla v(x) dx.
\]

If, in addition, \(Au = \tilde{f}\) in \(\Omega\), where \(\tilde{f} \in \tilde{H}^{-1}(\Omega)\), then according to the definition of \(T^+_a (\tilde{f}, u)\) in (8), the two-operator second Green identity can be written as

\[
\langle \tilde{f}, v \rangle_{\Omega} - \int_\Omega u(x) Bv(x) dx
\]

\[
+ \int_\Omega [a(x) - b(x)] \nabla u(x) \cdot \nabla v(x) dx
\]

\[
= \langle T^+_a (\tilde{f}, u), \gamma^+ v \rangle_{\partial \Omega} - \langle T^+_b v, \gamma^+ u \rangle_{\partial \Omega}. \quad (17)
\]

Moreover, for \(u, v \in H^{1,0}(\Omega; A) = H^{1,0}(\Omega; B)\) (17) becomes

\[
\int_\Omega [v(x) Au(x) - u(x) Bv(x)] dx
\]

\[
+ \int_\Omega [a(x) - b(x)] \nabla u(x) \cdot \nabla v(x) dx
\]

\[
= \langle T^+_a u, \gamma^+ v \rangle_{\partial \Omega} - \langle T^+_b v, \gamma^+ u \rangle_{\partial \Omega}.
\]

**Parametrix, remainder and potential type operators.**

**Definition 4.** We will say, a function \(R_b(x, y)\) of two variables \(x, y \in \Omega\) is a parametrix (Levi function) for the operator \(B(x, \partial_x)\) in \(\mathbb{R}^2\) (Miranda, 1970; Hellwig, 1977; Pomp, 1998a,b; Mikhailov, 2002) if

\[
B(x, \partial_x) R_b(x, y) = \delta(x - y) + R_b(x, y), \quad (18)
\]

where \(\delta\) is the Dirac-delta distribution, while \(R_b(x, y)\) is a remainder possessing at most a weak singularity at \(x = y\).
For some positive constant \( r_0 \) and \( x, y \in \mathbb{R}^2 \), the parametrix and hence the corresponding remainder in 2D can be chosen as in Mikhailov (2002),

\[
P_b(x, y) = \frac{1}{2\pi b(y)} \ln \left( \frac{|x - y|}{r_0} \right), \tag{19}
\]

\[
R_b(x, y) = \frac{2}{3\pi b(y)} \frac{x - y}{|x - y|^2} \frac{\partial b(x)}{\partial x}, \tag{20}
\]

The parametrix \( P_b(x, y) \) in (19) is the fundamental solution to the operator \( B(y, \partial_x) := b(y) \Delta_x \) with “frozen” coefficient \( b(x) = b(y) \), and

\[
B(y, \partial_x) P_b(x, y) = \delta(x - y). \tag{21}
\]

Let \( b \in C^\infty(\mathbb{R}^2) \) and \( b(x) > 0 \) a.e. in \( \mathbb{R}^2 \). For some scalar function \( g \) the parametrix-based Newtonian and the remainder potential operators, corresponding to the parametrix (19) and the remainder (20) are given by

\[
P_b g(y) := \int_{\mathbb{R}^2} P_b(x, y)g(x)dx, \quad y \in \mathbb{R}^2, \tag{22}
\]

\[
P_b g(y) := \int_{\Omega} P_b(x, y)g(x)dx, \quad y \in \Omega, \tag{23}
\]

\[
R_b g(y) := \int_{\Omega} R_b(x, y)g(x)dx, \quad y \in \Omega. \tag{24}
\]

For \( g \in H^s(\mathbb{R}^2), \ s \in \mathbb{R} \), (22) is understood as \( P_b g = \frac{1}{b} P \Delta g \), where the Newtonian potential operator \( P_\Delta \) for Laplacian \( \Delta \) is well defined in terms of the Fourier transform (i.e., as pseudodifferential operator), on any space \( H^s(\mathbb{R}^2) \). For \( g \in H^s(\Omega) \), and any \( s \in \mathbb{R} \), definitions in (23) and (24) can be understood as

\[
P_b g = \frac{1}{b} r^2 \Omega \Delta g, \quad P_b g = \frac{a}{b} r^2 \Omega P_\Delta g, \tag{25}
\]

\[
R_b g = -\frac{1}{b} r^2 \Omega \nabla \cdot \Delta g (\nabla g),
\]

while for \( g \in H^s(\Omega), -\frac{1}{2} < s < \frac{1}{2} \), as (25) with \( g \) replaced by \( \hat{E} g \), where \( \hat{E} : H^s(\Omega) \to \hat{H}^s(\Omega), -\frac{1}{2} < s < \frac{1}{2} \), is the unique extension operator related with the operator \( \hat{E} \) of extension by zero, cf. (Mikhailov, 2011, Theorem 16).

For \( y \notin \partial \Omega \), the single layer and the double layer surface potential operators, corresponding to the parametrix (19) are defined as

\[
V_b g(y) := -\int_{\partial \Omega} P_b(x, y)g(x)ds_x, \tag{26}
\]

\[
W_b g(y) := -\int_{\partial \Omega} [T_b(x, n(x), \partial_x) P_b(x, y)]g(x)ds_x, \tag{27}
\]

where \( g \) is some scalar density function. The integrals are understood in the distributional sense if \( g \) is not integrable, while \( V_\Delta \) and \( W_\Delta \) are the single layer and double layer potentials corresponding to the Laplacian \( \Delta \). The corresponding boundary integral (pseudodifferential) operators of direct surface values to the single and the double layer potentials, \( V_b \) and \( W_b \) when \( y \in \partial \Omega \), are

\[
V_b g(y) := -\int_{\partial \Omega} P_b(x, y)g(x)ds_x, \tag{28}
\]

\[
W_b g(y) := -\int_{\partial \Omega} T_b(x, n(x), \partial_x) P_b(x, y)g(x)ds_x, \tag{29}
\]

where \( V_\Delta \) and \( W_\Delta \) are respectively the direct values of the single and double layer potentials corresponding to the Laplacian \( \Delta \).

We can also calculate at \( y \in \partial \Omega \) the co-normal derivatives, associated with the operator \( A \), of the single layer potential and of the double layer potential:

\[
T_\Delta^+ V_b g(y) = \frac{a(y)}{b(y)} T_b^+ V_b g(y), \tag{30}
\]

\[
L_{\Delta b}^+ g(y) := T_\Delta^+ W_b g(y) = \frac{a(y)}{b(y)} T_b^+ W_b g(y). \tag{31}
\]

The direct value operators associated with (30) are:

\[
W_{ab} g(y) := -\int_{\partial \Omega} [T_a(y, n(y), \partial_y) P_b(x, y)]g(x)ds_x, \tag{32}
\]

\[
W_{b} g(y) := -\int_{\partial \Omega} [T_b(y, n(y), \partial_y) P_b(x, y)]g(x)ds_x, \tag{33}
\]

From equations (22)-(33) we deduce representations of the parametrix-based surface potential boundary operators in terms of their counterparts for \( b = 1 \), that is, associated with the fundamental solution \( P_\Delta = \frac{1}{2\pi} \ln \left( \frac{|x-y|}{r_0} \right) \) of the Laplace operator \( \Delta \).

\[
P_a g = \frac{1}{a} P_\Delta g, \quad P_b g = \frac{1}{b} P_\Delta g, \tag{34}
\]

\[
P_a g = \frac{1}{a} P_\Delta g, \quad P_b g = \frac{1}{b} P_\Delta g.
\]

\[
\frac{a}{b} V_a g = V_b g = \frac{1}{b} V_\Delta g, \tag{35}
\]

\[
\frac{a}{b} W_a \left( \frac{b g}{a} \right) = W_b g = \frac{1}{b} W_\Delta (bg),
\]

\[
\frac{a}{b} V_a g = V_b g = \frac{1}{b} V_\Delta g, \tag{36}
\]

\[
\frac{a}{b} W_a \left( \frac{b g}{a} \right) = W_b g = \frac{1}{b} W_\Delta (bg).
\]
\[ W'_{abg} = \frac{a}{b} W'_{bg} \]
\[ = \frac{a}{b} \left\{ W'_\Delta g + \left[ b \frac{\partial}{\partial n} \left( \frac{1}{b} \right) \right] V_\Delta g \right\}, \tag{37} \]
\[
\mathcal{L}^+_a g = \frac{a}{b} \mathcal{L}^+_b g = \frac{a}{b} \left\{ \mathcal{L}_\Delta (bg) + \left[ b \frac{\partial}{\partial n} \left( \frac{1}{b} \right) \right] \mathcal{L}^+_\Delta (bg) \right\}, \tag{38} \]
\[
\mathcal{L}^+_b g := T^+_\Delta W_\Delta (bg) = T^+_\Delta W_\Delta (bg) = \mathcal{L}_\Delta (bg) \quad \text{on} \quad \partial \Omega. \tag{39} \]

It is taken into account that and its derivatives are continuous in \( \mathbb{R}^2 \) and
\[
\mathcal{L}_\Delta (bg) := \mathcal{L}^+_\Delta (bg) = \mathcal{L}^-_\Delta (bg)
\]
by the Liapunov-Tauber theorem. Hence,
\[
\Delta (bV_\Delta g) = 0, \quad \Delta (bW_\Delta g) = 0 \quad \text{in} \quad \Omega,
\]
\[
\forall g \in H^s(\partial \Omega), \quad \forall s \in \mathbb{R}, \tag{40} \]
\[
\Delta (bP_\Delta g) = g \quad \text{in} \quad \Omega,
\]
\[
\forall g \in \tilde{H}^s(\Omega), \quad \forall s \in \mathbb{R}. \tag{41} \]

The mapping properties of the operators (22)-(33) follow from relations (34)-(39) and are described in detail in (Ayele Tsegaye and Mikhailov, 2011, Appendix A). Particularly, we have the following jump relations:

**Theorem 1.** For \( g_1 \in H^{-\frac{1}{2}}(\partial \Omega) \), and \( g_2 \in H^{\frac{1}{2}}(\partial \Omega) \). Then there hold the following relations on \( \partial \Omega \),
\[
\gamma^\pm V_\Delta g_1 = V_\Delta g_1, \tag{42} \]
\[
\gamma^\pm W_\Delta g_2 = \pm \left( \frac{a}{b} \right) g_2 + W_\Delta g_2, \tag{43} \]
\[
T^+_\Delta V_\Delta g_1 = \pm \left( \frac{a}{b} \right) g_1 + W'_{abg_1}. \tag{44} \]

**The two-operator third Green identity and integral relations**

Applying some limiting procedures (Miranda, 1970) and (Hellwig, 1977, Section 3.8), we obtain the parametricity based third Green identities.

**Theorem 2.** (i) If \( u \in H^1(\Omega) \), then the following third Green identity holds,
\[
u + Z_b u + \mathcal{R}_b u + W_b \gamma^+ u = \mathcal{P}_b \mathcal{A} u \quad \text{in} \quad \Omega, \tag{45} \]
where the operator \( \mathcal{A} \) is defined in (7), and for \( u \in C^{1}(\Omega) \),
\[
\mathcal{P}_b \mathcal{A} u(y) := \langle \mathcal{A} u, P_b(u, y) \rangle_{\Omega} = -E_a(u, P_b(u, y)) \]
\[
= - \int_{\Omega} [a(x) - b(x)] \nabla u(x) \cdot \nabla P_b(x, y) dx \tag{46} \]
and
\[
Z_b u = - \int_{\Omega} [a(x) - b(x)] \nabla_x P_b(x, y) \cdot \nabla u(x) dx = \frac{1}{b(y)} \sum_{j=1}^{2} \partial_j \mathcal{P}_\Delta [(a - b) \partial_j u] \quad \text{in} \quad \Omega. \tag{47} \]

(ii) If \( Au = r_q \tilde{f} \quad \text{in} \quad \Omega, \) where \( \tilde{f} \in \tilde{H}^{-1}(\Omega) \),
then recalling the definition of \( T^+_a (\tilde{f}, u) \), in (8), we arrive at the generalised two-operator third Green identity in the following form,
\[
\langle T^+_a (\tilde{f}, u), P_b(x, y) \rangle_{\Omega} = -V_b T^+_a (\tilde{f}, u) + W_b \gamma^+ u = \mathcal{P}_b \tilde{f} \quad \text{in} \quad \Omega, \tag{48} \]
where it was taken into account that
\[
\langle T^+_a (\tilde{f}, u), P_b(x, y) \rangle_{\partial \Omega} = -V_b T^+_a (\tilde{f}, u) \quad \text{and} \quad \mathcal{P}_b \tilde{f}. \tag{49} \]

**Proof.** (i) Let first \( u \in D(\Omega) \). Let \( y \in \Omega \), \( B_\epsilon(y) \subset \Omega \) be a ball centred at \( y \) with sufficiently small radius \( \epsilon \), and \( \Omega_\epsilon := \Omega \backslash \overline{B_\epsilon(y)} \). For the fixed \( y \), evidently, \( P_b(., y) \in D(\tilde{\Omega}) \subset H^{1,0}(A; \partial \Omega) \) and has the coinciding classical and canonical co-normal derivatives on \( \partial \Omega_\epsilon \). Then from (19) and the first Green identity (16) employed for \( \Omega_\epsilon \) with \( v = P_b(., y) \) we obtain
\[
- \int_{\partial B_\epsilon(y)} T^+_x P_b(x, y) \gamma^+ u(x) ds_x
\]
\[
- \int_{\tilde{\Omega}} T_x P_b(x, y) \gamma^+ u(x) dx
\]
\[
+ \int_{\Omega_\epsilon} u(x) R_b(x, y) dx
\]
\[
= - \int_{\Omega} b(x) \nabla u(x) \cdot \nabla P_b(x, y) dx,
\]
which we rewrite as
\[
- \int_{\partial B_\epsilon(y)} T^+_x P_b(x, y) \gamma^+ u(x) ds_x
\]
\[
- \int_{\tilde{\Omega}} T_x P_b(x, y) \gamma^+ u(x) dx
\]
\[
- \int_{\Omega_\epsilon} [a(x) - b(x)] \nabla u(x) \cdot \nabla P_b(x, y) dx \tag{50} \]
\[
+ \int_{\Omega_\epsilon} u(x) R_b(x, y) dx
\]
\[
= - \int_{\Omega} a(x) \nabla u(x) \cdot \nabla P_b(x, y) dx.
\]

Taking the limit as \( \epsilon \to 0 \), (50) reduces to the third Green identity (45)–(46) for any \( u \in D(\Omega) \). Taking into account the density of \( D(\tilde{\Omega}) \) in \( H^1(\Omega) \), and the mapping properties of
the integral potentials, see (Ayele Tsegaye and MikhaiIov, 2011, Appendix A), we obtain that (45)–(46) hold true also for any \( u \in H^1(\Omega) \).

(ii) Let \( \{f_k\} \in D(\Omega) \) be a sequence of converging to \( f \) in \( H^{-1}(\Omega) \) as \( k \to \infty \). Then, according to (Mikhailov, 2015, Theorem B.1) there exists a sequence \( \{u_k\} \in D(\Omega) \) converging to \( u \) in \( H^1(\Omega) \) such that \( Au_k = \gamma_0 f_k \) and \( \nabla y T^+_a(u_k) = T^+_a(f_k, u_k) \) converge to \( T^+_a(f, u) \) in \( H^{-\frac{1}{2}}(\partial \Omega) \). For such \( u_k \) by (46) and (8), we have

\[
P_b \hat{A} u_k(y) = \frac{1}{b(y)} \nabla y \int_{\Omega} a(x) P_\Delta(x, y) \nabla u_k(x) dx
- \frac{1}{b(y)} \lim_{\epsilon \to 0} \int_{\Omega} a(x) \nabla u_k(x) P_{\Delta B}(x, y) dx
- \lim_{\epsilon \to 0} \int_{\Omega} P_b(x, y) \nabla y T^+_a u_k(x) ds(x)
+ \lim_{\epsilon \to 0} \int_{\epsilon B(y)} P_b(x, y) T^+_a u_k(x) ds(x)
= P_b \hat{f} + V_b T^+_a u_k(y). \quad (51)
\]

Taking the limits as \( k \to \infty \) in (51), we obtain

\[
P_b \hat{A} u(y) = P_b \hat{f} + V_b T^+_a(u, f, u),
\]

and substitution to (45) gives (48).

Below we shall state and prove Corollary 3 from Ayele Tsegaye and Bekele Solomon (2023) for completeness.

**Corollary 3.** Using the Gauss divergence theorem, we can rewrite \( Z_b u(y) \) in the form that does not involve derivatives of \( u \),

\[
Z_b u(y) := \left[ \frac{a(y)}{b(y)} - 1 \right] u(y) + \hat{Z}_b u(y), \quad (52)
\]

\[
\hat{Z}_b u(y) := \frac{a(y)}{b(y)} W_a \gamma^+ u(y) - W_b \gamma^+ u(y)
+ \frac{a(y)}{b(y)} R_a u(y) - R_b u(y), \quad (53)
\]

which allows to call \( Z_b \) integral operator in spite of its integro-differential representation (47).

**Proof.** As in the proof of Theorem 2 item (i), let first \( u \in D(\Omega) \). Let \( y \in \Omega, B(y) \subset \Omega \) be a ball centred at \( y \) with sufficiently small radius \( \epsilon \), and \( \Omega_\epsilon := \Omega \setminus B(y) \). For the fixed \( y \), evidently, \( P_b(\cdot, y) \in D(\Omega_\epsilon) \subset H^{1,0}(A; \Omega_\epsilon) \) and has the coinciding classical and canonical conormal derivatives on \( \partial \Omega_\epsilon = \partial \Omega \cap \partial B(y) \). Further, let us denote

\[
Z_b^\epsilon u(y) = - \int_{\Omega_\epsilon} [a(x) - b(x)] \nabla x P_b(x, y) \cdot \nabla u(x) dx,
\]

which can be rewritten as

\[
Z_b^\epsilon u(y) = \int_{\Omega_\epsilon} \left[ \nabla (a(x) - b(x)) \cdot \nabla x P_b(x, y) \right] u(x) dx
- \int_{\Omega_\epsilon} \nabla \left[ (a(x) - b(x)) u(x) \right] \cdot \nabla x P_b(x, y) dx.
\]

Observe that

\[
I_1(y, \epsilon) = \int_{\Omega_\epsilon} \left[ \nabla (a(x) - b(x)) \cdot \nabla x P_b(x, y) \right] u(x) dx
= \int_{\Omega_\epsilon} \left[ \nabla b(x) \cdot \nabla x P_b(x, y) \right] u(x) dx
- \frac{a(y)}{b(y)} \int_{\Omega_\epsilon} \nabla x P_a(x, y) u(x) dx
- \frac{a(y)}{b(y)} \int_{\partial B(y)} a(x) \nabla x P_a(x, y) \cdot n(x) u(x) ds(x)
\]

\[
= \int_{\Omega_\epsilon} \left[ \nabla b(x) \cdot \nabla x P_b(x, y) \right] u(x) dx
- \frac{a(y)}{b(y)} \int_{\Omega_\epsilon} \nabla x P_a(x, y) u(x) dx
- \frac{a(y)}{b(y)} \int_{\partial B(y)} a(x) \nabla x P_a(x, y) \cdot n(x) u(x) ds(x)
\]

and

\[
I_2(y, \epsilon) = - \int_{\Omega_\epsilon} \nabla \left[ (a(x) - b(x)) u(x) \right] \nabla x P_b(x, y) dx
= \int_{\Omega_\epsilon} \left[ a(x) - b(x) \right] u(x) \nabla x P_b(x, y) dx
- \frac{a(y)}{b(y)} \int_{\Omega_\epsilon} \nabla x P_a(x, y) u(x) dx
- \frac{a(y)}{b(y)} \int_{\partial B(y)} a(x) \nabla x P_a(x, y) \cdot n(x) u(x) ds(x)
\]

\[
= \int_{\Omega_\epsilon} \left[ a(x) - b(x) \right] u(x) \nabla x P_b(x, y) dx
- \frac{a(y)}{b(y)} \int_{\Omega_\epsilon} \nabla x P_a(x, y) u(x) dx
- \frac{a(y)}{b(y)} \int_{\partial B(y)} a(x) \nabla x P_a(x, y) \cdot n(x) u(x) ds(x)
\]

Taking the limit as \( \epsilon \to 0 \) we obtain

\[
Z_b u(y) = \lim_{\epsilon \to 0} Z_b^\epsilon u(y) = \lim_{\epsilon \to 0} [I_1(y, \epsilon) + I_2(y, \epsilon)]
= - \frac{a(y)}{b(y)} \nabla R_a u(y) - \nabla R_b u(y) + a(y) W_a \gamma^+ u(y)
- W_b \gamma^+ u(y),
\]

which is as in (52) and (53).

Note that the operator \( Z_b \) does not vanish unless operators \( A \) and \( B \) are equal. For some
functions \( \tilde{f}, \Psi, \Phi \) let us consider a more general "indirect" integral relation, associated with \((48)\):

\[ u + \mathcal{Z}_0 u + \mathcal{R}_b u - V_b \Psi + W_b \Phi = \mathcal{P}_b \tilde{f} \text{ in } \Omega. \tag{54} \]

**Lemma 4.** Let \( u \in H^1(\Omega), \Psi \in H^{-\frac{1}{2}}(\partial \Omega), \Phi \in H^\ast(\partial \Omega) \) and \( \tilde{f} \in \tilde{H}^{-1}(\Omega) \) satisfy \((54)\). Then

\[ Au = r_\Omega \tilde{f} \text{ in } \Omega, \tag{55} \]

\[ r_\Omega V_b(\Psi) - T_a^+ (\tilde{f}, u) - r_\Omega W_b(\Phi) - \gamma^+ u = 0 \text{ in } \Omega, \tag{56} \]

\[ \gamma^+ u + \gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u - \frac{1}{2} \Phi + \mathcal{W}_b \Phi = \gamma^+ \mathcal{P}_b \tilde{f} \text{ on } \partial \Omega. \tag{57} \]

\[ T_a^+ (\tilde{f}, u) + T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u - \frac{a}{2b} \Psi - \mathcal{W}'_{ab} \Psi + \mathcal{L}^+_b \Phi = T_a^+ (\tilde{f} + \mathcal{E} \mathcal{R}_a^b \tilde{f}, \mathcal{P}_b \tilde{f}) \text{ on } \partial \Omega, \tag{58} \]

where

\[ \mathcal{R}_a^b \tilde{f}(y) := -\sum_{j=1}^{2} \hat{c}_{ij} (\hat{c}_{j,0}) \mathcal{P}_b \tilde{f}. \tag{59} \]

**Proof.** Subtracting \((54)\) from identity \((45)\), we obtain

\[ V_b \Psi(y) - W_b(\Phi - \gamma^+ u)(y) = \mathcal{P}_b[Au(y) - \tilde{f}(y)], \quad y \in \Omega. \tag{60} \]

Multiplying equality \((60)\) by \(b(y)\), applying the Laplace operator \(\Delta\) and taking into account equations \((40)\) and \((41)\), we get \(r_\Omega \tilde{f} = r_\Omega (\tilde{A}u) = Au \text{ in } \Omega. \) This means \( \tilde{f} \) is an extension of the distribution \(Au \in H^{-1}(\Omega)\) to \(\tilde{H}^{-1}(\Omega)\), and \(u\) satisfies \((5)\). Then \((8)\) implies

\[ \mathcal{P}_b[Au - \tilde{f}](y) = \langle Au - \tilde{f}, \mathcal{P}_b(., y) \rangle_{\Omega} = \langle T_a^+ (\tilde{f}, u), \mathcal{P}_b(., y) \rangle_{\partial \Omega} \tag{61} \]

\[ = V_b T_a^+ (\tilde{f}, u), \quad y \in \Omega. \]

Substituting \((61)\) into \((60)\) leads to \((56)\). Equation \((57)\) follows from \((54)\) and jump relations in \((42)\) and \((43)\). To prove \((58)\), let us first remark that for \(u \in H^1(\Omega)\), we have \(H^1(\Omega; A) = H^1(\Omega; \Delta) = H^1(\Omega; B)\) and

\[ B \mathcal{P}_b \tilde{f} = \tilde{f} + \mathcal{R}_b^b \tilde{f} \text{ in } \Omega, \tag{62} \]

due to \((55)\), which implies \(B(\mathcal{P}_b \tilde{f} - u) = \mathcal{R}_b^b \tilde{f} \) in \(\Omega\), with \(\mathcal{R}_b^b \tilde{f}\) given by \((59)\), and thus \(\mathcal{R}_b^b \tilde{f} \in L_2(\Omega)\). Then \(B(\mathcal{P}_b \tilde{f} - u)\) can be canonically extended (by zero) to

\[ \tilde{B}(\mathcal{P}_b \tilde{f} - u) = \mathcal{E} \mathcal{R}_a^b \tilde{f} \in \tilde{H}^0(\Omega) \subset \tilde{H}^{-1}(\Omega). \]

Thus there exists a canonical co-normal derivative \(T_b^+ (\mathcal{P}_b \tilde{f} - u)\) written as (see, e.g., \(\text{Mikhailov, 2015, Eq. (4.14)}, \) \(\text{Mikhailov, 2018, Eq. (4.23)})\)

\[ T_b^+ (\mathcal{P}_b \tilde{f} - u) = T_b^+ (\tilde{f} + \mathcal{E} \mathcal{R}_a^b \tilde{f}, \mathcal{P}_b \tilde{f}) - T_b^+ (\tilde{f}, u), \tag{63} \]

and hence

\[ T_a^+ (\mathcal{P}_b \tilde{f} - u) = a \mathcal{P}_b (\tilde{f} - u) = a [T_b^+ (\tilde{f} + \mathcal{E} \mathcal{R}_a^b \tilde{f}, \mathcal{P}_b \tilde{f}) - T_b^+ (\tilde{f}, u)] \tag{64} \]

\[ = T_a^+ (\tilde{f} + \mathcal{E} \mathcal{R}_a^b \tilde{f}, \mathcal{P}_b \tilde{f}) - T_a^+ (\tilde{f}, u). \]

From \((54)\) it follows that

\[ \mathcal{P}_b \tilde{f} - u = \mathcal{Z}_b u + \mathcal{R}_b u - V_b \Psi + W_b \Phi \text{ in } \Omega. \]

Substituting this on the left-hand side of \((63)\) and taking into account \((38)\) and the jump relation \((44)\), we arrive at \((58)\).

**Remark 5.** If \( \tilde{f} \in \tilde{H}^{-\frac{1}{2}}(\Omega) \subset \tilde{H}^{-1}(\Omega) \), then \( \tilde{f} + \mathcal{E} \mathcal{R}_a^b \tilde{f} \in \tilde{H}^{-\frac{1}{2}}(\Omega) \) as well, which implies \( \tilde{f} + \mathcal{E} \mathcal{R}_a^b \tilde{f} = A \mathcal{P}_b \tilde{f} \) and

\[ T_a^+ (\tilde{f} + \mathcal{E} \mathcal{R}_a^b \tilde{f}, \mathcal{P}_b \tilde{f}) = T_a^+ (\tilde{B} \mathcal{P}_b \tilde{f}, \mathcal{P}_b \tilde{f}) \tag{65} \]

Furthermore, if the hypotheses of Lemma 4 are satisfied, then \((55)\) implies \(u \in H^{1-\frac{1}{2}}(\Omega; A)\) and \(T_a^+ (\tilde{f}, u) = T_a^+ (\tilde{A} u, u) = T_a^+ u. \) Henceforth \((58)\), takes the familiar form, cf. \(\text{(Ayele Tsegaye and Mikhailov, 2011, equation (3.29)})\),

\[ T_a^+ u + T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u - \frac{a}{2b} \Psi - \mathcal{W}'_{ab} \Psi + \mathcal{L}^+_b \Phi = T_a^+ \mathcal{P}_b \tilde{f} \text{ on } \partial \Omega. \tag{58} \]

**Remark 6.** Let \( \tilde{f} \in \tilde{H}^{-1}(\Omega) \) and a sequence \(\{\phi_i\} \in \tilde{H}^{-1}(\Omega)\) converge to \( \tilde{f} \) in \(\tilde{H}^{-1}(\Omega)\). By the continuity of operators \(\text{Mikhailov, 2015, C.1 and C.2)}, \) estimate \((9)\) and relation \((65)\) for \(\phi_i\), we obtain

\[ T_a^+ (\tilde{f} + \mathcal{E} \mathcal{R}_a^b \tilde{f}, \mathcal{P}_b \tilde{f}) = \lim_{i \to \infty} T_a^+ (\phi_i + \mathcal{E} \mathcal{R}_a^b \phi_i, \mathcal{P}_b \phi_i) = \lim_{i \to \infty} T_a^+ \mathcal{P}_b \phi_i, \quad \text{in } H^{-\frac{1}{2}}(\partial \Omega), \tag{Mikhailov, 2015, Theorem B.1}. \]

Lemma 4 and the third Green identity \((48)\) imply, the following assertion.
Corollary 7. If $u \in H^1(\Omega)$ and $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ are such that $Au = r_0 \tilde{f}$ in $\Omega$, then

$$\frac{1}{2} \gamma^+ u + \gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u - \mathcal{V}_b T_a^+(\tilde{f}, u) + \mathcal{W}_b \gamma^+ u = \gamma^+ \mathcal{P}_b \tilde{f}, \text{ on } \partial \Omega,$$  

(66)

$$
\left(1 - \frac{a}{2b}\right) T_a^+(\tilde{f}, u) + T_a^+ \mathcal{Z}_a u
+ \mathcal{T}_a^+ \mathcal{R}_a u - \mathcal{W}_a^+ \mathcal{T}_a^+(\tilde{f}, u) + \mathcal{L}_a^+ \gamma^+ u
= T_a^+(\tilde{f} + \mathcal{E} \mathcal{R}_b^\ast \mathcal{P}_b \tilde{f}) \text{ on } \partial \Omega. \tag{67}
$$

Note that if $\mathcal{P}_b$ is not only the parametrix but also the fundamental solution of the operator $B$, then the remainder operator $\mathcal{R}_b$ vanishes in (48) and (66)-(67) (and everywhere in the paper), while the operator $\mathcal{Z}_b$ stays unless $A = B$. The following statement is proved in (Mikhailov, 2015, Lemma 4.6).

**Theorem 8.** Let $\tilde{f} \in \tilde{H}^{-1}(\Omega)$. A function $u \in H^1(\Omega)$ is a solution of PDE $Au = r_0 \tilde{f}$ in $\Omega$ if and only if it is a solution of BDIDE (48).

Proof. If $u \in H^1(\Omega)$ solves PDE $Au = r_0 \tilde{f}$ in $\Omega$, then it satisfies (48). On the other hand, if $u$ solves BDIDE (48), then using Lemma 4 for $\Psi = T_a^+(\tilde{f}, u)$, $\Phi = \gamma^+ u$ completes the proof. □

**Invertibility of single layer potential operator**

The boundary integral operator, $\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial \Omega) \to H^{\frac{1}{2}}(\partial \Omega)$ is a Fredholm operator of index zero, see, e.g., (McLean, 2000, Theorem 7.6). Thus the first relation in (36) leads to the same result for the single layer potential $\mathcal{V}_b$. For the case of 3D, (Ayele Tsegaye and Mikhailov, 2011, Lemma 3.2(i)) asserts that for $\Psi^* \in H^{-\frac{1}{2}}(\partial \Omega)$, if $\mathcal{V}_b \Psi^* = 0$ in $\Omega$, then $\Psi^* = 0$ in $\Omega$. Implying the invertibility of single layer potential operator $\mathcal{V}_b$ mapping from $H^{-\frac{1}{2}}(\partial \Omega)$ to $H^{\frac{1}{2}}(\partial \Omega)$. But this is not the case for 2D. It is well-known, see e.g., (Constanda, 2000, Remark 1.42(ii)) and (Steinbach, 2007, Theorem 6.22) that for some 2D domains the kernel of the operator $\mathcal{V}_\Delta$ is nontrivial, thus due to the first relation in (36), the kernel of operator $\mathcal{V}_b$ is nontrivial as well for the same domains. To ensure the invertibility of the single layer potential operator in 2D, for $s \in \mathbb{R}$, let us define the subspace of $H^s(\partial \Omega)$, cf. e.g., (Steinbach, 2007, p. 147),

$$H^s_{\ast s}(\partial \Omega) := \{g \in H^s(\partial \Omega) : \langle g, 1 \rangle_{\partial \Omega} = 0\}. \tag{68}$$

The following result is proved in (Dufera Tamirat and Mikhailov, 2015, Theorem 4), see also (Ayele Tsegaye and Bekele Solomon, 2019, Theorem 1).

**Theorem 9.** If $\psi \in H^{-\frac{1}{2}}(\partial \Omega)$ satisfies $\mathcal{V}_b \psi = 0$ on $\partial \Omega$, then $\psi = 0$.

Following (McLean, 2000, Theorem 8.15), there exists a unique real-valued distribution $\psi_{eq} \in H^{-\frac{1}{2}}(\partial \Omega)$ called equilibrium density for $\partial \Omega$ such that $\mathcal{V}_\Delta \psi_{eq}$ is constant on $\partial \Omega$, and $(1, \psi_{eq})_{\partial \Omega} = 1$. For $n = 2$ the constant $\mathcal{V}_\Delta \psi_{eq}$ is not always positive and one introduces the logarithmic capacity, $\text{Cap}_{\partial \Omega}$ using the relation

$$\mathcal{V}_\Delta \psi_{eq} = \frac{1}{2\pi} \ln \left(\frac{r_0}{\text{Cap}_{\partial \Omega}}\right),$$

for some positive constant $r_0$ as in equation (19). In particular $\mathcal{V}_\Delta \psi_{eq} = 0$ if and only if $r_0 = \text{Cap}_{\partial \Omega}$.

The following statement is proved in (McLean, 2000, Theorem 8.16).

**Theorem 10.** Let $r_0$ be some positive constant.

(i) The operator $\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial \Omega) \to H^{\frac{1}{2}}(\partial \Omega)$, is $H^{-\frac{1}{2}}(\partial \Omega)$-elliptic, i.e.,

$$\langle \mathcal{V}_\Delta \psi, \psi \rangle_{\partial \Omega} \geq c\|\psi\|_{H^{-\frac{1}{2}}(\partial \Omega)}^2$$

for all $\psi \in H^{-\frac{1}{2}}(\partial \Omega)$, if and only if $r_0 > \text{Cap}_{\partial \Omega}$.

(ii) The operator $\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial \Omega) \to H^{\frac{1}{2}}(\partial \Omega)$, has a bounded inverse if and only if $r_0 \neq \text{Cap}_{\partial \Omega}$.

The following theorem ensures the invertibility of the single layer potential operator $\mathcal{V}_b$ in 2D.

**Theorem 11.** Let $\Omega \subset \mathbb{R}^2$ with $r_0 > \text{diam}(\Omega)$. Then the single layer potential $\mathcal{V}_b : H^{-\frac{1}{2}}(\partial \Omega) \to H^{\frac{1}{2}}(\partial \Omega)$ is invertible.

Proof. Since $\text{Cap}_{\partial \Omega} \leq \text{diam}(\Omega)$, see, (Yan and Sloan, 1988, p.553, properties 1 and 3), then $r_0 > \text{diam}(\Omega)$ implies $r_0 > \text{Cap}_{\partial \Omega}$. For the case $a = b$ the assertion is proved in (Dufera Tamirat and Mikhailov, 2015, Theorem 5). Due to the first relation in (36) and Theorem 10(ii) follows the invertibility of the single layer potential operator $\mathcal{V}_b$ for the case $a \neq b$ as well, see also (Ayele Tsegaye and Bekele Solomon, 2019, Theorem 2). □

As in Ayele Tsegaye and Bekele Solomon (2023), we shall restrict ourselves to Theorem 11 while discussing about the invertibility of single layer potential $\mathcal{V}_b$ in 2D. On the other hand, choosing an appropriate parameter $r_0$, one can
get the zero kernel for $\mathcal{V}_b$ not only on the subspace $H^{-1/2}_c(\partial \Omega)$ but also on the entire space $H^{-1/2}(\partial \Omega)$. The proof to the following result is due to (Ayele Tsegaye and Bekele Solomon, 2019, Lemma 1) and (Ayele Tsegaye, 2021b, Lemma 2).

**Lemma 12.**
(i) Let $r_0 > \text{diam}(\Omega)$. If $\Psi^* \in H^{-\frac{1}{2}}(\partial \Omega)$ and $r_1 V_b \Psi^* = 0$ in $\Omega$, then $\Psi^* = 0$.

(ii) If $\Phi^* \in H^{\frac{1}{2}}(\partial \Omega)$ and $r_1 W_b \Phi^* = 0$ in $\Omega$, then $\Phi^* = 0$.

**Two-operator BDIE systems for Dirichlet problem**

Let $\Omega$ be a domain in $\mathbb{R}^2$ bounded by a smooth curve $\partial \Omega$. We shall derive and investigate the two-operator BDIE systems for the following Dirichlet problem: for $\varphi_0 \in H^{\frac{1}{2}}(\partial \Omega)$ and $f \in H^{-1}(\Omega)$, find a function $u \in H^{1}(\Omega)$ satisfying

$$Au = f \text{ in } \Omega, \quad \gamma^+ u = \varphi_0 \text{ on } \partial \Omega. \quad (69)$$

Here equation (69) is understood in the distributional sense (4) and the Dirichlet boundary condition (70) is understood in the trace sense. The following assertion is well-known and can be proved e.g. using variational settings and the Lax-Milgram lemma.

**Theorem 13.** The Dirichlet problem (69)-(70) is uniquely solvable in $H^{1}(\Omega)$. The solution is $u = (A^D)^{-1}(f, \varphi_0)$, where the inverse operator, $(A^D)^{-1} : H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial \Omega) \to H^{1}(\Omega)$, to the left-hand-side operator, $A^D : H^{1}(\Omega) \to H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial \Omega)$, of the Dirichlet problem (69)-(70), is continuous.

**BDIE system formulation to the Dirichlet problem.** Following similar procedure as in Mikhailov (2015), let us reduce the Dirichlet problem (69)-(70) with $f \in H^{-1}(\Omega)$, for $u \in H^{1}(\Omega)$, to two different systems of segregated two-operator BDIEs.

Let $f \in \hat{H}^{-1}(\Omega)$ be an extension of $f$ in $H^{-1}(\Omega)$ (i.e., $f = r_1 \hat{f}$), which always exists, see, (Mikhailov, 2015, Lemma 2.15 and Theorem 2.16). We represent in (48), (66) and (67) the generalized conormal derivative and the trace of the function $u$ as

$$T^+_a(\hat{f}, u) = \psi, \quad \gamma^+ u = \varphi_0$$

respectively, and will regard the new unknown function $\psi \in H^{-\frac{1}{2}}(\partial \Omega)$ as formally segregated of $u$. Thus we will look for the couple $(u, \psi) \in H^{1}(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega)$.

**BDIE system (D1).** To reduce BVP (69)-(70) to one of BDIE systems we will use equation (48) in $\Omega$ and equation (66) on $\partial \Omega$. Then we arrive at the system of BDIEs (D1),

$$u + Z_b u + R_b u - V_b \psi = F^D_1 \text{ in } \Omega, \quad (71)$$

$$\gamma^+ Z_b u + \gamma^+ R_b u - V_b \psi = F^D_2 \text{ on } \partial \Omega, \quad (72)$$

where $F_0 := P_b \hat{f} - W_b \varphi_0$ and

$$F^{D1} := \left[ \begin{array}{c} F^D_1 \\ F^D_2 \end{array} \right] = \left[ \begin{array}{c} F_0 \\ \gamma^+ F_0 - \varphi_0 \end{array} \right]. \quad (73)$$

For $\varphi_0 \in H^{\frac{1}{2}}(\partial \Omega)$, we have the inclusion $F^{D1} \in F^{D1} \in H^{1}(\Omega) \times H^{\frac{1}{2}}(\partial \Omega)$.

**Remark 14.** Let $\hat{f} \in \hat{H}^{-1}(\Omega)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial \Omega)$, Then $F^{D1} = 0$ if and only if $\hat{f}, \varphi_0) = 0$.

**Proof.** The later equality implies the former. Conversely, let $F^{D1} = 0$, that is, $F_0 = P_b \hat{f} - W_b \varphi_0 = 0$ in $\Omega$ and $\gamma^+ F_0 - \varphi_0 = 0$ on $\partial \Omega$. Then $\varphi_0 = 0$ on $\partial \Omega$ and $P_b \hat{f} = 0$ in $\Omega$. Multiplying the later by $b$, we get $P_b \hat{f} = 0$ in $\Omega$ and applying Laplace operator gives $\hat{f} = 0$ in $\mathbb{R}^2$.

**BDIE system (D2).** To obtain a BDIE system of the second kind, we will use equation (48) in $\Omega$ and equation (67) on $\partial \Omega$. Then we arrive at the system of BDIEs (D2),

$$u + Z_b u + R_b u - V_b \psi = P_b \hat{f} - W_b \varphi_0 \text{ in } \Omega, \quad (74)$$

$$(1 - \frac{a}{2b}) \psi + T^+_a Z_b u + T^+_a R_b u - W'_b \psi = T^+_a(\hat{f} + E \mathcal{R}_b \hat{f}, P_b \hat{f}) - \mathcal{L}^+_a \varphi_0$$

on $\partial \Omega, \quad (75)$$

where

$$F^{D2} := \left[ \begin{array}{c} F^D_1 \\ F^D_2 \end{array} \right] = \left[ \begin{array}{c} P_b \hat{f} - W_b \varphi_0 \\ T^+_a(\hat{f} + E \mathcal{R}_b \hat{f}, P_b \hat{f}) - \mathcal{L}^+_a \varphi_0 \end{array} \right]. \quad (76)$$

Due to the mapping properties of operators involved in (76), we have the inclusion $F^{D2} \in H^{1}(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega)$. In similar way as in Remark 14, we can prove the following remark.
Remark 15. Let $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ and $\varphi_0 \in H^\frac{1}{2}(\partial \Omega)$. Then $F_{D2} = \emptyset$ if and only if $(\tilde{f}, \varphi_0) = 0$.

Proof. The later equality implies the former. Conversely, let $F_{D2} = \emptyset$, that is, $F_0 = \mathcal{P}_b \tilde{f} - W_b \varphi_0 = 0$ in $\Omega$ and $T_a^+(\tilde{f} + E_\gamma \mathcal{P}_b \tilde{f}, \mathcal{P}_b \tilde{f}) - L_{ab} \varphi_0$ on $\partial \Omega$. Multiplying the first relation by $b$, we get $\mathcal{P}_b \tilde{f} - W_b \varphi_0 = 0$ in $\Omega$. Taking into account that $bW_b(\varphi_0) = W_b(\varphi_0)$ is harmonic and applying Laplace operator gives $\tilde{f} = 0$ in $\mathbb{R}^2$, and hence $W_b \varphi_0 = 0$ in $\Omega$. Then by Lemma 12(ii), $\varphi_0 = 0$ on $\partial \Omega$. □

BDIE systems equivalence to the Dirichlet problem.

Theorem 16. Let $\varphi_0 \in H^\frac{1}{2}(\partial \Omega), \tilde{f} \in H^{-1}(\Omega)$ and $\tilde{f} \in \tilde{H}^{-1}(\Omega)$ is such that $\gamma_\Omega \tilde{f} = f$. Then

(i) If $u \in H^1(\Omega)$ solves the BVP (69)-(70), then the couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\Omega)$, where

$$\psi = T_a^+(\tilde{f}, u), \quad \text{on } \partial \Omega, \quad (77)$$

solves the BDIE systems (D1) and (D2).

(ii) If a couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega)$ solves BDIE system (D1) and $r_0 > \text{diam}(\Omega)$, then this solution is unique and solves BDIEs (D2), while $u$ solves the Dirichlet problem (69)-(70), and $\psi$ satisfies (77).

(iii) If a couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega)$ solves BDIE system (D2), then this solution is unique and solves BDIEs (D1), while $u$ solves the Dirichlet problem (69)-(70), and $\psi$ satisfies (77).

Proof. (i) Let $u \in H^1(\Omega)$ be a solution to BVP (69)-(70). Due to Theorem 13 it is unique. Setting $\psi$ by (77) evidently implies, $\psi \in H^{-\frac{1}{2}}(\partial \Omega)$.

From Theorem 8 and relations (66)-(67) follows that the couple $(u, \psi)$ satisfies the BDIE systems (D1) and (D2), with the right-hand sides (73) and (76) respectively, which completes the proof of item (i).

Let now the couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega)$ solve BDIE system (D1) or (D2). Due to Theorem 8, the hypothesis of Lemma 4 are satisfied implying that $u$ solves PDE (69) in $\Omega$, while relations in (55) and (56) also hold.

(ii) Let the couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega)$ solve BDIE system (D1). Taking trace of (71) on $\partial \Omega$ and subtracting (72) from it we obtain

$$\gamma^+ u = \varphi_0, \quad \text{on } \partial \Omega, \quad (78)$$

that is, $u$ satisfies the Dirichlet condition (70). (71) and Lemma 4 with $\Psi = \psi$, $\Phi = \varphi_0$ imply that $V_0 \Psi^* - W_0 \Phi^* = 0$, in $\Omega$, where $\Psi^* = \psi - T_a^+(\tilde{f}, u)$ and $\Phi^* = \varphi_0 - \gamma^+ u$. Due to (78), $\Phi^* = 0$. Then Lemma 12(ii) implies $\Psi^* = 0$, which proves condition (77). Thus $u$ obtained from the solution of BDIE system (D1) solves the Dirichlet problem and hence, by item (i) of the theorem, $(u, \psi)$ solve also BDIE system (D2).

(iii) Now let the couple $(u, \psi) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega)$ solve BDIE system (D2). Taking generalized conormal derivative of (74) and subtracting (75) from it, we get $\psi = T_a^+(\tilde{f}, u)$ on $\Omega$. Then substituting this in (56) gives $W_b(\varphi_0 - \gamma^+ u) = 0$ in $\Omega$ and Lemma 12(ii) then implies $\varphi_0 = \gamma^+ u$ on $\partial \Omega$. Due to (73), the BDIE system (71)-(72) with zero right-hand side can be considered as obtained for $\tilde{f} = 0, \varphi_0 = 0$, where $\tilde{f} \in \tilde{H}(\Omega)$ is an extension of $f \in H^{-1}(\Omega)$, that is, $\tilde{f} = r_\Omega \tilde{f}$, implying that its solution is given by a solution of the homogeneous problem (69)-(70), which is zero by Theorem 13. This implies uniqueness of the solution of the inhomogeneous BDIE system (D1). Similar arguments work for the BDIE system (D2). □

BDIE system operators invertibility for the Dirichlet problem. The BDIE systems (D1) and (D2) can be written as

$$D^1 \mathcal{U}^D = F_{D1} \quad \text{and} \quad D^2 \mathcal{U}^D = F_{D2},$$

respectively. Here $\mathcal{U}^D : = (u, \psi)^\top \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega)$.

$$D^1 := \begin{bmatrix} I + Z_b + R_b & -V_b \\ \gamma^+ Z_b + \gamma^+ R_b & -V_b \end{bmatrix}, \quad (79)$$

$$D^2 := \begin{bmatrix} I + Z_b + R_b & -V_b \\ T_a^+ Z_b + T_a^+ R_b (1 - \frac{a}{2b}) I - W_b' \end{bmatrix},$$

while $F_{D1}$ and $F_{D2}$ are given by (73) and (76) respectively. Due to the mapping properties of the operators involved in the definitions of the operators $D^1$ and $D^2$ as well as the right-hand sides $F_{D1}$ and $F_{D2}$ (see, e.g., (Ayele Tsegaye and Mikhailev, 2011, Appendix A), we have $F_{D1} \in H^1(\Omega) \times H^\frac{1}{2}(\partial \Omega), F_{D2} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega)$, while the operators

$$D^1 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega) \to H^1(\Omega) \times H^\frac{1}{2}(\partial \Omega), \quad (81)$$

$$D^2 : H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega) \to H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega) \quad (82)$$
are continuous. Due to Theorem 16(ii)-(iii), operators (81) and (82) are injective.

**Lemma 17.** For any couple \((F_1, F_2) \in H^1(\Omega) \times H^{-1/2}(\partial \Omega)\), there exists a unique couple \((\tilde{f}_{**}, \Phi_*) \in \tilde{H}^{-1}(\Omega) \times H^{1/2}(\partial \Omega)\) such that

\[
F_1 = \mathcal{P}_b \tilde{f}_{**} - W_b \Phi_* \tag{83}
\]

\[
F_2 = T_a^+(\tilde{f}_{**} + \tilde{E} \mathcal{R}_a \tilde{f}_{**}) - \mathcal{L}_a^+ \Phi_* \tag{84}
\]

Moreover, \((\tilde{f}_{**}, \Phi_*) = C_{**}(F_1, F_2)\) with \(C_{**} : H^1(\Omega) \times H^{-1/2}(\partial \Omega) \to \tilde{H}^{-1}(\Omega) \times H^{1/2}(\partial \Omega)\) a linear continuous operator given by

\[
\tilde{f}_{**} = \Delta(b F_1) + \gamma^*(F_2 + (\gamma^+ F_1) \partial_n b) \tag{85}
\]

\[
\Phi_* = \frac{1}{b} \left( - \frac{1}{2} I + \Omega \Delta \right)^{-1} \gamma^+ \left\{ -b F_1 + \mathcal{P}_a \left[ \Delta(b F_1) \right] + \gamma^* \left( \frac{b}{a} F_2 + (\gamma^+ F_1) \partial_n b \right) \right\} \tag{86}
\]

where \(\Delta(b F_1) = \nabla \cdot \tilde{E} \nabla(b F_1)\).

**Proof.** Let us first assume that there exist \((\tilde{f}_{**}, \Phi_*) \in \tilde{H}^{-1}(\Omega) \times H^{1/2}(\partial \Omega)\) satisfying equations (83)-(84) and find their expression in terms of \(F_1\) and \(F_2\). Let us rewrite (83) as

\[
F_1 - \mathcal{P}_b \tilde{f}_{**} = - W_b \Phi_* \quad \text{in } \Omega. \tag{87}
\]

Multiplying (87) by \(b\) and applying Laplacian to it, we obtain,

\[
\Delta(b F_1 - \mathcal{P}_a \tilde{f}_{**}) = \Delta(b F_1) - \tilde{f}_{**}

= -\Delta(W \partial_*(b \Phi_*)) = 0 \quad \text{in } \Omega, \tag{88}
\]

which means

\[
\Delta(b F_1) = r \tilde{f}_{**} \quad \text{in } \Omega, \tag{89}
\]

and \(b F_1 - \mathcal{P}_a \tilde{f}_{**} \in H^{1,0}(\Omega, \Delta)\) and hence \(F_1 - \mathcal{P}_b \tilde{f}_{**} \in H^{1,0}(\Omega, B) = H^{1,0}(\Omega, A)\). The latter imply that the canonical co-normal derivatives \(T_b^+(F_1 - \mathcal{P}_b \tilde{f}_{**})\) and \(T_a^+(F_1 - \mathcal{P}_b \tilde{f}_{**})\) are well defined and can be also written in terms of their general co-normal derivatives

\[
-\frac{b}{a} T_a^+(F_1 - \mathcal{P}_b \tilde{f}_{**}) = T_b^+(F_1 - \mathcal{P}_b \tilde{f}_{**})

= T_b^+(\tilde{B}(F_1 - \mathcal{P}_b \tilde{f}_{**}), F_1 - \mathcal{P}_b \tilde{f}_{**})

= T_b^+(\tilde{E} \nabla \cdot (b \nabla (F_1 - \mathcal{P}_b \tilde{f}_{**})), F_1 - \mathcal{P}_b \tilde{f}_{**})

= T_b^+(\tilde{E} \nabla \cdot (b F_1 - \mathcal{P}_a \tilde{f}_{**}))

= \tilde{E} \nabla \cdot ((F_1 - \mathcal{P}_b \tilde{f}_{**}) \nabla b), F_1 - \mathcal{P}_b \tilde{f}_{**})

= T_b^+(-\tilde{E} \nabla \cdot (F_1 \nabla b) - \tilde{E} \mathcal{R}_a \tilde{f}_{**}, F_1 - \mathcal{P}_b \tilde{f}_{**})
\]

and therefore,

\[
T_a^+(F_1 - \mathcal{P}_b \tilde{f}_{**}) = T_a^+(-\tilde{E} \nabla \cdot (F_1 \nabla b)

- \tilde{E} \mathcal{R}_a \tilde{f}_{**}, F_1 - \mathcal{P}_b \tilde{f}_{**}) \tag{90}
\]

where (62) and (89) were taken into account. Applying the co-normal derivative operator \(T_a^+\) to both sides of equation (87), substituting their (90), taking into account (11), we obtain,

\[
T_a^+(-\tilde{E} \nabla \cdot (F_1 \nabla b), F_1)

- T_a^+(\tilde{f}_{**} + \tilde{E} \mathcal{R}_a \tilde{f}_{**}, F_1)

= -\mathcal{L}_a^+ \Phi_*, \text{ on } \partial \Omega. \tag{91}
\]

Subtracting (91) from (84), we get,

\[
F_2 = T_a^+(\tilde{f}_{**} - \tilde{E} \nabla \cdot (F_1 \nabla b), F_1)

\text{ on } \partial \Omega. \tag{92}
\]

Due to (89), we can represent

\[
\tilde{f}_{**} = \tilde{f}(b F_1) + \tilde{f}_1 = \nabla \cdot \tilde{E} \nabla(b F_1)

\text{ on } \partial \Omega. \tag{93}
\]

where \(\tilde{f}_1 \in H^{-1/2}(\partial \Omega)\) is defined by (2) and hence, due to e.g. (Mikhailov, 2011, Theorem 2.10) can be in turn represented as \(\tilde{f}_1 = -\Psi_{**}, \text{ with some } \Psi_{**} \in H^{-1/2}(\partial \Omega)\). Then (89) is satisfied and

\[
\tilde{f}_{**} \in H^{-1/2}(\partial \Omega) \text{ and hence, due to e.g. (Mikhailov, 2011, Theorem 2.10) can be in turn represented as } \tilde{f}_1 = -\Psi_{**}, \text{ with some } \Psi_{**} \in H^{-1/2}(\partial \Omega). \tag{94}
\]

because

\[
\langle (\gamma^{-1})^* \tilde{E} \nabla(b F_1 \nabla b) - \gamma^* \Psi_{**} - \tilde{E} \nabla \cdot (F_1 \nabla b), w \rangle_{\partial \Omega}

= \langle \tilde{E} \nabla \cdot (F_1 \nabla b) - \gamma^* \Psi_{**} - \tilde{E} \nabla \cdot (F_1 \nabla b), w \rangle_{\partial \Omega}

= \langle \tilde{E} \nabla \cdot (F_1 \nabla b), \gamma^{-1} w \rangle_{\partial \Omega}

- \gamma^* \Psi_{**} - \langle \tilde{E} \nabla \cdot (F_1 \nabla b), \gamma^{-1} w \rangle_{\partial \Omega}

= \langle \tilde{E} \nabla \cdot (F_1 \nabla b), \gamma^{-1} w \rangle_{\partial \Omega}

= \langle \tilde{E} \nabla \cdot (F_1 \nabla b), \gamma^{-1} w \rangle_{\partial \Omega}

= \langle \tilde{E} \nabla \cdot (F_1 \nabla b), \gamma^{-1} w \rangle_{\partial \Omega}

= -\langle \gamma^{-1} (F_1 \nabla b), w \rangle_{\partial \Omega} - \Psi_{**}. \tag{95}
\]
Hence (92) reduces to
\[ \Psi_{**} = \frac{b}{a} - F_2 - (\gamma^+ F_1) \partial_n b = -T_0^+ F_1 - (\gamma^+ F_1) \partial_n b, \] (96)
and (93) to (85).

Now (87) can be written in the form
\[ W_\Delta(b\Phi_*) = F_\Delta \text{ in } \Omega, \] (97)
where
\[ F_\Delta := -bF_1 + P_\Delta \tilde{f}_{**} = -bF_1 + P_\Delta \left[ \tilde{\Delta}(bF_1) + \gamma^* \left( \frac{b}{a} F_2 + (\gamma^+ F_1) \partial_n b \right) \right] \] (98)
is harmonic function in \( \Omega \) due to (88). The trace of Eq. (98) gives
\[ -\frac{1}{2} b\Phi_* + W_\Delta(b\Phi_*) = \gamma^+ F_\Delta \text{ on } \partial\Omega. \] (99)
It is well known that the operator \([-\frac{1}{2} I + W_\Delta]\) is an isomorphism, see (Steinbach, 2007, Lemmas 6.10 and 6.11), this implies
\[ \Phi_* = \frac{1}{b} \left( -\frac{1}{2} I + W_\Delta \right)^{-1} \gamma^+ F_\Delta = \frac{1}{b} \left( -\frac{1}{2} I + W_\Delta \right)^{-1} \gamma^+ \left\{ -bF_1 + P_\Delta \left[ \tilde{\Delta}(bF_1) \right] + \gamma^* \left( \frac{b}{a} F_2 + (\gamma^+ F_1) \partial_n b \right) \right\}, \]
which is Eq. (86). Relations (85), (86) can be written as \((\tilde{f}_{**}, \Phi_*) = C_{**}(F_1, F_2), \) where \( C_{**} : H^1(\Omega) \times H^\frac{1}{2}(\partial\Omega) \rightarrow H^{-1}(\Omega) \times H^\frac{1}{2}(\partial\Omega) \) is a linear continuous operator, as required. We still have to check that the functions \( \tilde{f}_{**} \) and \( \Phi_*, \) given by (85) and (86), satisfy equations (83) and (84). Indeed, \( \Phi_* \) given by (86) satisfies equation (99) and thus \( \gamma^+ W_\Delta(a\Phi_*) = \gamma^+ F_\Delta. \) Since both \( W_\Delta(a\Phi_*) \) and \( F_\Delta \) are harmonic functions, this implies (97)-(98) and by (85) also (83). Finally, (85) implies by (94) that (92) is satisfied, and adding (91) to it leads to (84). Let us prove that the operator \( C_{**} \) is unique. Indeed, let a couple \((\tilde{f}_{**}, \Phi_*) \in H^{-1}(\Omega) \times H^\frac{1}{2}(\partial\Omega) \) be a solution of linear system (83)-(84) with \( F_1 = 0 \) and \( F_2 = 0. \) Then (89) implies that \( r_{**} \tilde{f}_{**} = 0 \) in \( \Omega, \) that is \( \tilde{f}_{**} \in H^{-1}_{\Omega} \subset H^{-1}(\Omega). \) Hence (92) reduces to
\[ 0 = T_0^+ (\tilde{f}_{**}, 0) \text{ on } \partial\Omega. \] (100)
By the first Green identity (10), this gives,
\[ 0 = \langle T_0^+ (\tilde{f}_{**}, 0), \gamma^+ v \rangle_{\partial\Omega} = \langle \tilde{f}_{**}, v \rangle_{\Omega}, \quad \forall v \in H^1(\Omega), \] (101)
which implies \( \tilde{f}_{**} = 0 \in \mathbb{R}^2. \) Finally, (86) gives \( \Phi_* = 0. \) Hence any solution of non-homogeneous linear system (83) – (84) has only one solution, which implies the uniqueness of the operator \( C_{**}. \)

The following assertion is (Mikhailov, 2005b, Lemma 19) generalized to a wider space in 2D.

**Lemma 18.** For any couple \((\tilde{F}_1, \tilde{F}_2) \in H^1(\Omega) \times H^\frac{1}{2}(\partial\Omega), \) there exists a unique couple \((\tilde{f}_{**}, \Phi_*) \in H^{-1}(\Omega) \times H^\frac{1}{2}(\partial\Omega) \) such that
\[ \tilde{F}_1 = P_b \tilde{f}_{**} - W_b \Phi_* \] (102)
\[ \tilde{F}_2 = \gamma^+ (P_b \tilde{f}_{**} - W_b \Phi_*) \] (103)
Moreover, \((\tilde{f}_{**}, \Phi_*) = \tilde{C}_{**}(\tilde{F}_1, \tilde{F}_2), \) with \( \tilde{C}_{**} : H^1(\Omega) \times H^\frac{1}{2}(\partial\Omega) \rightarrow H^{-1}(\Omega) \times H^\frac{1}{2}(\partial\Omega) \) a linear continuous operator is given by
\[ \tilde{f}_{**} = \tilde{\Delta}(b\tilde{F}_1) + \gamma^* (T_0^+ \tilde{F}_1 + \tilde{F}_2) \partial_n b \] (104)
\[ \Phi_* = \frac{1}{b} \left( -\frac{1}{2} I + W_\Delta \right)^{-1} \left( -bF_2 + \gamma^+ P_\Delta (\tilde{\Delta}(b\tilde{F}_1)) + \gamma^* (T_0^+ \tilde{F}_1 + \tilde{F}_2) \partial_n b \right) \] (105)
where \( \tilde{\Delta}(b\tilde{F}_1) = \nabla \cdot \hat{E} \nabla (b\tilde{F}_1). \)

**Proof.** Let us first assume that there exist \((\tilde{f}_{**}, \Phi_*) \in H^{-1}(\Omega) \times H^\frac{1}{2}(\partial\Omega) \) satisfying equations (102)-(103) and find their expression in terms of \( \tilde{F}_1 \) and \( \tilde{F}_2. \) Let us re write (102) as
\[ \tilde{F}_1 - P_b \tilde{f}_{**} = -W_b \Phi_* \text{ in } \Omega. \] (106)
Multiplying (106) by \( b \) and applying Laplacian to it, we obtain,
\[ \Delta (b\tilde{F}_1 - P_\Delta \tilde{f}_{**}) = \Delta (b\tilde{F}_1) - \tilde{f}_{**} = -\Delta (W_\Delta(b\Phi_*)) = 0 \text{ in } \Omega, \] (107)
which means
\[ \Delta (b\tilde{F}_1) = r_{**} \tilde{f}_{**} \text{ in } \Omega, \] (108)
and \( b\tilde{F}_1 - P_\Delta \tilde{f}_{**} \in H^{1,0}(\Omega, \Delta), \) while \( \tilde{F}_1 - P_b \tilde{f}_{**} \in H^{1,0}(\Omega, B) = H^{1,0}(\Omega, A). \) The latter imply that the canonical co-normal derivatives \( T_{\tilde{F}_1}^+(\tilde{F}_1 - P_b \tilde{f}_{**}) \) and \( T_{\tilde{F}_1}^+(\tilde{F}_1 - P_\Delta \tilde{f}_{**}) \) are well defined and \( T_{\tilde{F}_1}^+(\tilde{F}_1 - P_b \tilde{f}_{**}) = \frac{b}{a} T_{\tilde{F}_1}^+(\tilde{F}_1 - P_b \tilde{f}_{**}). \)
Due to (108) and using \( \tilde{f}_{**} = -\gamma^* \Psi_{**} \) with some \( \Psi_{**} \in H^{-\frac{1}{2}}(\partial\Omega) \) as in (96), we can represent
\[ \tilde{f}_{**} = \tilde{\Delta}(b\tilde{F}_1) + \tilde{f}_{1*} = -\nabla \cdot \hat{E} \nabla (b\tilde{F}_1) - \gamma^* \Psi_{**} \] (109)
where \( \tilde{f}_{1*} \in H^{-\frac{1}{2}}(\partial\Omega). \) Then (108) is satisfied. Replacing \( \tilde{F}_2 \) by \( T_0^+(\tilde{F}_1, u) \) in Lemma 17, relation
Let \( r_0 > \text{diam(}\Omega) \). The operators (81) and (82) are continuous and continuously invertible.

**Proof.** The continuity of operators (81) and (82) is proved above. To prove the invertibility of operator (81), let us consider the BDIE system (D1) with arbitrary right-hand side

\[
\mathcal{F}^{D1}_s = (\mathcal{F}^{D1}_{s_1}, \mathcal{F}^{D1}_{s_2})^T \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega).
\]

Take \( \tilde{\mathcal{F}}_1 = \mathcal{F}^{D1}_{s_1} \) and \( \Phi_s = \gamma^+ \mathcal{F}^{D1}_{s_1} - \mathcal{F}^{D1}_{s_2} \) in Lemma 18, to obtain the representation of \( \mathcal{F}^{D1}_s \) as:

\[
\mathcal{F}^{D1}_{s_1} = \tilde{\mathcal{F}}_1, \quad \mathcal{F}^{D1}_{s_2} = \gamma^+ \tilde{\mathcal{F}}_1 - \Phi_s
\]

where the couple

\[
(\hat{f}_s, \Phi_s) = \tilde{\mathcal{C}}_{**}(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2) \in H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)
\]

is linear and continuous. If \( r_0 > \text{diam(}\Omega) \), then taking into account (Mikhailov, 2015, Remark 5.3) and applying Theorem 13 with \( f = r_0 \hat{f} = r_0 \hat{\mathcal{F}}_1, \Phi_s = \varphi_0 \), we obtain that BDIE system (D1) is uniquely solvable and its solution is:

\[
U_1 = (A^D)^{-1}(r_1 \hat{f}, \varphi_0)^T, \quad U_2 = \gamma^+ U_1 - \varphi_0,
\]

where the inverse operator, \((A^D)^{-1} : H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{-1}(\Omega) \), to the left-hand side operator, \( A^D : H^1(\Omega) \rightarrow H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \), of the Dirichlet problem (69)-(70), is continuous. Representation (115) and continuity of the operator (116) imply invertibility of (81). To prove the invertibility of operator (82), let us consider the BDIE system (D2) with arbitrary right-hand side

\[
\mathcal{F}^{D2}_s = (\mathcal{F}^{D2}_{s_1}, \mathcal{F}^{D2}_{s_2})^T \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega).
\]

Take \( \mathcal{F}_1 = \mathcal{F}^{D2}_{s_1} \) and \( \mathcal{F}_2 = T^a(F_1, u) = \mathcal{F}^{D2}_{s_2} \) in Lemma 17 to represent \( \mathcal{F}^{D2}_s \) as:

\[
\mathcal{F}^{D2}_{s_1} = \tilde{\mathcal{F}}_1, \quad \mathcal{F}^{D2}_{s_2} = T^a(F_1, u) = \mathcal{F}_2
\]

and the couple

\[
(\hat{f}_s, \Phi_s) = \tilde{\mathcal{C}}_{**}(\mathcal{F}_1, \mathcal{F}_2) \in H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)
\]

is linear and continuous. Taking into account (Mikhailov, 2015, Remark 5.3) and applying Theorem 13 with \( \hat{f} = \hat{f}_s, \Phi_s = \varphi_0 \), we obtain that BDIE system (D2) is uniquely solvable and its solution is:

\[
U_1 = (A^D)^{-1}(r_1 \hat{f}, \varphi_0)^T, \quad U_2 = T^a(r_0 \hat{f}, U_1),
\]

where the inverse operator, \((A^D)^{-1} : H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)\)
Theorem 20. (i) The homogeneous Neumann problem (118)-(119) admits only linearly independent solution \( u^0 = 1 \) in \( H^1(\Omega) \).

(ii) The nonhomogeneous Neumann problem (118)-(119) is solvable if and only if the following solvability condition is satisfied.

\[
\langle \tilde{f}, u^0 \rangle_\Omega - \langle \psi_0, \gamma^+ u^0 \rangle_\Omega = 0
\]

BDIE system formulation for the Neumann problem. We explore different possibilities of reducing the Neumann problem (118)-(119) with \( \tilde{f} \in \tilde{H}^{-1}(\Omega) \), for \( u \in H^1(\Omega) \), to two different segregated boundary-domain integral equations (BDIE) systems. Corresponding formulations for the Neumann problem for \( u \in H^1(\Omega, \Delta) \) with \( f \in L_2(\Omega) \) in 2D were introduced and analysed in Ayele Tsegaye et al. (2017). Let us represent in (48), (66) and (67) the generalised conormal derivative and the trace of the function \( u \) as

\[
T^+_a(\tilde{f}, u) = \psi_0, \quad \gamma^+ u = \varphi,
\]

and will regard the new unknown function \( \varphi \in H^\frac{1}{2}(\partial \Omega) \) as formally segregated of \( u \). Thus we will look for the couple \( (u, \varphi) \in H^1(\Omega) \times H^\frac{1}{2}(\partial \Omega) \).

BDIE system (N1). To reduce BVP (118)-(119) to a BDIE system in this section we will use equation (48) in \( \Omega \) and equation (67) on \( \partial \Omega \).

Then we arrive at the following system, (N1), of two boundary-domain integral equations for the couple of unknowns, \((u, \varphi)\),

\[
u + \mathcal{Z}_b u + \mathcal{R}_b u + W_b \varphi = \mathcal{F}^1_{N1}, \quad \text{in} \; \Omega, \tag{121}
\]

\[
T^+_a \mathcal{Z}_b u + T^+_a \mathcal{R}_b u + \mathcal{L}^+_{ab} \varphi = \mathcal{F}^2_{N1}, \quad \text{on} \; \partial \Omega, \tag{122}
\]

where

\[
\mathcal{F}^N := \begin{bmatrix}
\mathcal{F}^1_{N1} \\
\mathcal{F}^2_{N1}
\end{bmatrix} = \begin{bmatrix}
\mathcal{P}_b \tilde{f} + V_b \psi_0 \\
T^+_a (\tilde{f} + \tilde{E} \mathcal{R}^a_b \tilde{f}, \mathcal{P}_b \tilde{f}) - \psi_0 + \frac{\gamma^+}{\gamma} \psi_0 + W_{ab} \psi_0
\end{bmatrix}
\]

Due to the mapping properties of operators involved in (123) we have \( \mathcal{F}^N_1 \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega) \) and \( \mathcal{F}^N_0 := \mathcal{P}_b \tilde{f} + V_b \psi_0 \in H^1(\Omega) \).

Remark 21. Let \( \tilde{f} \in \tilde{H}^{-1}(\Omega), \psi_0 \in H^{-\frac{1}{2}}(\partial \Omega) \) and \( r_0 > \text{diam}(\Omega) \). Then \( \mathcal{F}^N_1 = 0 \) if and only if \( (\tilde{f}, \psi_0) = 0 \).

Proof. The later equality implies the former. Conversely, let \( \mathcal{F}^N_1 = 0 \), that is, \( \mathcal{P}_b \tilde{f} + V_b \psi_0 = 0 \) in \( \Omega \) and \( T^+_a (\tilde{f} + \tilde{E} \mathcal{R}^a_b \tilde{f}, \mathcal{P}_b \tilde{f}) - \psi_0 + \frac{\gamma^+}{\gamma} \psi_0 + W_{ab} \psi_0 = 0 \) on \( \partial \Omega \). Multiplying the first relation by \( b \) gives \( \mathcal{P}_b \tilde{f} + V_b \psi_0 = 0 \) in \( \Omega \). Further, taking into account that \( b V_b \psi_0 = V_b \psi_0 \) is harmonic and applying Laplace operator we get \( \tilde{f} = 0 \) in \( \mathbb{R}^2 \) and hence \( V_b \psi_0 = 0 \) in \( \Omega \). Then due to Lemma 12(i), we get \( \psi_0 = 0 \) on \( \partial \Omega \).
Remark 22. Let $\tilde{f} \in \tilde{H}^{-1}(\Omega)$, $\psi_0 \in H^{-\frac{1}{2}}(\partial \Omega)$ and $r_0 > \text{diam}(\Omega)$. Then $\mathcal{F}^{N^2} = 0$ if and only if $(\tilde{f}, \psi_0) = 0$.

Proof. The later equality implies the former. Conversely, let $\mathcal{F}^{N^2} = 0$, that is, $\mathcal{P}_b \tilde{f} + V_\gamma \psi_0 = 0$ in $\Omega$ and $\gamma^+ \mathcal{P}_b \tilde{f} + V_T \psi_0 = 0$ on $\partial \Omega$. Multiplying the first relation by $b$ gives $\mathcal{P}_\Delta \tilde{f} + V_\Delta \psi_0 = 0$ in $\Omega$. Further, taking into account that $b \tilde{f} = 0$ in $\mathbb{R}^2$ and hence $V_\Delta \psi_0 = 0$ in $\Omega$. Then due to Lemma 12(i), we get $\psi_0 = 0$ on $\partial \Omega$. □

BDIE systems equivalence to the Neumann problem.

Theorem 23. Let $\psi_0 \in H^{-\frac{1}{2}}(\partial \Omega)$ and $\tilde{f} \in \tilde{H}^{-1}(\Omega)$.

(i) If a function $u \in H^1(\Omega)$ solves the BVP (118)-(119), then the couple $(u, \varphi)$, where

$$\varphi = \gamma^+ u \tag{127}$$

solves the BDIE systems (N1) and (N2).

(ii) If a couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega)$ solves the BDIE system (N1), then the $u$ solves BDIE system (N2) and $u$ solves the Neumann problem (118)-(119) and $\varphi$ satisfies (127).

(iii) If a couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega)$ solves the BDIE system (N2) and $r_0 > \text{diam}(\Omega)$, then the $u$ solves BDIE system (N1) and Neumann problem (118)-(119) and $\varphi$ satisfies (127).

(iv) The homogeneous BDIE systems (N1) and (N2) have unique linearly independent solution spanned by $\Phi_0 = (u_0, \varphi_0)^\top = (1, 1)^\top$ in $H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega)$. Condition (120) is necessary and sufficient for solvability of the nonhomogeneous BDIE systems (N1) and, if $r_0 > \text{diam}(\Omega)$, also of the system (N2), in $H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega)$.

Proof. (i) Let $u \in H^1(\Omega)$ be a solution to the Neumann BVP (118)-(119). It immediately follows from Theorem 48 and relations (66)-(67) that the couple $(u, \varphi)$ with $\varphi = \gamma^+ u$ satisfies the BDIE systems (N1) and (N2), which proves item (i).

(ii) Let now a couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega)$ solve BDIE system (N1) or (N2). Due to the first equations in the BDIE systems, the hypotheses of Lemma 4 are satisfied implying that $u$ is a solution of equation (118) in $\Omega$, and equations (55)-(58) hold for $\Psi = \psi_0$ and $\Phi = \varphi$.

If a couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega)$ solve the system (N1) then subtracting (58) from (122) gives $T^+_a (\tilde{f}, u) = \psi_0$ on $\partial \Omega$. Thus Neumann (119) is satisfied. Further, from (56) we derive $W_b \gamma^+ u - \varphi = 0$ in $\Omega$, where $\gamma^+ u = \varphi$ on $\partial \Omega$ by Lemma 12 completing the proof of item (ii).

(iii) Let now couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega)$ solve BDIE system (N2). Further, taking the trace of (124) on $\partial \Omega$ and comparing the results with (125), we easily derive that $\gamma^+ u = \varphi$ on $\partial \Omega$. Lemma 4 for equation (124) implies that $u$ is a solution of equation (118), while equations (55)-(58) hold for $\Psi = \psi_0$ and $\Phi = \varphi$. Further, from (56) we derive

$$V_b (\psi_0 - T_a (\tilde{f}, u)) = 0 \quad \text{in} \quad \Omega,$$

whence $T_a (\tilde{f}, u) = \psi_0$ on $\partial \Omega$ due to Lemma 12 (i) and $u$ solves Neumann problem (118)-(119) which completes the proof of item (iii).

(iv) Theorem 20 along with items (i) and (ii) imply the claims of item (iii) for BDIE system (N2) and (N1). □

Properties of BDIE system operators for the Neumann problem. BDIE systems (N1) and (N2) can be written respectively, as

$$\mathcal{R}^1 U^N = \mathcal{F}^{N_1}, \quad \mathcal{R}^2 U^N = \mathcal{F}^{N^2},$$

where $U^N = (u, \varphi)^T \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega_D)$, while $\mathcal{F}^{N_1}$ and $\mathcal{F}^{N^2}$ are given by Eqs. (123) and (126) respectively. Due to the mapping properties of potentials in (123) and (126), $\mathcal{F}^{N_1} \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega) \text{ and } \mathcal{F}^{N^2} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega)$.

$$\mathcal{R}^1 := \begin{bmatrix} I + Z_b + R_b & W_b \\ T^+ a Z_b + T^+ a R_b & \mathcal{L}_{ab}^+ \end{bmatrix},$$

$$\mathcal{R}^2 := \begin{bmatrix} I + Z_b + R_b & W_b \\ \gamma^+ Z_b + \gamma^+ R_b & \frac{1}{2} I + W_b \end{bmatrix}.$$
one-dimensional and the couple \((u^0, \varphi^0) = (1, 1)\) belong to \(\ker \mathcal{N}^1 = \ker \mathcal{N}^2\) directly follows from Theorem 23(iii).

To describe in more details the range of operators (129) and (130), i.e., to give more information about the co-kernels of these operators, we will need several auxiliary assertions. First of all, let us remark that for any \(v \in H^{s-\frac{2}{3}}(\partial \Omega), s < \frac{3}{2}\) the single layer potential can be defined as follows:

\[
V_b v(y) := -\langle \gamma P_b(\cdot, y), v \rangle_{\partial \Omega} = -\langle P_b(\cdot, y), \gamma^* v \rangle_{\mathbb{R}^3} = -\mathcal{P}_b \gamma^* v(y), \quad y \in \mathbb{R}^2 \setminus \partial \Omega. \tag{131}
\]

where \(\gamma^* : H^{s-\frac{2}{3}}(\partial \Omega) \rightarrow H^{2-s}(\partial \Omega), s < \frac{3}{2}\), is the operator adjointed to the trace operator \(\gamma : H^1(\partial \Omega) \rightarrow H^{\frac{1}{2}}(\partial \Omega)\), and the space \(H^2_{\partial \Omega}\) is defined as (2).

**Lemma 25.** Let \(\tilde{f} \in \tilde{H}^{s-2}(\Omega), s > \frac{1}{2}\) and \(r_0 > \text{diam} (\Omega)\). If

\[
r_0 \mathcal{P}_b \tilde{f} = 0 \quad \text{in} \quad \Omega, \tag{132}
\]

then \(\tilde{f} = 0\) in \(\mathbb{R}^2\).

**Proof.** Multiplying (132) by \(b\), taking into account the first relation in (34) and applying the Laplace operator, we obtain \(r_0 \tilde{f} = 0\), which means \(\tilde{f} \in \tilde{H}^{s-2}_{\partial \Omega}\). If \(s \geq \frac{3}{2}\), then \(\tilde{f} = 0\) by (Mikhailov, 2011, Theorem 2.10). If \(\frac{1}{2} < s < \frac{3}{2}\), then by the same theorem there exists \(v \in H^{s-\frac{2}{3}}(\partial \Omega)\) such that \(\tilde{f} = \gamma^* v\). This gives \(\mathcal{P}_b \tilde{f} = \mathcal{P}_b \gamma^* v = -V_b v\) in \(\mathbb{R}^2\). Then (132) reduces to \(V_b v = 0\) in \(\Omega\), which by Lemma 12(i) (for \(s = 1\), which can be generalized to \(\frac{1}{2} < s < \frac{3}{2}\)) implies \(v = 0\) on \(\partial \Omega\) and thus \(\tilde{f} = 0\) in \(\mathbb{R}^2\).

**Theorem 26.** Let \(\frac{1}{2} < s < \frac{3}{2}\) and \(r_0 > \text{diam} (\Omega)\). The operator

\[
\mathcal{P}_b : \tilde{H}^{s-2}(\Omega) \rightarrow H^s(\Omega) \tag{133}
\]

and its inverse

\[
(\mathcal{P}_b)^{-1} : H^s(\Omega) \rightarrow \tilde{H}^{s-2}(\Omega) \tag{134}
\]

are continuous and

\[
(\mathcal{P}_b)^{-1} g = \left[ \Delta \tilde{E}(I-r_0 V_{\Delta}^{-1} \gamma^*) + \gamma^* V_{\Delta}^{-1} \gamma^* \right] (bg) \quad \text{in} \quad \mathbb{R}^2, \quad \forall g \in H^s(\Omega). \tag{135}
\]

**Proof.** The continuity of equation (133) follows from (Chkadua et al., 2009, Theorem 3.8). By Lemma 25 operator (133) is injective. Let us prove its surjectivity. To this end, for arbitrary \(g \in H^s(\Omega)\) let us consider the following equation with respect to \(\tilde{f} \in \tilde{H}^{s-2}(\Omega)\),

\[
\mathcal{P}_\Delta \tilde{f} = g \quad \text{in} \quad \Omega. \tag{136}
\]

Let \(g_1 \in H^s(\Omega)\) be the (unique) solution of the following Dirichlet problem:

\[
\Delta g_1 = 0 \quad \text{in} \quad \Omega, \quad \gamma^* g_1 = \gamma^* g, \tag{137}
\]

which by (Ayele Tsegaye and Bekele Solomon, 2019, Theorem 2) the single layer potential \(V_{\Delta}^{-1}\) exists and due to Costabel (1988) or (Mikhailov, 2011, Lemma 2.6) can be particularly presented as \(g_1 = V_{\Delta}^{-1} \gamma^* g\). Let \(g_0 := g - g_1\). Then \(g_0 \in H^s(\Omega)\) and \(\gamma^* g_0 = 0\) and thus \(g_0\) can be uniquely extended to \(\tilde{E} g_0 \in \tilde{H}^s(\Omega)\), where \(\tilde{E}\) is the operator of extension by zero outside \(\Omega\). Thus by (131), equation (136) takes form

\[
r_0 \mathcal{P}_\Delta \tilde{f} + \gamma^* V_{\Delta}^{-1} \gamma^* \tilde{g} = g_0 \quad \text{in} \quad \Omega. \tag{138}
\]

Any solution \(\tilde{f} \in \tilde{H}^{s-2}(\Omega)\) of the corresponding equation on \(\mathbb{R}^2\)

\[
\mathcal{P}_\Delta \tilde{f} + \gamma^* V_{\Delta}^{-1} \gamma^* \tilde{g} = \tilde{E} g_0 \quad \text{in} \quad \mathbb{R}^2, \tag{139}
\]

solves (137). If \(\tilde{f}\) solves (138) then acting with the Laplace operator on (138), we obtain

\[
\tilde{f} = \tilde{Q} g := \Delta \tilde{E} g_0 - \gamma^* V_{\Delta}^{-1} \gamma^* g
\]

\[
= \Delta \tilde{E} (g - r_0 V_{\Delta}^{-1} \gamma^* g) - \gamma^* V_{\Delta}^{-1} \gamma^* g \quad \text{in} \quad \mathbb{R}^2. \tag{139}
\]

On the other hand, substituting \(\tilde{f}\) given by (139) to (138) and taking into account that \(\mathcal{P}_\Delta \tilde{h} = \tilde{h}\) for any \(\tilde{h} \in \tilde{H}^s(\Omega), \tilde{h} \in \mathbb{R}\), we obtain that \(\tilde{Q} g\) is indeed a solution of equation (138) and thus (137). By Lemma 25 the solution of (138) is unique, which means that the operator \(\tilde{Q}\) is inverse to operator (133), i.e., \(\tilde{Q} = (r_0 \mathcal{P}_b)^{-1}\).

Since \(\Delta\) is a continuous operator from \(\tilde{H}^s(\Omega)\) to \(\tilde{H}^{s-2}(\Omega)\), equation (85) implies that operator \((r_0 \mathcal{P}_b)^{-1} : H^s(\Omega) \rightarrow \tilde{H}^{s-2}(\Omega)\) is continuous. The relations \(\mathcal{P}_b = \frac{1}{r_0} \mathcal{P}_\Delta\) and \(b(x) > c > 0\) then imply invertibility of operator (133) and anstutz (135).

**Theorem 27.** The co-kernel of operator (129) is spanned over the functional

\[
g^{u^1} := (\gamma^+) \mathcal{C}_0 b, 1 \tag{140}
\]

in \(\tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial \Omega)\), that is, \(g^{u^1}(F_1, F_2) = (\gamma^+ F_1) \mathcal{C}_0 b + F_2, \gamma^+ u^0 \rangle_{\partial \Omega}, \) where \(u^0 = 1\).

**Proof.** The proof follows from the proof of (Mikhailov, 2015, Theorem 6.7) and Lemma 17. Indeed, let us consider the first equation in (128), i.e. the equation \(\mathcal{R}^\mathcal{U} = (F_1, F_2)^T\)
representing the BDIE system (N1)
\[ u + Z_b u + \mathcal{R}_b u + W_b \varphi = \mathcal{F}_1 \quad \text{in} \quad \Omega, \quad (141) \]
\[ T_a^+ Z_b u + T_a^+ \mathcal{R}_b u + \mathcal{L}_{ab}^+ (\varphi + \Phi_a) = \mathcal{F}_2 \quad \text{on} \quad \partial \Omega, \quad (142) \]
with arbitrary right hand side \((\mathcal{F}_1, \mathcal{F}_2)^T \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega)\), for \((u, \varphi) \in H^1(\Omega) \times H^\frac{1}{2}(\partial \Omega)\).
By Lemma 17 the right-hand side of the system has the form (83)-(84), that is, system (141)-(142) reduces to
\[ u + Z_b u + \mathcal{R}_b u + W_b (\varphi + \Phi_a) = \mathcal{P}_b \tilde{f}_{**} \quad \text{in} \quad \Omega, \quad (143) \]
where \((\tilde{f}_{**}, \Phi_a) \in \tilde{H^{-1}}(\Omega) \times H^{\frac{1}{2}}(\partial \Omega)\) is given by (83)-(84). Up to the notations (143)-(144) is the same as in (123) with \(\psi_0 = 0\).
Then Theorems 23(iii) and 26 imply that the BDIE system (143)-(144) and hence (141)-(142) is solvable if and only if
\[ \langle \tilde{f}_{**}, u_0^0 \rangle_{\Omega} = \langle (\tilde{\Delta} \mathcal{F}_1) + \gamma^2 (\mathcal{F}_2 + (\gamma^+ \mathcal{F}_1) \tilde{c}_a b), u_0^0 \rangle_{\Omega} \]
\[ = \langle (\nabla \cdot \tilde{E} \nabla (b \mathcal{F}_1) + \gamma^2 (\mathcal{F}_2 + (\gamma^+ \mathcal{F}_1) \tilde{c}_a b), u_0^0 \rangle_{\Omega} \]
\[ = \langle (\mathcal{F}_2 + (\gamma^+ \mathcal{F}_1) \tilde{c}_a b), \gamma^2 u_0^0 \rangle_{\Omega} = 0, \]
where we took into account that \(\nabla u_0^0 = 0\) in \(\mathbb{R}^2\). Thus the functional \(g^{*1}\) defined by (140) generates the necessary and sufficient solvability condition for the first equation in (128). Hence \(g^{*1}\) is basis of the co-kernel of \(\mathfrak{R}_1\).

**Theorem 28.** Let \(r_0 > \text{diam}(\Omega)\). Then the co-kernel of operator (130) is spanned over
\[ g^{*2} := \begin{pmatrix} -b \gamma^+ (\frac{1}{2} + W_\Delta) V_\Delta^{-1} \gamma^+ u_0^0 \\ -b (\frac{1}{2} - W_\Delta) V_\Delta^{-1} \gamma^+ u_0^0 \end{pmatrix} \quad (146) \]
in \(\tilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega)\), that is,
\[ g^{*2}(\mathcal{F}_1, \mathcal{F}_2) = \langle -b \gamma^+ (\frac{1}{2} + W_\Delta) V_\Delta^{-1} \gamma^+ u_0^0, \mathcal{F}_1 \rangle_{\Omega} \]
\[ + \langle b (\frac{1}{2} - W_\Delta) V_\Delta^{-1} \gamma^+ u_0^0, \mathcal{F}_2 \rangle_{\partial \Omega}, \]
where \(u_0^0 = 1\).

**Proof.** The proof follows from the proof of (Mikhailov, 2015, Theorem 6.8), (Ayele Tsegaye and Bekele Solomon, 2019, Theorem 2) and Lemma 17. Indeed, let us consider the first equation in (128), i.e. the equation \(\mathfrak{U}^1 \mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^T\), representing the BDIE system (N1)
\[ u + Z_b u + \mathcal{R}_b u + W_b \varphi = \mathcal{F}_1 \quad \text{in} \quad \Omega, \quad (147) \]
\[ \frac{1}{2} \varphi + \gamma^+ Z_b u + \gamma^+ \mathcal{R}_b u + W_b \varphi = \mathcal{F}_2 \quad \text{on} \quad \partial \Omega, \quad (148) \]
with arbitrary \((\mathcal{F}_1, \mathcal{F}_2)^T \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega)\), for \((u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial \Omega)\).
Introducing the new variable \(\varphi' = \varphi - (\mathcal{F}_2 - \gamma^+ \mathcal{F}_1)\), BDIE system (147)-(148) takes the form
\[ \frac{1}{2} \varphi' + \gamma^+ Z_b u + \gamma^+ \mathcal{R}_b u + W_b \varphi' = \mathcal{F}_1' \quad \text{in} \quad \Omega, \quad (149) \]
\[ = \mathcal{F}_2' \quad \text{on} \quad \partial \Omega, \quad (150) \]
where
\[ \mathcal{F}_1' = \mathcal{F}_1 - W_b (\mathcal{F}_2 - \gamma^+ \mathcal{F}_1) \in H^1(\Omega). \]
Let us recall that \(\mathcal{P}_b = r_0 \mathcal{P}_b : \tilde{H}^{s-2}(\Omega) \rightarrow H^s(\Omega)\) and then by Theorem 26, the operator \(\mathcal{P}_b^{-1} : (\mathcal{P}_b)^{-1} \circ H^s(\Omega) \rightarrow \tilde{H}^{s-2}(\Omega)\) is continuous for \(\frac{1}{2} < s < \frac{3}{2}\), while \(\mathcal{V}_{\Delta}^{-1}\) exists (Ayele Tsegaye and Bekele Solomon, 2019, Theorem 2).
Hence we always represent \(\mathcal{F}_1 = \mathcal{P}_b \tilde{f}_s\), with
\[ \tilde{f}_s = \mathcal{E}(I - r_0 V_{\Delta} \mathcal{V}_{\Delta}^{-1} \gamma^+) \]
\[ - \gamma^+ \mathcal{V}_{\Delta}^{-1} \gamma^+ [(b \mathcal{F}_2') \in \tilde{H}^{-1}(\Omega). \]
For \(\mathcal{F}_1' = \mathcal{P}_b \tilde{f}_s\), the right hand side of BDIE system (149)-(150) is the same as in (126) with \(f = \tilde{f}_s\) and \(\psi_0 = 0\). Then Theorems 23(iii) implies that the BDIE system (149)-(150) and hence (147)-(148) is solvable if and only if
\[ \langle \tilde{f}_s, u_0^0 \rangle_{\Omega} = \]
\[ = \langle \mathcal{E}(I - r_0 V_{\Delta} \mathcal{V}_{\Delta}^{-1} \gamma^+) - \gamma^+ \mathcal{V}_{\Delta}^{-1} \gamma^+ \rangle (b \mathcal{F}_2'), u_0^0 \rangle_{\mathbb{R}^2} \]
\[ = \langle \mathcal{E}(I - r_0 V_{\Delta} \mathcal{V}_{\Delta}^{-1} \gamma^+) - \gamma^+ \mathcal{V}_{\Delta}^{-1} \gamma^+ \rangle (b \mathcal{F}_2'), u_0^0 \rangle_{\mathbb{R}^2} \]
\[ - \langle (\gamma^+ \mathcal{V}_{\Delta}^{-1} \gamma^+) (b \mathcal{F}_2'), u_0^0 \rangle_{\mathbb{R}^2} \]
\[ - \langle \gamma^+ (b \mathcal{F}_2'), \mathcal{V}_{\Delta}^{-1} \gamma^+ u_0^0 \rangle_{\partial \Omega} = \]
\[ - \langle \frac{1}{2} [\gamma^+ (b \mathcal{F}_2') - W_{\Delta} (b \mathcal{F}_2' - \gamma^+ \mathcal{F}_1)], \mathcal{V}_{\Delta}^{-1} \gamma^+ u_0^0 \rangle_{\partial \Omega} \]
\[ = \langle \gamma^+ (b \mathcal{F}_2') \mathcal{V}_{\Delta}^{-1} \gamma^+ u_0^0, \partial \Omega \rangle_{\partial \Omega} = 0. \quad (151) \]
Thus the functional $g^{*2}$ defined by (146) generates the necessary and sufficient solvability condition of the equation $\mathfrak{R}^2 \mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^T$. Hence $g^{*2}$ is basis of the cokernel of $\mathfrak{R}^2$.

**Perturbed segregated BDIE systems for Neumann problem.** Theorem 23 implies, that even when the solvability condition (120) is satisfied, the solutions of both BDIE systems, (N1) and (N2), are not unique. By Theorem 24, in turn, the BDIE left hand side operators, $\mathfrak{R}^1$ and $\mathfrak{R}^2$, have non-zero kernels and thus are not invertible. To find a solution $(u, \varphi)$ from uniquely solvable BDIE system with continuously invertible left hand side operators, let us consider, following Mikhailov (1999), some BDIE systems obtained form (N1) and (N2) by finite-dimensional operator perturbations (cf. Ayele Tsegaye (2021b) for the three-dimensional case). Below we use the notations $\mathcal{U} = (u, \varphi)^T$ and $|\partial \Omega| := \int_{\partial \Omega} dS$.

### 0.0.1. Perturbation of BDIE system (N1).

Let us introduce the perturbed counterparts of the BDIE system (N1),

$$\hat{\mathfrak{R}}^1 \mathcal{U}^N = \mathcal{F}^N_1,$$

where

$$\hat{\mathfrak{R}}^1 := \mathfrak{R}^1 + \hat{\mathfrak{R}}^1$$

and

$$\hat{\mathfrak{R}}^1 \mathcal{U}^N(y) := \hat{g}^0(\mathcal{U}^N) \mathcal{G}^1(y)$$

that is,

$$\hat{g}^0(\mathcal{U}^N) := \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \varphi(x) dS \begin{pmatrix} b^{-1}(y) \\ 0 \end{pmatrix},$$

For the functional $g^{*1}$ given by (140) in Theorem 27, $g^{*1}(\mathcal{F}^N_1) = |\partial \Omega|$, while $g^{0}(\mathcal{U}^N) = 1$.

Hence (Mikhailov, 2015, Theorem D.1 in Appendix) and (Ayele Tsegaye et al., 2017) imply the following assertion.

**Theorem 29.** Let $r_0 > \text{diam}(\Omega)$, then

(i) The operator $\hat{\mathfrak{R}}^1 : H^1(\Omega) \times H^{1/2}(\partial \Omega) \to H^1(\Omega) \times H^{-1/2}(\partial \Omega)$ is continuous and continuously invertible.

(ii) If condition $g^{*1}(\mathcal{F}^N_1) = 0$ or condition (120) for $\mathcal{F}^N_1$ in form (129) is satisfied, then the unique solution of perturbed BDIE system (152) gives a solution of original BDIE system (N1) such that

$$g^0(\mathcal{U}^N) = \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \gamma^+ u dS = \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \varphi dS = 0.$$  

**Perturbation of BDIE system (N2).** Let us introduce the perturbed counterparts of the BDIE system (N2)

$$\hat{\mathfrak{R}}^2 \mathcal{U}^N = \mathcal{F}^N_2,$$

where

$$\hat{\mathfrak{R}}^2 := \mathfrak{R}^2 + \hat{\mathfrak{R}}^2$$

and

$$\hat{\mathfrak{R}}^2 \mathcal{U}^N(y) := g^0(\mathcal{U}^N) \mathcal{G}^2(y)$$

that is,

$$g^0(\mathcal{U}^N) := \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \varphi(x) dS,$$

$$\mathcal{G}^2(y) := \begin{pmatrix} (b^{-1}u^0)(y) \\ (b^{-1}u^0)^\gamma(y) \end{pmatrix}.$$  

For the functional $g^{*2}$ given by (146) in Theorem 28, since the operator $\mathcal{V}^{-1}_{\partial \Omega} : H^1(\partial \Omega) \to H^{-1}(\partial \Omega)$ is positive definite (with additional condition $r_0 > \text{diam}(\Omega)$) and $u^0(x) = 1$, there exists a positive constant $C$ such that

$$g^{*2}(\mathcal{G}^2) =$$

$$\langle -b\gamma^{1+} + \mathcal{V} \mathcal{V}^{-1}_{\partial \Omega} \mathcal{V}^{1-} \gamma^+ u^0, b^{-1}u^0 \rangle_{\partial \Omega} +$$

$$\langle -b\gamma^{1+} + \mathcal{V} \mathcal{V}^{-1}_{\partial \Omega} \mathcal{V}^{1-} \gamma^+ (b^{-1}u^0) \rangle_{\partial \Omega}$$

$$\leq -C\|\gamma^+ u^0\|_{H^{1/2}(\partial \Omega)}^2,$$

$$\leq -C\|\gamma^+ u^0\|_{H^{1/2}(\partial \Omega)}^2 = -C|\partial \Omega|^2 < 0.$$  

Due to (154) and $g^0(\mathcal{U}^0) = 1$, (Mikhailov, 2015, Theorem D.1) and (Ayele Tsegaye et al., 2017, Theorem 7) imply the following assertion.

**Theorem 30.** Let $r_0 > \text{diam}(\Omega)$, then

(i) The operator $\hat{\mathfrak{R}}^2 : H^1(\Omega) \times H^{1/2}(\partial \Omega) \to H^1(\Omega) \times H^{-1/2}(\partial \Omega)$ is continuous and continuously invertible.

(ii) If condition $g^{*2}(\mathcal{F}^N_2) = 0$ or condition (120) for $\mathcal{F}^N_2$ in form (130) is satisfied, then the unique solution of perturbed BDIE system (153) gives a solution of original BDIE system (N2) such that

$$g^0(\mathcal{U}^N) = \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \gamma^+ u dS = \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \varphi dS = 0.$$
ACKNOWLEDGMENTS

The authors thank the anonymous reviewers for their valuable comments and suggestions. This work was supported by the International Science Program (ISP), Sweden. The first author thanks the Simons Foundation based at Botswana International University of Science and Technology (BIUST) for partial support.

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