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DISTANCE BASED INDICES OF GENERALIZED TRANSFORMATION GRAPHS

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ABSTRACT: In this paper, the expressions for the Wiener index, Gutman index, degree distance, eccentric connectivity index and eccentric distance sum of the generalized transformation graphs G^{+-} and G^{-+} are obtained in terms of the parameters of underline graphs.

Keywords/phrases: degree distance, eccentric connectivity index,eccentric distance sum, generalized transformation graphs, gutman index, wiener index.

Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The order n of a graph G is the number of vertices of G and the size m of a graph G is the number of edges of G. The degree of a vertex $v \in V(G)$, $d_G(v)$, is the number of edges incident with v. The distance $d_G(u,v)$ between two vertices $u, v \in V(G)$ is the number of edges in a shortest path connecting them. The minimum and the maximum degree of a graph G are denoted by $\delta(G)$ and $\triangle(G)$ respectively. The eccentricity of v, $ecc_G(v)$, is the distance between v and any vertex which is furthest from v in G. A triangle-free graph is an undirected graph in which no three vertices form a triangle of edges. The line graph $L(G)$ of a graph G is the graph with vertex set $E(G)$ in which $e, f \in E(G)$ are adjacent as vertices in $L(G)$ if and only if they are adjacent as edges in G.

Topological indices have been extensively studied due to their chemical importance. Some of these topological indices are based on degrees of vertices and the most common such indices are the first Zagreb index ($M_1(G)$) and second Zagreb index ($M_2(G)$) of a graph G are defined, respectively, as

$$M_1(G) = \sum_{v \in V(G)} d^2(v) \text{ and } M_2(G) = \sum_{uv \in E(G)} d(u)d(v).$$

For any simple graph G of size m, we can easily observe that

$$||L(G)|| = \frac{1}{2}M_1(G) - m.$$

Let us present definitions of well-known distance-based topological indices of graphs. The Wiener index of a graph G,

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v),$$

the degree distance

$$DD(G) = \sum_{\{u,v\} \subseteq V(G)} (deg(u) + deg(v))d(u,v),$$

the Gutman index

$$Gut(G) = \sum_{\{u,v\} \subseteq V(G)} deg(u)deg(v)d(u,v)$$

the eccentric connectivity index

$$ECI(G) = \sum_{v \in V(G)} ecc(v)d(v)$$

and the eccentric distance sum

$$EDS(G) = \sum_{v \in V(G)} ecc(v)D(v)$$

where $D(v) = \sum_{u \in V(G)} d(u,v)$.

More studied distance based topological indices are: Wiener, degree distance, eccentric connectivity and eccentric distance sum (Agrawal*et al.* (2000); Dankelmann *et al.* (2009); Morgan *et al.* (2011); Hua *et al.* (2011); Ilić *et al.* (2011)). The generalized transformation graph G^{xy}, introduced recently by Basavanagoud *et al.*(2015), is

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a graph whose vertex set is $V(G) \cup E(G)$, and for $\alpha, \beta \in V(G^{xy})$, the vertices α and β are adjacent in G^{xy} if and only if (a) and (b) holds:
(a) $\alpha, \beta \in V(G)$, α, β are adjacent in G if $x = +$ and α, β are not adjacent in G if $x = -$.
(b) $\alpha \in$V(G) and $\beta \in$ E(G), α, β are incident in G if $y = +$ and α, β are not incident in G if $y = -$.

One can obtain the four graphical transformations of graphs as G^{++}, G^{+-}, G^{-+} and G^{--}. Note that G^{++} is just the semitotal-point graph of G, which was introduced by Sampathkumar and Chikkodimath *et al.*(1973). The vertex v of G^{xy} corresponding to a vertex v of G is referred to as a point vertex. The vertex e of G^{xy} corresponding to an edge e of G is referred to as a line vertex.

In Ramane *et al.*(2018), the harmonic index and Randic index of generalized transformation graphs were obtained.

We study the Wiener index, the degree distance, the Gutman index, the eccentric connectivity index and the eccentric distance sum of the generalized transformation graphs G^{+-} and G^{-+}.

By definition of simple graph an edge is a subset of two element set of the vertex set of V(G). That is, if e is an edge between vertices u and v, then we write $e = \{u, v\}$. So for any edge e and a vertex u of a graph, we write $u \in e$ to mean u and e are incident. If u and e are not incident, we write $u \notin e$. For two edges e_1 and e_2, we write $e_1 \cap e_2 \neq \emptyset$ to mean the edges are adjacent, and $e_1 \cap e_2 = \emptyset$ to mean the edges are not adjacent.

Distance based indices of the generalized transformation graph G^{+-}

In this section, we determine the expressions for the Wiener index, Gutman index, degree distance, eccentric connectivity index and eccentric distance sum of the generalized transformation graph G^{+-} in terms of the order and size of the underline graphs. In Basavanagoud *et al.*(2015), the authors found the degrees of all vertices of G^{xy}.

Proposition 1. Let G be a graph with n vertices and m edges. Let $u \in V(G)$ and $e \in E(G)$. Then the degrees of point and line vertices in G^{xy} are
(i). $d_{G^{++}}(u) = 2d_G(u)$ and $d_{G^{++}}(e) = 2$.
(ii). $d_{G^{+-}}(u) = m$ and $d_{G^{+-}}(e) = n - 2$.
(iii). $d_{G^{-+}}(u) = n - 1$ and $d_{G^{-+}}(e) = 2$.
(iv). $d_{G^{--}}(u) = n + m - 1 - 2d_G(u)$ and $d_{G^{--}}(e) = n - 2$.

Now let us first determine the distance between any two vertices of the generalized transformed graph G^{+-}.

Proposition 2. Let G be a graph of order $n \geq 5$ and $\delta(G) \geq 2$. Then for $u, v \in V(G)$ and $e, e_1, e_2 \in E(G)$, the distance between any two vertices in G^{+-} is given by

$$(i).\ d_{G^{+-}}(u, v) = \begin{cases} 1 & \text{if } uv \in E(G) \\ 2 & \text{if } uv \notin E(G) \end{cases}$$

$$(ii).\ d_{G^{+-}}(u, e) = \begin{cases} 1 & \text{if } u \notin e, \\ 2 & \text{if } u \in e \end{cases}$$

$$(iii).\ d_{G^{+-}}(e_1, e_2) = 2$$

$$(iv).\ ec_{G^{+-}}(u) = 2 = ec_{G^{+-}}(e)$$

Proof. (i). Let $u, v \in V(G)$. By the definition of G^{+-}, u and v are adjacent in G if and only if they are adjacent in G^{+-}. That is, if $uv \in E(G)$, then $uv \in E(G^{+-})$ and hence $d_{G^{+-}}(u, v) = 1$. Suppose $uv \notin E(G)$. Then $d_G(u, v) \geq 2$. If $d_G(u, v) = 2$, then $d_{G^{+-}}(u, v) = 2$. Suppose $d_G(u, v) \geq 3$. Let $u = u_1 u_2 \ldots u_k = v$ be a (u, v)-path of length at least 3 in G. Let $e_i = u_i u_{i+1}$. Then clearly e_2 is not incident with both u and v in G. Thus by the definition of G^{+-}, e_2 is adjacent with both u and v in G^{+-}. That is, ue_2v is a (u, v)-path of length 2 in G^{+-} and hence $d_{G^{+-}}(u, v) = 2$.

(ii). Let $u \in V(G)$ and $e \in E(G)$. By the definition of G^{+-}, u and e are not incident in G if and only if u and e are adjacent in G^{+-}. That is, if $u \notin e$, then $ue \in E(G^{+-})$ and hence $d_{G^{+-}}(u, e) = 1$. Suppose $u \in e$. Then $e = uv$ for some $v \in V(G)$. Since $\delta(G) \geq 2$ there exists a vertex w in G such that $uw \in E(G)$ and $w \neq v$. This implies that e and w are not incident in G and hence $we \in E(G^{+-})$. This implies that, uwe is a (u, e)-path in G^{+-} and hence $d_{G^{+-}}(u, e) = 2$.

(iii). Let $e_1, e_2 \in E(G)$. By the definition of G^{+-}, no two edges are adjacent in G^{+-}. That is,

$d_{G^{+-}}(e_1, e_2) \geq 2$. Let $e_1 = ab$ and $e_2 = cd$. Since G has at least five vertices there exists a vertex $u \in V(G) \setminus \{a, b, c, d\}$. Then u is not incident with e_1 and e_2. This implies that u is a djacent with both e_1 and e_2 in G^{+-}. That is, e_1ue_2 is a (e_1, e_2)-path in G^{+-} and hence $d_{G^{+-}}(e_1, e_2) = 2$. (iv).Follows from (i), (ii) and (iii).

From propositions (1) and (2), we have the following results.

Theorem 1. *Let G be a graph of order $n \geq 5$ and size m. If $\delta(G) \geq 2$, then*

(i) $W(G^{+-}) = n^2 + m^2 + mn - n$
(ii) $DD(G^{+-}) = 3nm^2 + 3mn^2 - 4mn - 4m^2$
(iii) $Gut(G^{+-}) = 3(mn)^2 - 5nm^2 - mn^2 - m^3 + 4mn - 4m$
(iv) $ECI(G^{+-}) = 4m(n-1)$
(v) $EDS(G^{+-}) = 4n^2 + 4m^2 + 4mn - 4n$.

Proof. (i). $W(G^{+-})$

$$\begin{aligned}
&= \sum_{\{u,v\}\subseteq V(G^{+-})} d_{G^{+-}}(u,v) \\
&= \sum_{\{u,v\}\subseteq V(G)} d_{G^{+-}}(u,v) + \sum_{\substack{u\in V(G)\\ e\in E(G)}} d_{G^{+-}}(u,e) \\
&\quad + \sum_{\{e_1,e_2\}\subseteq E(G)} d_{G^{+-}}(e_1,e_2) \\
&= \sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\in E(G)}} d_{G^{+-}}(u,v) + \sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\notin E(G)}} d_{G^{+-}}(u,v) \\
&\quad + \sum_{\substack{u\in V(G)\\ e\in E(G), u\in e}} d_{G^{+-}}(u,e) + \sum_{\substack{u\in V(G)\\ e\in E(G), u\notin e}} d_{G^{+-}}(u,e) \\
&\quad + \sum_{\{e_1,e_2\}\subseteq E(G)} d_{G^{+-}}(e_1,e_2) \\
&= \sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\in E(G)}} 1 + \sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\notin E(G)}} 2 + \sum_{\substack{u\in V(G)\\ e\in E(G), u\in e}} 2 \\
&\quad + \sum_{\substack{u\in V(G)\\ e\in E(G), u\notin e}} 1 + \sum_{\{e_1,e_2\}\subseteq E(G)} 2 \\
&= m + 2\left(\binom{n}{2} - m\right) + 2(2m) + nm \\
&\quad - 2m + 2\binom{m}{2} \\
&= 2\binom{n}{2} + 2\binom{m}{2} + nm + m \\
&= n^2 + m^2 + nm - n.
\end{aligned}$$

(ii). $DD(G^{+-})$

$$\begin{aligned}
&= \sum_{\{u,v\}\subseteq V(G^{+-})} (d_{G^{+-}}(u) + d_{G^{+-}}(v)) d_{G^{+-}}(u,v) \\
&= \sum_{\{u,v\}\subseteq V(G)} (d_{G^{+-}}(u) + d_{G^{+-}}(v)) d_{G^{+-}}(u,v) \\
&\quad + \sum_{\substack{u\in V(G)\\ e\in E(G)}} (d_{G^{+-}}(u) + d_{G^{+-}}(e)) d_{G^{+-}}(u,e) \\
&\quad + \sum_{\{e_1,e_2\}\subseteq E(G)} (d_{G^{+-}}(e_1) + d_{G^{+-}}(e_2)) d_{G^{+-}}(e_1,e_2) \\
&= \sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\in E(G)}} (d_{G^{+-}}(u) + d_{G^{+-}}(v)) d_{G^{+-}}(u,v) \\
&\quad + \sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\notin E(G)}} (d_{G^{+-}}(u) + d_{G^{+-}}(v)) d_{G^{+-}}(u,v) \\
&\quad + \sum_{\substack{u\in V(G)\\ e\in E(G), u\in e}} (d_{G^{+-}}(u) + d_{G^{+-}}(e)) d_{G^{+-}}(u,e) \\
&\quad + \sum_{\substack{u\in V(G)\\ e\in E(G)\\ u\notin e}} (d_{G^{+-}}(u) + d_{G^{+-}}(e)) d_{G^{+-}}(u,e) \\
&\quad + \sum_{\{e_1,e_2\}\subseteq E(G)} (d_{G^{+-}}(e_1) + d_{G^{+-}}(e_2)) d_{G^{+-}}(e_1,e_2) \\
&= \sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\in E(G)}} 2m + \sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\notin E(G)}} 4m + \sum_{\substack{u\in V(G)\\ e\in E(G), u\in e}} 2(m+n-2) \\
&\quad + \sum_{\substack{u\in V(G)\\ e\in E(G), u\notin e}} (m+n-2) + \sum_{\{e_1,e_2\}\subseteq E(G)} 4(n-2) \\
&= 2m^2 + 4m\left(\binom{n}{2} - m\right) + 4m(m+n-2) \\
&\quad + (m+n-2)(nm-2m) + 4(n-2)\binom{m}{2} \\
&= 3nm^2 + 3mn^2 - 4mn - 4m^2.
\end{aligned}$$

(iii). $Gut(G^{+-})$

$$\begin{aligned}
&= \sum_{\{u,v\}\subseteq V(G^{+-})} (d_{G^{+-}}(u) d_{G^{+-}}(v)) d_{G^{+-}}(u,v) \\
&= \sum_{\{u,v\}\subseteq V(G)} (d_{G^{+-}}(u) d_{G^{+-}}(v)) d_{G^{+-}}(u,v) \\
&\quad + \sum_{\substack{u\in V(G)\\ e\in E(G)}} (d_{G^{+-}}(u) d_{G^{+-}}(e)) d_{G^{+-}}(u,e)
\end{aligned}$$

$$+\sum_{\{e_1,e_2\}\subseteq E(G)}(d_{G^{+-}}(e_1)d_{G^{+-}}(e_2))d_{G^{+-}}(e_1,e_2)$$

$$=\sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\in E(G)}}(d_{G^{+-}}(u)d_{G^{+-}}(v))d_{G^{+-}}(u,v)$$

$$+\sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\notin E(G)}}(d_{G^{+-}}(u)d_{G^{+-}}(v))d_{G^{+-}}(u,v)$$

$$+\sum_{\substack{u\in V(G)\\ e\in E(G),u\in e}}(d_{G^{+-}}(u)d_{G^{+-}}(e))d_{G^{+-}}(u,e)$$

$$+\sum_{\substack{u\in V(G)\\ e\in E(G),u\notin e}}(d_{G^{+-}}(u)d_{G^{+-}}(e))d_{G^{+-}}(u,e)$$

$$+\sum_{\{e_1,e_2\}\subseteq E(G)}(d_{G^{+-}}(e_1)d_{G^{+-}}(e_2))d_{G^{+-}}(e_1,e_2)$$

$$=\sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\in E(G)}}m^2+\sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\notin E(G)}}2m^2+\sum_{\substack{u\in V(G)\\ e\in E(G),u\in e}}2m(n-2)$$

$$+\sum_{\substack{u\in V(G)\\ e\in E(G),u\notin e}}m(n-2)+\sum_{\{e_1,e_2\}\subseteq E(G)}2(n-2)^2$$

$$=m^3+2m^2\left(\binom{n}{2}-m\right)+4m^2(n-2)$$

$$+m(n-2)(nm-2m)+2(n-2)^2\binom{m}{2}$$

$$=3(nm)^2-5nm^2-mn^2-m^3+4mn-4m.$$

(iv). $ECI(G^{+-})$

$$=\sum_{u\in V(G^{+-})}ec_{G^{+-}}(u)d_{G^{+-}}(u)$$

$$=\sum_{u\in V(G)}ec_{G^{+-}}(u)d_{G^{+-}}(u)+\sum_{e\in E(G)}ec_{G^{+-}}(u)d_{G^{+-}}(u)$$

$$=\sum_{u\in V(G)}2m+\sum_{e\in E(G)}2(n-2)$$

$$=4mn-4m.$$

(v). Let $u\in V(G)$. Then

$D_{G^{+-}}(u)$

$$=\sum_{v\in V(G^{+-})}d_{G^{+-}}(u,v)$$

$$=\sum_{v\in V(G)}d_{G^{+-}}(u,v)+\sum_{e\in E(G)}d_{G^{+-}}(u,e)$$

$$=\sum_{\substack{v\in V(G)\\ v\in N_G(u)}}d_{G^{+-}}(u,v)+\sum_{\substack{v\in V(G)\\ v\notin N_G(u)}}d_{G^{+-}}(u,v)$$

$$+\sum_{\substack{e\in E(G)\\ e\in u}}d_{G^{+-}}(u,e)+\sum_{\substack{e\in E(G)\\ e\notin u}}d_{G^{+-}}(u,e)$$

$$=d_G(u)+2(n-1-d_G(u))+2d_G(u)+m-d_G(u)$$

$$=2(n-1)+m.$$

Let $e\in E(G)$. Then

$$D_{G^{+-}}(e)=\sum_{v\in V(G^{+-})}d_{G^{+-}}(e,v)$$

$$=\sum_{v\in V(G)}d_{G^{+-}}(e,v)+\sum_{f\in E(G)}d_{G^{+-}}(e,f)$$

$$=\sum_{\substack{v\in V(G)\\ v\in e}}d_{G^{+-}}(e,v)+\sum_{\substack{v\in V(G)\\ v\notin e}}d_{G^{+-}}(e,v)$$

$$+\sum_{f\in E(G)}d_{G^{+-}}(e,f)$$

$$=2(2)+n-2+2(m-1)$$

$$=n+2m.$$

$EDS(G^{+-})$

$$=\sum_{v\in V(G^{+-})}ecc_{G^{+-}}(v)D_{G^{+-}}(v)$$

$$=\sum_{v\in V(G)}ecc_{G^{+-}}(v)D_{G^{+-}}(v)+\sum_{e\in E(G)}ecc_{G^{+-}}(e)D_{G^{+-}}(e)$$

$$=\sum_{v\in V(G)}2(m+2(n-1))+\sum_{e\in E(G)}2(n+2m)$$

$$=4n^2+4m^2+4nm-4n.$$

Distance based indices of the generalized transformation graph G^{-+}

In this section, we determine the expressions for the Wiener index, Gutman index, degree distance, eccentric connectivity index and eccentric distance sum of the generalized transformation graph G^{-+} of a triangle free graph in terms of the order and size of the underline graphs. First let us determine the distance between any two vertices of the generalized transformed graph G^{-+}.

Proposition 3. Let G be a triangle free graph of order n and size m. Then for all $u,v\in V(G)$ and $e,e_1,e_2\in E(G)$, the distance between any two vertices of G^{-+} is given by

$$(i).\ d_{G^{-+}}(u,v)=\begin{cases}2 & \text{if } uv\in E(G)\\ 1 & \text{if } uv\notin E(G)\end{cases}$$

$$(ii).\ d_{G^{-+}}(u,e) = \begin{cases} 1 & \text{if } u \in e, \\ 2 & \text{if } u \notin e \end{cases}$$

$$(iii).\ d_{G^{-+}}(e_1,e_2) = \begin{cases} 2 & \text{if } e_1 \cap e_2 \neq \emptyset \\ 3 & \text{if } e_1 \cap e_2 = \emptyset \end{cases}$$

$$(iv).\ ec_{G^{-+}}(u) = 2$$
$$(v).\ 2 \leq ec_{G^{-+}}(e) \leq 3$$

Proof. (i). Let $u, v \in V(G)$. By the definition of G^{-+}, u and v are adjacent in G if and only if they are not adjacent in G^{-+}. If $uv \notin E(G)$, then $uv \in E(G^{-+})$ and hence $d_{G^{-+}}(u,v) = 1$. Suppose $e = uv \in E(G)$. Then $uv \notin E(G^{-+})$ and $ue, ve \in E(G^{-+})$. Thus uev is a (u,v)-path in G^{-+} and hence $d_{G^{-+}}(u,v) = 2$.

(ii). Let $u \in V(G)$ and $e \in E(G)$. By the definition of G^{-+}, u and e are incident in G if and only if u and e are adjacent in G^{-+}. If $u \in e$, then $ue \in E(G^{-+})$ and hence $d_{G^{-+}}(u,e) = 1$. Suppose $u \notin e$. Then $e = vw$ for some $v, w \in V(G) \setminus \{u\}$. Since G is triangle free graph we have $uv \notin E(G)$ or $uw \notin E(G)$. By the definition of G^{-+}, $uv \in E(G^{-+})$ or $uw \in E(G^{-+})$. That is, either uve or uwe is a (u,e)-path of length 2 and hence $d_{G^{-+}}(u,e) = 2$.

(iii). Let $e_1, e_2 \in E(G)$. By the definition of G^{-+}, no two edges are adjacent in G^{-+}. That is, $d_{G^{-+}}(e_1,e_2) \geq 2$. Let $e_1 = ab$ and $e_2 = cd$. If e_1 and e_2 are adjacent in G, then e_1we_2 is a (e_1,e_2)-path of lenth 2, where w is the vertex which is incident with both e_1 and e_2. That is, $d_{G^{-+}}(e_1,e_2) = 2$. Suppose e_1 and e_2 are not adjacent in G. Then $\{a,b\} \cap \{c,d\} = \emptyset$. Since G is triangle free graph we have $ac \notin E(G)$ or $ad \notin E(G)$ or $bc \notin E(G)$ or $bd \notin E(G)$. Without loss of generality assume that $ac \notin E(G)$. Then $ac \in E(G^{-+})$ and hence e_1ace_2 is a (e_1,e_2)-path of lenth 3 in G^{-+}. That is, $d_{G^{-+}}(e_1,e_2) = 3$.
(iv) and (v) follows from (i), (ii) and (iii).
Using propositions (1) and (3), we have the following results on distance based indices of G^{-+}.

Theorem 2. Let G be a triangle free graph of order $n \geq 5$ and size m. Then

$$(i).\ WG^{-+}) = 3\binom{m}{2} + \binom{n}{2} + 2nm - M_1(G)$$

$$(ii).\ DD(G^{-+}) = n^3 - 2n^2 + 6m^2 + 2mn^2$$
$$+2mn + n - 6m - 4M_1(G)$$

$$(iii).\ Gut(G^{-+}) = \binom{n}{2}(n-1)^2 + 5m(n-1)^2$$
$$+6m^2 - 2m - 4M_1(G)$$

$$(iv).\ ECI(G^{-+})$$
$$= \begin{cases} 2n^2 + 4m - 2n & \text{if } \triangle(L(G)) = m-1, \\ 2n^2 + 6m - 2n & \text{if } \triangle(L(G)) \leq m-2 \end{cases}$$

(v). If $\triangle(L(G)) = m-1$, then $EDS(G^{-+}) = 2n^2 + 6m^2 + 8mn - 2n - 6m - 2M_1(G)$ and if $\triangle(L(G)) \leq m-2$, then $EDS(G^{-+}) = 2n^2 + 9m^2 + 10mn - 2n - 9m - 3M_1(G)$, where $M_1(G)$ is the first Zagreb index of G and $L(G)$ is the line graph of G.

Proof. (i). $W(G^{-+})$

$$= \sum_{\{u,v\} \subseteq V(G^{-+})} d_{G^{-+}}(u,v)$$

$$= \sum_{\{u,v\} \subseteq V(G)} d_{G^{-+}}(u,v) + \sum_{\substack{u \in V(G) \\ e \in E(G)}} d_{G^{-+}}(u,e)$$
$$+ \sum_{\{e_1,e_2\} \subseteq E(G)} d_{G^{-+}}(e_1,e_2)$$

$$= \sum_{\substack{\{u,v\} \subseteq V(G) \\ uv \in E(G)}} d_{G^{-+}}(u,v) + \sum_{\substack{\{u,v\} \subseteq V(G) \\ uv \notin E(G)}} d_{G^{-+}}(u,v)$$
$$+ \sum_{\substack{u \in V(G) \\ e \in E(G), u \in e}} d_{G^{-+}}(u,e) + \sum_{\substack{u \in V(G) \\ e \in E(G), u \notin e}} d_{G^{-+}}(u,e)$$
$$+ \sum_{\substack{\{e_1,e_2\} \subseteq E(G) \\ e_1 \cap e_2 = \emptyset}} d_{G^{-+}}(e_1,e_2) + \sum_{\substack{\{e_1,e_2\} \subseteq E(G) \\ e_1 \cap e_2 \neq \emptyset}} d_{G^{-+}}(e_1,e_2)$$

$$= \sum_{\substack{\{u,v\} \subseteq V(G) \\ uv \in E(G)}} 2 + \sum_{\substack{\{u,v\} \subseteq V(G) \\ uv \notin E(G)}} 1 + \sum_{\substack{u \in V(G) \\ e \in E(G), u \in e}} 1$$
$$+ \sum_{\substack{u \in V(G) \\ e \in E(G), u \notin e}} 2 + \sum_{\substack{\{e_1,e_2\} \subseteq E(G) \\ e_1 \cap e_2 = \emptyset}} 3 + \sum_{\substack{\{e_1,e_2\} \subseteq E(G) \\ e_1 \cap e_2 \neq \emptyset}} 2$$

$$= 2m + \binom{n}{2} - m + 2m + 2(nm - 2m) + 2||L(G)||$$
$$+3\left(\binom{m}{2} - ||L(G)||\right)$$

$$= \binom{n}{2} + 3\binom{m}{2} + 2nm - m - ||L(G)||$$

$$= \binom{n}{2} + 3\binom{m}{2} + 2nm - \frac{1}{2}M_1(G).$$

From the last equality we can easily observe, a simple relation between the Wiener index of the general transformed graph G^{-+} and the first Zagreb index of the underline graph G, which is,

$$W(G^{-+})+\frac{1}{2}M_1(G)=\binom{n}{2}+3\binom{m}{2}+2nm.$$

(ii). $DD(G^{-+})$

$$\begin{aligned}
&= \sum_{\{u,v\}\subseteq V(G^{-+})} (d_{G^{-+}}(u)+d_{G^{-+}}(v))d_{G^{-+}}(u,v)\\
&= \sum_{\{u,v\}\subseteq V(G)} (d_{G^{-+}}(u)+d_{G^{-+}}(v))d_{G^{-+}}(u,v)\\
&\quad+\sum_{\substack{u\in V(G)\\ e\in E(G)}} (d_{G^{-+}}(u)+d_{G^{-+}}(e))d_{G^{-+}}(u,e)\\
&\quad+\sum_{\{e_1,e_2\}\subseteq E(G)} (d_{G^{-+}}(e_1)+d_{G^{-+}}(e_2))d_{G^{-+}}(e_1,e_2)\\
&= \sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\in E(G)}} (d_{G^{-+}}(u)+d_{G^{-+}}(v))d_{G^{-+}}(u,v)\\
&\quad+\sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\notin E(G)}} (d_{G^{-+}}(u)+d_{G^{-+}}(v))d_{G^{-+}}(u,v)\\
&\quad+\sum_{\substack{u\in V(G)\\ e\in E(G),u\in e}} (d_{G^{-+}}(u)+d_{G^{-+}}(e))d_{G^{-+}}(u,e)\\
&\quad+\sum_{\substack{u\in V(G)\\ e\in E(G),u\notin e}} (d_{G^{-+}}(u)+d_{G^{-+}}(e))d_{G^{-+}}(u,e)\\
&\quad+\sum_{\substack{\{e_1,e_2\}\subseteq E(G)\\ e_1\cap e_2\neq\emptyset}} (d_{G^{-+}}(e_1)+d_{G^{-+}}(e_2))(d_{G^{-+}}(e_1,e_2)\\
&\quad+\sum_{\substack{\{e_1,e_2\}\subseteq E(G)\\ e_1\cap e_2=\emptyset}} (d_{G^{-+}}(e_1)+d_{G^{-+}}(e_2))(d_{G^{-+}}(e_1,e_2)\\
&= \sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\in E(G)}} 4(n-1)+\sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\notin E(G)}} 2(n-1)\\
&\quad+\sum_{\substack{u\in V(G)\\ e\in E(G),u\in e}} (n+1)+\sum_{\substack{u\in V(G)\\ e\in E(G),u\notin e}} 2(n+1)\\
&\quad+\sum_{\substack{\{e_1,e_2\}\subseteq E(G)\\ e_1\cap e_2\neq\emptyset}} 8+\sum_{\substack{\{e_1,e_2\}\subseteq E(G)\\ e_1\cap e_2=\emptyset}} 12\\
&= 4m(n-1)+2(n-1)\left(\binom{n}{2}-m\right)+2m(n+1)\\
&\quad+2(n+1)(nm-2m)+8||L(G)||+12\left(\binom{m}{2}-||L(G)||\right)\\
&= -2M_1(G)+n^3-2n^2+2n^2m+6m^2+2mn+n-6m.
\end{aligned}$$

From the last equality we can easily observe, a simple relation between the degree distance of the general transformed graph G^{-+} and the first Zagreb index of the underline graph G, which is,

$$\begin{aligned}
DD(G^{-+})+2M_1(G)&=n^3-2n^2+2n^2m+6m^2\\
&\quad+2mn+n-6m.
\end{aligned}$$

(iii). $Gut(G^{+-})$

$$\begin{aligned}
&= \sum_{\{u,v\}\subseteq V(G^{-+})} (d_{G^{-+}}(u)d_{G^{-+}}(v))d_{G^{-+}}(u,v)\\
&= \sum_{\{u,v\}\subseteq V(G)} (d_{G^{-+}}(u)d_{G^{-+}}(v))d_{G^{-+}}(u,v)\\
&\quad+\sum_{\substack{u\in V(G)\\ e\in E(G)}} (d_{G^{-+}}(u)d_{G^{-+}}(e))d_{G^{-+}}(u,e)\\
&\quad+\sum_{\{e_1,e_2\}\subseteq E(G)} (d_{G^{-+}}(e_1)d_{G^{-+}}(e_2))d_{G^{-+}}(e_1,e_2)\\
&= \sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\in E(G)}} (d_{G^{-+}}(u)d_{G^{-+}}(v))d_{G^{-+}}(u,v)\\
&\quad+\sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\notin E(G)}} (d_{G^{-+}}(u)d_{G^{-+}}(v))d_{G^{-+}}(u,v)\\
&\quad+\sum_{\substack{u\in V(G)\\ e\in E(G),u\in e}} (d_{G^{-+}}(u)d_{G^{-+}}(e))d_{G^{-+}}(u,e)\\
&\quad+\sum_{\substack{u\in V(G)\\ e\in E(G),u\notin e}} (d_{G^{-+}}(u)d_{G^{-+}}(e))d_{G^{-+}}(u,e)\\
&\quad+\sum_{\substack{\{e_1,e_2\}\subseteq E(G)\\ e_1\cap e_2\neq\emptyset}} (d_{G^{-+}}(e_1)d_{G^{-+}}(e_2))d_{G^{-+}}(e_1,e_2)\\
&\quad+\sum_{\substack{\{e_1,e_2\}\subseteq E(G)\\ e_1\cap e_2=\emptyset}} (d_{G^{-+}}(e_1)d_{G^{-+}}(e_2))d_{G^{-+}}(e_1,e_2)\\
&= \sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\in E(G)}} 2(n-1)^2+\sum_{\substack{\{u,v\}\subseteq V(G)\\ uv\notin E(G)}} (n-1)^2
\end{aligned}$$

$$+\sum_{\substack{u\in V(G)\\ e\in E(G), u\in e}} 2(n-1) + \sum_{\substack{u\in V(G)\\ e\in E(G), u\notin e}} 4(n-1)$$
$$+\sum_{\substack{\{e_1,e_2\}\subseteq E(G)\\ e_1\cap e_2\neq\emptyset}} 8 + \sum_{\substack{\{e_1,e_2\}\subseteq E(G)\\ e_1\cap e_2=\emptyset}} 12$$
$$= 2m(n-1)^2+(n-1)^2\left(\binom{n}{2}-m\right)+4m(n-1)$$
$$+4(n-1)(nm-2m)+8||L(G)||$$
$$+12\left(\binom{m}{2}-||L(G)||\right)$$
$$= -2M_1(G)+\binom{n}{2}(n-1)^2$$
$$+5m(n-1)^2+6m^2-2m.$$

From the last equality we can easily observe, a simple relation between the Gutman index of the general transformed graph G^{-+} and the first Zagreb index of the underline graph G, which is,

$$Gut(G^{+-})+2M_1(G)=\binom{n}{2}(n-1)^2+5m(n-1)^2$$
$$+6m^2-2m.$$

(iv). $ECI(G^{-+})$

$$= \sum_{v\in V(G^{-+})} ecc_{G^{-+}}(v)d_{G^{-+}}(v)$$
$$= \sum_{v\in V(G)} ecc_{G^{-+}}(v)d_{G^{-+}}(v)$$
$$+\sum_{e\in E(G)} ecc_{G^{-+}}(e)d_{G^{-+}}(e).$$

If $\triangle(L(G)) = m-1$, then $ecc(e) = 2$ for all $e\in E(G)$. In this case,

$$\begin{aligned} ECI(G^{-+}) &= \sum_{v\in V(G)} ecc_{G^{-+}}(v)d_{G^{-+}}(v) \\ &\quad + \sum_{e\in E(G)} ecc_{G^{-+}}(e)d_{G^{-+}}(e) \\ &= \sum_{v\in V(G)} 2(n-1) + \sum_{e\in E(G)} 2(2) \\ &= 2n(n-1)+4m \\ &= 2n^2+4m-2n. \end{aligned}$$

If $\triangle(L(G)) \leq m-2$, then $ecc(e) = 3$ for all $e\in E(G)$. In this case,

$$\begin{aligned} ECI(G^{-+}) &= \sum_{v\in V(G)} ecc_{G^{-+}}(v)d_{G^{-+}}(v) \\ &\quad + \sum_{e\in E(G)} ecc_{G^{-+}}(e)d_{G^{-+}}(e) \\ &= \sum_{v\in V(G)} 2(n-1)+\sum_{e\in E(G)} 3(2) \\ &= 2n(n-1)+6m \\ &= 2n^2+6m-2n. \end{aligned}$$

(v). Let $u\in V(G)$. Then

$$D_{G^{-+}}(u)$$
$$= \sum_{v\in V(G^{-+})} d_{G^{-+}}(u,v)$$
$$= \sum_{v\in V(G)} d_{G^{-+}}(u,v) + \sum_{e\in E(G)} d_{G^{-+}}(u,e)$$
$$= \sum_{\substack{v\in V(G)\\ v\in N_G(u)}} d_{G^{-+}}(u,v) + \sum_{\substack{v\in V(G)\\ v\notin N_G(u)}} d_{G^{-+}}(u,v)$$
$$+\sum_{\substack{e\in E(G)\\ e\in u}} d_{G^{-+}}(u,e) + \sum_{\substack{e\in E(G)\\ e\notin u}} d_{G^{-+}}(u,e)$$
$$= 2d_G(u)+(n-1-d_G(u))+d_G(u)+2(m-d_G(u))$$
$$= 2m+n-1.$$

Let $e\in E(G)$. Then

$$D_{G^{-+}}(e)$$
$$= \sum_{v\in V(G^{-+})} d_{G^{-+}}(e,v)$$
$$= \sum_{v\in V(G)} d_{G^{-+}}(e,v)+\sum_{f\in E(G)} d_{G^{-+}}(e,f)$$
$$= \sum_{\substack{v\in V(G)\\ v\in e}} d_{G^{-+}}(e,v) + \sum_{\substack{v\in V(G)\\ v\notin e}} d_{G^{-+}}(e,v)$$
$$+\sum_{\substack{f\in E(G)\\ f\in N_{L(G)}(e)}} d_{G^{-+}}(e,f)+\sum_{\substack{f\in E(G)\\ f\notin N_{L(G)}(e)}} d_{G^{-+}}(e,f)$$
$$= 2(1)+2(n-2)+2d_{L(G)}(e)$$
$$+3(m-1-d_{L(G)}(e))$$
$$= 2n+3m-d_{L(G)}(e)-5.$$

$$
\begin{aligned}
&EDS(G^{-+})\\
&= \sum_{v\in V(G^{-+})} ecc_{G^{-+}}(v)D_{G^{-+}}(v)\\
&= \sum_{v\in V(G)} ecc_{G^{-+}}(v)D_{G^{-+}}(v)\\
&\quad + \sum_{e\in E(G)} ecc_{G^{-+}}(e)D_{G^{-+}}(e)\\
&= \sum_{v\in V(G)} ecc_{G^{-+}}(v)(2m+n-1)\\
&\quad + \sum_{e\in E(G)} ecc_{G^{-+}}(e)(2n+3m-d_{L(G)}(e)-5).
\end{aligned}
$$

We have two cases to be considered.

Case 1: $\triangle(L(G)) = m-1$. In this case, we have $ecc(e) = 2$ for all $e \in E(G)$. Thus

$$
\begin{aligned}
&EDS(G^{-+})\\
&= \sum_{v\in V(G)} ecc_{G^{-+}}(v)(2m+n-1)\\
&\quad + \sum_{e\in E(G)} ecc_{G^{-+}}(e)(2n+3m-d_{L(G)}(e)-5)\\
&= \sum_{v\in V(G)} 2(2m+n-1)\\
&\quad + \sum_{e\in E(G)} 2(2n+3m-d_{L(G)}(e)-5)\\
&= 2n(2m+n-1)\\
&\quad +2m(2n+3m-5)\\
&\quad -2 \sum_{e\in E(G)} d_{L(G)}(e)\\
&= 2n(2m+n-1)+2m(2n+3m-5)\\
&\quad -2 \sum_{uv\in E(G)} (d_G(u)+d_G(v)-2)\\
&= 2n(2m+n-1)+2m(2n+3m-5)+4m\\
&\quad -2 \sum_{v\in V(G)} d_G^2(v)\\
&= 2n^2+6m^2+8mn-2n-6m-2M_1(G).
\end{aligned}
$$

From the last equality we can easily observe, a simple relation between the eccentric distance sum of the general transformed graph G^{-+} and the first Zagreb index of the underline graph G, which is,
$EDS(G^{-+})+2M_1(G) = 2n^2+6m^2+8mn-2n-6m$ if $\triangle(L(G)) = m-1$.

Case 2: $\triangle(L(G)) \leq m-2$. In this case, we have $ecc(e) = 3$ for all $e \in E(G)$. Thus

$$
\begin{aligned}
&EDS(G^{-+})\\
&= \sum_{v\in V(G)} ecc_{G^{-+}}(v)(2m+n-1)\\
&\quad + \sum_{e\in E(G)} ecc_{G^{-+}}(e)(2n+3m-d_{L(G)}(e)-5)\\
&= \sum_{v\in V(G)} 2(2m+n-1)\\
&\quad + \sum_{e\in E(G)} 3(2n+3m-d_{L(G)}(e)-5)\\
&= 2n(2m+n-1)+3m(2n+3m-5)\\
&\quad -3 \sum_{e\in E(G)} d_{L(G)}(e)\\
&= 2n(2m+n-1)+3m(2n+3m-5)\\
&\quad -3 \sum_{uv\in E(G)} (d_G(u)+d_G(v)-2)\\
&= 2n(2m+n-1)+3m(2n+3m-5)+6m\\
&\quad -3 \sum_{v\in V(G)} d_G^2(v)\\
&= 2n^2+9m^2+10mn-2n-9m-3M_1(G).
\end{aligned}
$$

From the last equality we can easily observe, a simple relation between the eccentric distance sum of the general transformed graph G^{-+} and the first Zagreb index of the underline graph G, which is,
$EDS(G^{-+}) + 3M_1(G) = 2n^2 + 9m^2 + 10mn - 2n - 9m$ if $\triangle(L(G)) \leq m-2$.

Remark

(1) By Theorem (1), for all graphs G_1 and G_2, with $\delta(G_1) \geq 2$, $\delta(G_2) \geq 2$ having same number of vertices and same number of edges, we have
1.1: $W(G_1^{+-}) = W(G_2^{+-})$.
1.2: $DD(G_1^{+-}) = DD(G_2^{+-})$.
1.3: $Gut(G_1^{+-}) = Gut(G_2^{+-})$.
1.4: $ECI(G_1^{+-}) = ECI(G_2^{+-})$.
1.5: $EDS(G_1^{+-}) = EDS(G_2^{+-})$.

(2) By Theorem (2), for all triangle free graphs G_1 and G_2 of order at least five, having same number of vertices and same number of edges, we have
2.1: $W(G_1^{-+}) = W(G_2^{-+})$.
2.2: $DD(G_1^{-+}) = DD(G_2^{-+})$.
2.3: $Gut(G_1^{-+}) = Gut(G_2^{-+})$.
2.4: $ECI(G_1^{-+}) = ECI(G_2^{-+})$.
2.5: $EDS(G_1^{-+}) = EDS(G_2^{-+})$.

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