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ON A DEGENERATE GENERALIZED INTEGRAL TRANSFORM: PROPERTIES AND APPLICATIONS

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Abstract. In this work, we introduce a degenerate generalized integral transform that unifies classical transforms (Laplace, Sumudu, Elzaki, etc.) with their degenerate counterparts. The proposed transform is defined by

$$\mathtt{H}_\lambda\{\varphi(t)\}(r) = u(r)\int_0^\infty (1+\lambda t)^{-r^\vartheta/\lambda}\varphi(t)\,dt,$$

where $\lambda \in (0,\infty)$, $u(r) \neq 0$, $\vartheta \in \mathbb{Z}$, and $r \in \mathbb{C}$. In the limit $\lambda \to 0$, the classical transforms are recovered, and the degenerate Laplace, Sumudu, and Elzaki transforms are included as special cases. We establish fundamental properties including existence, linearity, differentiation, and convolution theorems. Additionally, we derive transform pairs for elementary functions and special functions such as the Fox-Wright function, the k-Gauss hypergeometric function, and the p-k-Mittag-Leffler function. Applications to variable coefficient initial value problems are demonstrated. The novelty of this work lies in the unification of classical and degenerate transforms within a single framework, providing researchers flexibility to select transforms tailored to specific problems while maintaining solution consistency.

Key words/phrases: Degenerate integral transforms; Generalized integral transform; Degenerate gamma function; Laplace transform

1. Introduction

For a long time, mathematicians and engineers have been interested in the concept of integral transforms. An integral transformation is a mathematical operation that uses integration with a given kernel to convert a function from its original domain into a different functional space. It is often simpler to analyze and manipulate some of the original properties of functions within this transformed space. An associated inverse transformation, which reconstructs the original function from its image, makes the process reversible. The role of integral transform in mathematics has grown recently, and many authors have studied them in detail [1, 2].

Formally, an integral transform acts on a function $\varphi(t)$ and produces a new function $\Phi(r)$ defined on $t_1 \leq t \leq t_2$, expressed as:

$$T\{\varphi(t)\}(r) = \Phi(r) = \int_{t_1}^{t_2} K(t,r)\varphi(t)\,dt, \tag{1.1}$$

provided the integral exists, where $K(t,r)$ is the kernel of the transform, which depends on both the integration variable t and a complex transform variable r, T is the integral transform operator, and $\Phi(r)$ is the image of $\varphi(t)$. The operator T is characterized by its specific kernel, and an inverse kernel enables the recovery of the original function. In order to find $\varphi(t)$ from the given $\Phi(r)$, we introduce the inverse operator T^{-1}:

$$T^{-1}\{\Phi(r)\}(t) = \varphi(t).$$

Based on (1.1), the Fourier and the Laplace transforms of a function $\varphi(t)$ can be written, respectively, as:

$$F\{\varphi(t)\}(r) = \int_{-\infty}^{\infty} e^{-irx}\varphi(t)\,dt \tag{1.2}$$

and

$$L\{\varphi(t)\}(r) = \int_{0}^{\infty} e^{-rt}\varphi(t)\,dt. \tag{1.3}$$

The existence of the integral in Equation (1.3) requires that the function $\varphi(t)$ be of exponential order C, such that for positive constants M and T:

$$|\varphi(t)| \leq Me^{Ct}, \quad \text{for all } t > T.$$

Several differential and integral problems, including ordinary, partial and fractional differential equations, have been approximated using these transformations. A number of scientific and engineering fields, including communication engineering, control engineering, electric analysis, quantum physics, partial differential equation solving, data mining, machine learning, signal processing, and integrated circuits, depend heavily on the concept of integral transformations.

Many mathematical equations, such as fractional kinetic equations, linear and nonlinear Volterra integral equations, partial differential equations, linear and nonlinear ordinary differential equations and many more, have been approximated using various kinds of integral transformations [3]. Integral transformations are necessary tools for solving mathematical and engineering problems. Integral transforms and their generalizations have numerous beneficial applications in a wide range of scientific fields, including applied sciences, physics, biology, chemistry, and engineering. Integral transformations are beneficial because they can simplify the problem-solving process and speed up the computation of solutions. To better understand nature, a number of mathematicians and engineers developed a number of analytical and numerical techniques that are applied in numerous scientific and technological fields. Specifically, it has been shown that the idea of integral transformation is a helpful mathematical tool for addressing a variety of issues in both pure and applied mathematics [4, 5]. It is crucial to recall that an integral transform occurs when a mathematical operator uses an integral to change a function from one function space to another.

The study of ordinary and partial differential equations has a strong link to the historical development of integral transforms. The development of integral transformations can be linked to the development of integral calculus and the need to solve difficult problems involving differential equations and other mathematical processes [6]. The integral transform is given its name to the famous mathematicians Laplace (1780) and Fourier (1822). Integral transforms were developed by Laplace, Euler, Fourier, Mellin, and others. The Laplace and Fourier integrals are the most commonly used transforms. In the fields of mathematics and engineering, integral transformations have proven essential for solving differential or integro-differential problems. The Mittag-Leffler function, the Struve function, the generalized Galué-type Struve function, the Hurwitz-Lerch zeta function, the extended hypergeometric function, and confluent hypergeometric functions are just a few of the special functions that researchers have recently discussed using integral transforms to solve fractional differential equations. In recent years, several alternative integrals have been proposed in addition to the Laplace, Sumudu, and Elzaki transforms. These include the Aboodh transform [7], Natural transform [8], Mohand transform [9], Pourreza transform [10], Kamal transform [11], Sawi transform [12], and Emad-Sara transform [13]. These integral transformations have found widespread applications in many disciplines, including mathematics, engineering, signal processing, wave propagation, control engineering, optics, and physics.

Modifications of integral transforms have been the subject of extensive recent research. For example, the authors of [14] developed certain relations and features of the modified Laplace transform. Duran et al. [15] introduced a modified Sumudu integral transform and listed many of its properties. The Elzaki transform was modified by the author of [16]. He et al. introduced a generalized integral transform in [17], which generalizes the Fourier, Laplace, Sumudu, and other transforms. In [18], Mustafa offered a modified iterative method and a modified Sumudu transform for solving partial differential equation systems numerically. When combined with semi-analytical methods like the variational iteration method, the homotopy perturbation method, and the Adomian decomposition method, these integral transformations can be applied to both linear and nonlinear problems [19].

In recent years, degenerate versions of existing integral transforms have also been studied. For instance, Kim and Kim [20] introduced the degenerate Laplace transform:

$$\begin{aligned} L_\lambda\{f(t)\}(s) = F_\lambda(s) &= \int_0^\infty e_\lambda^{-s}(t) f(t)\,dt \\ &= \int_0^\infty (1+\lambda t)^{-s/\lambda} f(t)\,dt, \end{aligned} \tag{1.4}$$

provided the integral exists, where $e_\lambda^{-s}(t)$ is the degenerate exponential function, and $\lim_{\lambda\to 0} L_\lambda\{f(t)\} = L\{f(t)\}$. For the degenerate

Laplace transform and degenerate gamma function, the authors provided lots of properties and interesting formulas. Following that, they defined a modified degenerate Laplace transform with different properties and a modified degenerate gamma function [21]. Many properties of the degenerate gamma function associated with analytic continuation as a meromorphic function on the complex plane were obtained by Kim and Kim [20], including the difference formula, values at nonnegative integers, an integral representation as an integral along a Hankel contour, and several expressions derived from the Euler and Weierstrass formulae for the ordinary gamma function. Upadhyaya [22] described the characteristics of a degenerate Laplace-type integral transform. The degenerate Sumudu transform was defined by Duran [23]:

(1.5)
$$S_\lambda[f(t)](s) = T_\lambda(s) = \frac{1}{s}\int_0^\infty e_\lambda^{-1/s}(t) f(t)\,dt,$$

provided the integral exists, and $\lim_{\lambda\to 0} S_\lambda[f(t)] = S[f(t)]$. Kalavathi et al. [24] defined the degenerate Elzaki transform:

(1.6)
$$E_\lambda[f(t)](s) = T_\lambda(s) = s\int_0^\infty e_\lambda^{-1/s}(t) f(t)\,dt,$$

provided the integral exists, and $\lim_{\lambda\to 0} E_\lambda[f(t)] = E[f(t)]$. For applications involving fractional kinetic equations, Yahya Almalki et al. [25] obtained significant results on the modified degenerate Laplace-type integral. For the solvability of partial differential equations involving Laplace operator please refer [33]

The paper is organized as follows: Section 2 presents the fundamental concepts and notations. Section 3 introduces the proposed degenerate generalized integral transform and establishes its basic properties. Section 4 provides applications to variable coefficient ordinary differential equations; and finally, Section 5 concludes the paper with a summary and future research directions.

2. Fundamental concepts

The notations and basic terms of the gamma function, degenerate gamma function, degenerate exponential function, degenerate Euler formula, and degenerate trigonometric functions that are necessary for this work are introduced in this section. These ideas act as the fundamental components for our degenerate generalized integral transform. We start with the classical gamma function, which is essential to the theory of special functions and integral transforms.

Definition 2.1 (Gamma Function [26]). The Euler gamma function is defined by:

(2.1)
$$\Gamma(\xi) = \int_0^\infty x^{\xi-1} e^{-x}\,dx, \quad \xi\in\mathbb{C},\ \Re(\xi) > 0.$$

It satisfies $\Gamma(\xi+1) = \xi\Gamma(\xi)$ and $\Gamma(n+1) = n!$ for $n\in\mathbb{N}$.

Definition 2.2 (Pochhammer Symbol [26]). The Pochhammer symbol $(\gamma)_\ell$ is given by:

(2.2)
$$(\gamma)_\ell = \gamma(\gamma+1)\cdots(\gamma+\ell-1) = \frac{\Gamma(\gamma+\ell)}{\Gamma(\gamma)}, \quad \gamma\in\mathbb{C},\ \Re(\gamma) > 0.$$

It satisfies $(\gamma)_0 = 1$.

Definition 2.3 (Degenerate Exponential Function [21]). For $\lambda\in(0,\infty)$ and $t\in\mathbb{R}$, the degenerate exponential function is defined as:

$$e_\lambda^t = (1+\lambda t)^{1/\lambda}. \tag{2.3}$$

Notably, $\lim_{\lambda\to 0^+} e_\lambda^t = \lim_{\lambda\to 0^+}(1+\lambda t)^{1/\lambda} = e^t$.

Definition 2.4 (Degenerate Euler Formula [21, 20]). The degenerate Euler formula is given by:

$$e_\lambda^{it} = (1+\lambda t)^{i/\lambda} = \cos_\lambda(t) + i\sin_\lambda(t), \tag{2.4}$$

where $i=\sqrt{-1}$. Consequently, $\lim_{\lambda\to 0^+}\cos_\lambda(t) = \cos t$ and $\lim_{\lambda\to 0^+}\sin_\lambda(t) = \sin t$. Therefore,

$$\lim_{\lambda\to 0^+} e_\lambda^{it} = \lim_{\lambda\to 0^+}(1+\lambda t)^{i/\lambda} = e^{it} = \cos t + i\sin t.$$

Definition 2.5 (Degenerate Gamma Function [20, 21]). For $\lambda\in(0,\infty)$ and $\xi\in\mathbb{C}$ with $0<\Re(\xi)<1/\lambda$, the degenerate gamma function is defined as:

(2.5)
$$\Gamma_\lambda(\xi) = \int_0^\infty (1+\lambda t)^{-1/\lambda} t^{\xi-1}\,dt = \int_0^\infty e_\lambda^{-t} t^{\xi-1}\,dt.$$

The degenerate gamma function satisfies the following properties [20]:

- $\lim_{\lambda\to 0^+}\Gamma_\lambda(\xi) = \Gamma(\xi)$, recovering the classical gamma function.
- For $m\in\mathbb{N}$,

$$\Gamma_\lambda(m) = \frac{(m-1)!}{(1-\lambda)(1-2\lambda)\cdots(1-m\lambda)}.$$

Definition 2.6 (Degenerate Trigonometric Functions [20, 21]). The degenerate sine and cosine functions are defined as:

(2.6)
$$\cos_\lambda(at) = \frac{e_\lambda^{ia}(t) + e_\lambda^{-ia}(t)}{2}, \quad \sin_\lambda(at) = \frac{e_\lambda^{ia}(t) - e_\lambda^{-ia}(t)}{2i}.$$

The degenerate hyperbolic sine and cosine functions are given by:

$$\cosh_\lambda(at) = \frac{e_\lambda^{a}(t) + e_\lambda^{-a}(t)}{2}, \tag{2.7}$$

$$\sinh_\lambda(at) = \frac{e_\lambda^{a}(t) - e_\lambda^{-a}(t)}{2}. \tag{2.8}$$

3. The Degenerate Generalized Integral Transform

He, J.-H., et al. [17] considered the generalized integral transform of an integrable function $\varphi(t)$ as follows:

$$\mathrm{H}\{\varphi(t)\}(r) = \mathrm{H}(r) = u(r)\int_0^\infty e^{-r^\vartheta t}\varphi(t)\,dt, \quad \text{where } r\in\mathbb{C},\ \vartheta\in\mathbb{Z},\ u(r)\neq 0. \tag{3.1}$$

Informed by the generalization given in Equation (3.1), we present a degenerate generalized integral transform incorporating various degenerate integral transforms and their properties. The operator $\mathrm{H}_\lambda\{\varphi(t)\}(r)$ represents the degenerate generalized integral transform throughout this article.

Definition 3.1. Let $\varphi(t)$ be an integrable function defined for $t\geq 0$, $u(r)\neq 0$, $r\in\mathbb{C}$, $\lambda\in(0,\infty)$, and $\vartheta\in\mathbb{Z}$. The *degenerate generalized integral transform* is defined as:

$$\mathrm{H}_\lambda\{\varphi(t)\}(r) = u(r)\int_0^\infty (1+\lambda t)^{-r^\vartheta/\lambda}\varphi(t)\,dt, \tag{3.2}$$

provided that the improper integral converges for some r^ϑ with $\Re(r)>0$ and $r^\vartheta>0$.

The degenerate exponential function $(1+\lambda t)^{-r^\vartheta/\lambda}$ is the kernel of the transformation, with r is the transform variable. Note that when $\lambda\to 0$, this transform reduces to Equation (3.1). That is,

$$\begin{aligned}
\lim_{\lambda\to 0}&\mathrm{H}_\lambda\{\varphi(t)\}(r)\\
&= u(r)\lim_{\lambda\to 0}\int_0^\infty (1+\lambda t)^{-r^\vartheta/\lambda}\varphi(t)\,dt\\
&= u(r)\int_0^\infty \left(\lim_{\lambda\to 0}(1+\lambda t)^{-r^\vartheta/\lambda}\right)\varphi(t)\,dt\\
&= u(r)\int_0^\infty e^{-r^\vartheta t}\varphi(t)\,dt\\
&= \mathrm{H}\{\varphi(t)\}(r).
\end{aligned}$$

This generalization is motivated by the degenerate exponential function $e_\lambda^t = (1+\lambda t)^{1/\lambda}$.

Definition 3.2 (Degenerate Exponential Order [20, 21]). A function $\varphi(t)$ is said to be of degenerate exponential order C if there exist constants $M>0$, $C>0$, and $T>0$ such that:

$$|\varphi(t)| \leq M(1+\lambda t)^{C/\lambda}, \quad \text{for all } t>T. \tag{3.3}$$

Theorem 3.3. *Let $\varphi(t)$ be a piecewise continuous function on $[0,\infty)$ of degenerate exponential order C. Then the degenerate generalized integral transform $\mathrm{H}_\lambda\{\varphi(t)\}(r)$ exists for all $r\in\mathbb{C}$ such that $r^\vartheta > C+\lambda$.*

Proof. Using the definition of the transform in Equation (3.2) and the triangle inequality:

$$\begin{aligned}
|\mathrm{H}_\lambda\{\varphi(t)\}(r)| &= \left|u(r)\int_0^\infty (1+\lambda t)^{-r^\vartheta/\lambda}\varphi(t)\,dt\right|\\
&\leq |u(r)|\int_0^\infty \left|(1+\lambda t)^{-r^\vartheta/\lambda}\right||\varphi(t)|\,dt.
\end{aligned} \tag{3.4}$$

Since $\varphi(t)$ is of degenerate exponential order C, there exists $M>0$ such that:

$$\begin{aligned}
|\mathrm{H}_\lambda\{\varphi(t)\}(r)| &\leq |u(r)|M\int_0^\infty (1+\lambda t)^{-(r^\vartheta-C)/\lambda}\,dt\\
&= \frac{|u(r)|M}{r^\vartheta-(\lambda+C)}, \quad r^\vartheta>\lambda+C.
\end{aligned} \tag{3.5}$$

This proves the existence of the transform in Equation (3.2). When $\lambda\to 0$, the degenerate exponential order reduces to the classical exponential order $|\varphi(t)|\leq Me^{Ct}$, and the existence condition becomes $r^\vartheta > C$, recovering the classical results of He et al. [17].

Theorem 3.4. *Let $\varphi(t)$ be piecewise continuous on $[0,\infty)$ of degenerate exponential order C, and let $F_\lambda(r) = \mathrm{H}_\lambda\{\varphi(t)\}(r)$. Then the inverse transform is given by:*

$$\varphi(t) = \frac{\vartheta}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{F_\lambda(r)}{u(r)}(1+\lambda t)^{r^\vartheta/\lambda-1}r^{\vartheta-1}\,dr,$$

where $\gamma > C+\lambda$ and $u(r)\neq 0$ on the contour.

Proof. Using $x=\frac{1}{\lambda}\ln(1+\lambda t)$ and $\psi(x) = \varphi\left(\frac{e^{\lambda x}-1}{\lambda}\right)e^{\lambda x}$, we have $F_\lambda(r) = u(r)L\{\psi(x)\}(r^\vartheta)$. Applying the Laplace inversion formula with $s=r^\vartheta$, $ds=\vartheta r^{\vartheta-1}dr$, and substituting back $\varphi(t) = e^{-\lambda x}\psi(x)$ yields

$$\varphi(t) = \frac{\vartheta}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{F_\lambda(r)}{u(r)}(1+\lambda t)^{r^\vartheta/\lambda-1}r^{\vartheta-1}\,dr.$$

Special Cases.

Corollary 3.5. *When $u(r)=1$ and $\vartheta=1$, the degenerate generalized transform in Equation (3.2) reduces to the degenerate Laplace transform of Kim and Kim [21, 20]:*

$$L_\lambda\{\varphi(t)\}(r) = \int_0^\infty (1+\lambda t)^{-r/\lambda}\varphi(t)\,dt. \tag{3.6}$$

The existence condition becomes $r > C+\lambda$, with the bound:

$$|L_\lambda\{\varphi(t)\}(r)| \leq \frac{M}{r-C-\lambda}. \tag{3.7}$$

Corollary 3.6. *When $\lambda\to 0$, $u(r)=1$, and $\vartheta=1$, we recover the classical Laplace transform:*

$$\lim_{\lambda\to 0^+} L_\lambda\{\varphi(t)\}(r) = L\{\varphi(t)\}(r) = \int_0^\infty e^{-rt}\varphi(t)\,dt. \tag{3.8}$$

Corollary 3.7. *When $u(r) = 1/r$ and $\vartheta = -1$, the degenerate generalized transform in Equation (3.2) reduces to the degenerate Sumudu transform of Duran [23]:*

$$S_\lambda\{\varphi(t)\}(r) = \frac{1}{r}\int_0^\infty (1+\lambda t)^{-1/(\lambda r)}\varphi(t)\,dt. \tag{3.9}$$

Corollary 3.8. *When $\lambda \to 0$, $u(r) = 1/r$, and $\vartheta = -1$, we recover the classical Sumudu transform:*

$$\begin{aligned}\lim_{\lambda\to 0^+} S_\lambda\{\varphi(t)\}(r) &= S\{\varphi(t)\}(r)\\ &= \frac{1}{r}\int_0^\infty e^{-t/r}\varphi(t)\,dt.\end{aligned} \tag{3.10}$$

Corollary 3.9. *When $u(r) = r$ and $\vartheta = -1$, the degenerate generalized transform (3.2) reduces to the degenerate Elzaki transform of Kalavathi et al. [24]:*

$$E_\lambda\{\varphi(t)\}(r) = r\int_0^\infty (1+\lambda t)^{-1/(\lambda r)}\varphi(t)\,dt. \tag{3.11}$$

Corollary 3.10. *When $\lambda \to 0$, $u(r) = r$, and $\vartheta = -1$, we recover the classical Elzaki transform:*

$$\begin{aligned}\lim_{\lambda\to 0^+} E_\lambda\{\varphi(t)\}(r) &= E\{\varphi(t)\}(r)\\ &= r\int_0^\infty e^{-t/r}\varphi(t)\,dt.\end{aligned} \tag{3.12}$$

Theorem 3.11. *For any $a, b \in \mathbb{R}$, and functions $\varphi(t)$ and $\varrho(t)$ whose transforms exist, the degenerate generalized integral transform has a linearity property:*

$$\mathtt{H}_\lambda\{a\varphi(t)+b\varrho(t)\} = a\mathtt{H}_\lambda\{\varphi(t)\}+b\mathtt{H}_\lambda\{\varrho(t)\}. \tag{3.13}$$

Using Definition 3.1:

$$\begin{aligned}&\mathtt{H}_\lambda\{a\varphi+b\varrho\}\\ &= u(r)\int_0^\infty (1+\lambda t)^{-r^\vartheta/\lambda}[a\varphi(t)+b\varrho(t)]\,dt\\ &= au(r)\int_0^\infty (1+\lambda t)^{-r^\vartheta/\lambda}\varphi(t)\,dt\\ &\quad + bu(r)\int_0^\infty (1+\lambda t)^{-r^\vartheta/\lambda}\varrho(t)\,dt\\ &= a\mathtt{H}_\lambda\{\varphi(t)\}+b\mathtt{H}_\lambda\{\varrho(t)\}.\end{aligned} \tag{3.14}$$

Therefore,

$$\lim_{\lambda\to 0}\mathtt{H}_\lambda\{a\varphi+b\varrho\} = a\lim_{\lambda\to 0}\mathtt{H}_\lambda\{\varphi(t)\}+b\lim_{\lambda\to 0}\mathtt{H}_\lambda\{\varrho(t)\}. \tag{3.15}$$

Using integration by parts:

$$\begin{aligned}\mathtt{H}_\lambda\{\varphi'(t)\} &= u(r)\int_0^\infty (1+\lambda t)^{-r^\vartheta/\lambda}\varphi'(t)\,dt\\ &= u(r)\lim_{\eta\to\infty}\Big[(1+\lambda t)^{-r^\vartheta/\lambda}\varphi(t)\Big]_0^\eta\\ &\quad + r^\vartheta u(r)\int_0^\infty (1+\lambda t)^{-r^\vartheta/\lambda-1}\varphi(t)\,dt\\ &= -u(r)\varphi(0)+r^\vartheta\mathtt{H}_\lambda\{(1+\lambda t)^{-1}\varphi(t)\}.\end{aligned} \tag{3.16}$$

For the second derivative, we apply the first formula to $\varphi'(t)$:

$$\mathtt{H}_\lambda\{\varphi''(t)\} = r^\vartheta\mathtt{H}_\lambda\{(1+\lambda t)^{-1}\varphi'(t)\}-u(r)\varphi'(0).$$

Now, to compute $\mathtt{H}_\lambda\{(1+\lambda t)^{-1}\varphi'(t)\}$, we set $\varrho(t) = (1+\lambda t)^{-1}\varphi(t)$. Then $\varrho'(t) = -\lambda(1+\lambda t)^{-2}\varphi(t)+(1+\lambda t)^{-1}\varphi'(t)$, so:

$$\begin{aligned}\mathtt{H}_\lambda\{(1+\lambda t)^{-1}\varphi'(t)\} &= \mathtt{H}_\lambda\{\varrho'(t)\}+\lambda\mathtt{H}_\lambda\{(1+\lambda t)^{-2}\varphi(t)\}\\ &= \Big[r^\vartheta\mathtt{H}_\lambda\{(1+\lambda t)^{-1}\varrho(t)\}-u(r)\varrho(0)\Big]\\ &\quad + \lambda\mathtt{H}_\lambda\{(1+\lambda t)^{-2}\varphi(t)\}\\ &= r^\vartheta\mathtt{H}_\lambda\{(1+\lambda t)^{-2}\varphi(t)\}-u(r)\varphi(0)\\ &\quad + \lambda\mathtt{H}_\lambda\{(1+\lambda t)^{-2}\varphi(t)\}\\ &= (r^\vartheta+\lambda)\mathtt{H}_\lambda\{(1+\lambda t)^{-2}\varphi(t)\}-u(r)\varphi(0).\end{aligned} \tag{3.17}$$

Substituting back:

$$\begin{aligned}\mathtt{H}_\lambda\{\varphi''(t)\} &= r^\vartheta\Big[(r^\vartheta+\lambda)\mathtt{H}_\lambda\{(1+\lambda t)^{-2}\varphi(t)\}-u(r)\varphi(0)\Big]-u(r)\varphi'(0)\\ &= r^\vartheta(r^\vartheta+\lambda)\mathtt{H}_\lambda\{(1+\lambda t)^{-2}\varphi(t)\}-u(r)[r^\vartheta\varphi(0)+\varphi'(0)].\end{aligned} \tag{3.18}$$

Continuing this process yields the general formula for $\varphi^{(m)}(t) = \frac{d^m}{dt^m}\varphi(t)$, $m \in \mathbb{N}$:

$$\begin{aligned}\mathtt{H}_\lambda\{\varphi^{(m)}(t)\} &= \left[\prod_{j=0}^{m-1}(r^\vartheta+j\lambda)\right]\mathtt{H}_\lambda\{(1+\lambda t)^{-m}\varphi(t)\}\\ &\quad - u(r)\sum_{i=0}^{m-1}\varphi^{(i)}(0)\prod_{j=0}^{m-i-1}(r^\vartheta+j\lambda)\end{aligned} \tag{3.19}$$

In order to solve more complicated problems, we will now derive the transform pairs for a few basic elementary functions. We start by calculating the transform of the constant function $\varphi(t) = 1$:

$$\mathtt{H}_\lambda\{1\}(r) = u(r)\int_0^\infty (1+\lambda t)^{-r^\vartheta/\lambda}\,dt = u(r)\lim_{R\to\infty}\int_0^R (1+\lambda t)^{-r^\vartheta/\lambda}\,dt.$$

Using the substitution $u = 1+\lambda t$, we obtain:

$$\mathtt{H}_\lambda\{1\}(r) = u(r)\frac{\lambda\Gamma(1+r^\vartheta/\lambda)}{r^\vartheta(-\lambda+r^\vartheta)\Gamma(r^\vartheta/\lambda)}, \quad \text{for } r^\vartheta > \lambda.$$

As a result:

$$\lim_{\lambda\to 0^+}\mathtt{H}_\lambda\{1\}(r) = u(r)\frac{\lambda\cdot(r^\vartheta/\lambda)\Gamma(r^\vartheta/\lambda)}{r^\vartheta(r^\vartheta-\lambda)\Gamma(r^\vartheta/\lambda)} = \frac{u(r)}{r^\vartheta}.$$

Similarly, for $\varphi(t) = t$, the transform yields:

$$\begin{aligned}\mathtt{H}_\lambda\{t\}(r) &= u(r)\int_0^\infty (1+\lambda t)^{-r^\vartheta/\lambda} t\,dt \\ &= u(r)\frac{\lambda\Gamma(1+r^\vartheta/\lambda)}{r^\vartheta(-2\lambda+r^\vartheta)(-\lambda+r^\vartheta)\Gamma(r^\vartheta/\lambda)}, \\ &\quad \text{for } r^\vartheta > 2\lambda.\end{aligned}$$

Thus,

$$\lim_{\lambda\to 0^+} \mathtt{H}_\lambda\{t\}(r) = \frac{u(r)}{r^{2\vartheta}}.$$

For $\varphi(t) = t^2$, we have:

$$\begin{aligned}\mathtt{H}_\lambda\{t^2\}(r) &= u(r)\int_0^\infty (1+\lambda t)^{-r^\vartheta/\lambda} t^2\,dt \\ &= \frac{2\lambda u(r)\Gamma(1+r^\vartheta/\lambda)}{\Gamma(r^\vartheta/\lambda)} \\ &\quad \times \frac{1}{r^\vartheta(r^\vartheta-\lambda)(r^\vartheta-2\lambda)(r^\vartheta-3\lambda)}, \\ &\quad \text{for } r^\vartheta > 3\lambda.\end{aligned}$$

In general, for $\varphi(t) = t^m$ with $m \in \mathbb{N}$, the transform is given by the closed-form expression:

$$\begin{aligned}\mathtt{H}_\lambda\{t^m\}(r) &= u(r)\int_0^\infty (1+\lambda t)^{-r^\vartheta/\lambda} t^m\,dt \\ &= \frac{\lambda\Gamma(m+1)\Gamma(1+r^\vartheta/\lambda)u(r)}{r^\vartheta\Gamma(r^\vartheta/\lambda)} \\ &\quad \times \frac{1}{\prod_{j=1}^{m+1}(r^\vartheta-j\lambda)}, \text{ for } r^\vartheta > (m+1)\lambda.\end{aligned}$$

For $\lambda \to 0$, we have:

$$\lim_{\lambda\to 0} \mathtt{H}_\lambda\{t^m\}(r) = \frac{u(r)\Gamma(m+1)}{r^{\vartheta(m+1)}}.$$

Theorem 3.12. *For $\alpha \in \mathbb{R}$ with $\alpha > -1$ and $r^\vartheta > (\alpha+1)\lambda$, the degenerate generalized integral transform of t^α satisfies:*

(3.20)

$$\mathtt{H}_\lambda\{t^\alpha\}(r) = u(r)\frac{\lambda^{\alpha+1}\Gamma(\alpha+1)}{r^{\vartheta(\alpha+1)}}\Gamma_{\lambda/r^\vartheta}(\alpha+1),$$

where Γ_λ is the degenerate gamma function defined by $\Gamma_\lambda(z) = \int_0^\infty (1+\lambda t)^{-1/\lambda} t^{z-1}\,dt$.

Proof. Using Definition (3.2) and the substitution $y = r^\vartheta t/\lambda$:

$$\mathtt{H}_\lambda\{t^\alpha\}(r) = u(r)\int_0^\infty (1+\lambda t)^{-r^\vartheta/\lambda} t^\alpha\,dt.$$

Let $y = \frac{r^\vartheta}{\lambda}t$. Then $t = \frac{\lambda}{r^\vartheta}y$ and $dt = \frac{\lambda}{r^\vartheta}dy$:

$$= u(r)\left(\frac{\lambda}{r^\vartheta}\right)^{\alpha+1}\int_0^\infty (1+\lambda y)^{-r^\vartheta/\lambda} y^\alpha\,dy.$$

Using the definition of the degenerate gamma function $\Gamma_{\lambda/r^\vartheta}(\alpha+1) = \int_0^\infty (1+\lambda y)^{-r^\vartheta/\lambda} y^\alpha\,dy$, we obtain:

$$\mathtt{H}_\lambda\{t^\alpha\}(r) = u(r)\frac{\lambda^{\alpha+1}}{r^{\vartheta(\alpha+1)}}\Gamma_{\lambda/r^\vartheta}(\alpha+1).$$

This completes the proof.

Corollary 3.13. *For $m \in \mathbb{N}_0$, the transform of t^m is:*

$$\mathtt{H}_\lambda\{t^m\}(r) = u(r)\frac{\lambda^{m+1}\Gamma(m+1)}{r^{\vartheta(m+1)}}\Gamma_{\lambda/r^\vartheta}(m+1).$$

Using $\Gamma_{\lambda/r^\vartheta}(m+1) = \frac{r^{\vartheta(m+1)}}{\lambda^{m+1}}\prod_{j=1}^{m+1}(r^\vartheta-j\lambda)^{-1}$, this reduces to the earlier power function formula:

$$\mathtt{H}_\lambda\{t^m\}(r) = \frac{u(r)\Gamma(m+1)}{\prod_{j=1}^{m+1}(r^\vartheta-j\lambda)}.$$

Next, we consider the transform of functions of the form $(1+\lambda t)^{-a/\lambda}$, which serve as the degenerate analogues of exponential functions:

$$\begin{aligned}\mathtt{H}_\lambda\{(1+\lambda t)^{-a/\lambda}\}(r) &= u(r)\int_0^\infty (1+\lambda t)^{-r^\vartheta/\lambda}(1+\lambda t)^{-a/\lambda}\,dt \\ &= \frac{u(r)}{r^\vartheta+a-\lambda}, \\ &\quad \text{for } r^\vartheta > \lambda - a.\end{aligned}$$

As $\lambda \to 0^+$, we have:

$$\lim_{\lambda\to 0^+} \mathtt{H}_\lambda\{(1+\lambda t)^{-a/\lambda}\}(r) = \mathtt{H}\{e^{-at}\}(r) = \frac{u(r)}{r^\vartheta+a}.$$

For $u(r) = 1$, $\vartheta = 1$:

$$L_\lambda\{(1+\lambda t)^{-a/\lambda}\}(r) = \frac{1}{r+a-\lambda}, \quad r > \lambda - a.$$

For $u(r) = 1/r$, $\vartheta = -1$, we have:

$$S_\lambda\{(1+\lambda t)^{-a/\lambda}\}(r) = \frac{1}{1+r(a-\lambda)}.$$

This result is particularly useful as it demonstrates how the degenerate kernel interacts with generalized exponential arguments. The degenerate generalized integral transform of the degenerate exponential function $e_\lambda^a(t) = (1+\lambda t)^{a/\lambda}$ is given by:

$$\begin{aligned}\mathtt{H}_\lambda\{(1+\lambda t)^{a/\lambda}\}(r) &= u(r)\int_0^\infty (1+\lambda t)^{-(r^\vartheta-a)/\lambda}\,dt \\ &= \frac{u(r)}{r^\vartheta-a-\lambda}, \\ &\quad \text{for } r^\vartheta > a+\lambda.\end{aligned}$$

As $\lambda \to 0^+$, we have:

$$\lim_{\lambda\to 0^+} \mathtt{H}_\lambda\{(1+\lambda t)^{a/\lambda}\}(r) = \mathtt{H}\{e^{at}\}(r) = \frac{u(r)}{r^\vartheta-a}.$$

Transform of Degenerate Trigonometric Functions. A significant advantage of the degenerate framework is the natural emergence of degenerate trigonometric functions. We now derive their transforms.

$$\begin{aligned}\mathtt{H}_\lambda\{\cos_\lambda(at)\} &= u(r)\int_0^\infty (1+\lambda t)^{-r^\vartheta/\lambda} \\ &\left[\frac{e_\lambda^{ia}(t)+e_\lambda^{-ia}(t)}{2}\right] dt \\ &= \frac{u(r)}{2}\int_0^\infty \Big[(1+\lambda t)^{-(r^\vartheta-ia)/\lambda} \\ &+ (1+\lambda t)^{-(r^\vartheta+ia)/\lambda}\Big] dt \\ &= \frac{u(r)}{2}\left[\frac{1}{r^\vartheta-ia-\lambda}+\frac{1}{r^\vartheta+ia-\lambda}\right] \\ &= \frac{u(r)(r^\vartheta-\lambda)}{(r^\vartheta-\lambda)^2+a^2}, \quad \text{for } r^\vartheta>\lambda.\end{aligned} \tag{3.21}$$

When $u(r)=r$ and $\vartheta=-1$, we have the following degenerate Elzaki transform of $\cos_\lambda(at)$:

$$E_\lambda\{\cos_\lambda(at)\} = \frac{r^2(1-\lambda r)}{(1-\lambda r)^2+a^2r^2}.$$

For $\lambda\to 0$, we have:

$$\lim_{\lambda\to 0^+}\mathtt{H}_\lambda\{\cos_\lambda(at)\}(r) = \mathtt{H}\{\cos(at)\}(r) = \frac{u(r)r^\vartheta}{r^{2\vartheta}+a^2}.$$

Similarly, for the degenerate sine function, we obtain:

$$\begin{aligned}\mathtt{H}_\lambda\{\sin_\lambda(at)\}(r) &= \frac{1}{2i}u(r)\int_0^\infty \Big((1+\lambda t)^{-\frac{r^\vartheta-ai}{\lambda}} \\ &- (1+\lambda t)^{-\frac{r^\vartheta+ai}{\lambda}}\Big) dt \\ &= \frac{1}{2i}u(r)(\frac{1}{r^\vartheta-\lambda-ai} \\ &- \frac{1}{r^\vartheta-\lambda+ai}) \\ &= u(r)\frac{a}{(r^\vartheta-\lambda)^2+a^2}.\end{aligned}$$

Thus, for $\lambda\to 0$, we have:

$$\lim_{\lambda\to 0^+}\mathtt{H}_\lambda\{\sin_\lambda(at)\}(r) = \mathtt{H}\{\sin(at)\}(r) = \frac{au(r)}{r^{2\vartheta}+a^2}.$$

When $u(r)=r$ and $\vartheta=-1$, we have the following degenerate Elzaki transform of $\sin_\lambda(at)$:

$$E_\lambda\{\sin_\lambda(at)\} = \frac{r^3a}{(1-\lambda r)^2+a^2r^2}.$$

For the degenerate hyperbolic cosine, we follow a similar procedure with real arguments:

$$\begin{aligned}\mathtt{H}_\lambda\{\cosh_\lambda(at)\}(r) &= \frac{1}{2}u(r)\int_0^\infty \Big((1+\lambda t)^{-\frac{r^\vartheta-a}{\lambda}} \\ &+ (1+\lambda t)^{-\frac{r^\vartheta+a}{\lambda}}\Big) dt \\ &= \frac{1}{2}u(r)\left(\frac{1}{r^\vartheta-\lambda-a}+\frac{1}{r^\vartheta-\lambda+a}\right) \\ &= u(r)\frac{r^\vartheta-\lambda}{(r^\vartheta-\lambda)^2-a^2}.\end{aligned}$$

For $\lambda\to 0$, we have:

$$\lim_{\lambda\to 0^+}\mathtt{H}_\lambda\{\cosh_\lambda(at)\}(r) = \mathtt{H}\{\cosh(at)\}(r) = \frac{u(r)r^\vartheta}{r^{2\vartheta}-a^2}.$$

When $u(r)=r$ and $\vartheta=-1$, we have the following degenerate Elzaki transform of $\cosh_\lambda(at)$:

$$E_\lambda\{\cosh_\lambda(at)\} = r^2\frac{1-\lambda r}{(1-\lambda r)^2-a^2r^2}.$$

Finally, for the degenerate hyperbolic sine function:

$$\begin{aligned}\mathtt{H}_\lambda\{\sinh_\lambda(at)\}(r) &= \frac{1}{2}u(r)\int_0^\infty \left((1+\lambda t)^{-\frac{r^\vartheta-a}{\lambda}} - (1+\lambda t)^{-\frac{r^\vartheta+a}{\lambda}}\right) dt \\ &= \frac{1}{2}u(r)\left(\frac{1}{r^\vartheta-\lambda-a}-\frac{1}{r^\vartheta-\lambda+a}\right) \\ &= u(r)\frac{a}{(r^\vartheta-\lambda)^2-a^2},\end{aligned}$$

and for $\lambda\to 0$, we have:

$$\lim_{\lambda\to 0^+}\mathtt{H}_\lambda\{\sinh_\lambda(at)\}(r) = \mathtt{H}\{\sinh(at)\}(r) = \frac{au(r)}{r^{2\vartheta}-a^2}.$$

When $u(r)=r$ and $\vartheta=-1$, we have the following degenerate Elzaki transform of $\sinh_\lambda(at)$:

$$E_\lambda\{\sinh_\lambda(at)\} = \frac{ar^3}{(1-\lambda r)^2-a^2r^2}.$$

When $u(r)=r$ and $\vartheta=-1$, we have the following degenerate Elzaki transform of $\sinh_\lambda(at)$:

$$E_\lambda\{\sinh_\lambda(at)\} = \frac{ar^3}{(1-\lambda r)^2-a^2r^2}.$$

These transform pairs for degenerate trigonometric and hyperbolic trigonometric functions are essential for solving differential equations involving oscillatory and hyperbolic behaviors within the degenerate framework. Notably, as $\lambda\to 0$, all these expressions reduce to their classical transform $\mathtt{H}\{\varphi(t)\}$ [17], confirming the consistency of our generalized approach.

Remark 3.14. All the transform pairs derived above reduce to their classical counterparts as $\lambda\to 0$. For instance,

$$\lim_{\lambda\to 0}\mathtt{H}_\lambda\{\sin_\lambda(at)\}(r) = \mathtt{H}\{\sin(at)\}(r) = u(r)\frac{a}{r^{2\vartheta}+a^2},$$

TABLE 1. Comparison of Transforms for Elementary Functions

Function $\varphi(t)$	$\mathtt{H}_\lambda\{\varphi(t)\}$	$L_\lambda\{\varphi(t)\}$ ($u(r)=1,\ \vartheta=1$) [20]	$\mathbf{S}_\lambda\{\varphi(t)\}$ ($u(r)=1/r,\ \vartheta=-1$) [23]
t^m	$\dfrac{u(r)\Gamma(m+1)}{\prod_{j=1}^{m+1}(r^\vartheta-j\lambda)}$	$\dfrac{\Gamma(m+1)}{\prod_{j=1}^{m+1}(r-j\lambda)}$	$\dfrac{\Gamma(m+1)r^m}{\prod_{j=1}^{m+1}(1-j\lambda r)}$
$\sin_\lambda(at)$	$u(r)\dfrac{a}{(r^\vartheta-\lambda)^2+a^2}$	$\dfrac{a}{(r-\lambda)^2+a^2}$	$\dfrac{ar}{(1-\lambda r)^2+a^2r^2}$
$\cos_\lambda(at)$	$u(r)\dfrac{r^\vartheta-\lambda}{(r^\vartheta-\lambda)^2+a^2}$	$\dfrac{r-\lambda}{(r-\lambda)^2+a^2}$	$\dfrac{1-\lambda r}{(1-\lambda r)^2+a^2r^2}$
$\sinh_\lambda(at)$	$u(r)\dfrac{a}{(r^\vartheta-\lambda)^2-a^2}$	$\dfrac{a}{(r-\lambda)^2-a^2}$	$\dfrac{ar}{(1-\lambda r)^2-a^2r^2}$
$\cosh_\lambda(at)$	$u(r)\dfrac{r^\vartheta-\lambda}{(r^\vartheta-\lambda)^2-a^2}$	$\dfrac{r-\lambda}{(r-\lambda)^2-a^2}$	$\dfrac{1-\lambda r}{(1-\lambda r)^2-a^2r^2}$

TABLE 2. Reduction of $\mathtt{H}_\lambda$ to Classical and Degenerate Transforms

Transform	$u(r)$	ϑ	**Resulting Transform**
Degenerate Laplace	1	1	$L_\lambda\{\varphi(t)\}(r)=\int_0^\infty(1+\lambda t)^{-r/\lambda}\varphi(t)\,dt$
Classical Laplace	1	$1,\ \lambda\to 0$	$L\{\varphi(t)\}(r)=\int_0^\infty e^{-rt}\varphi(t)\,dt$
Degenerate Sumudu	$\frac{1}{r}$	-1	$\mathbf{S}_\lambda\{\varphi(t)\}(r)=\frac{1}{r}\int_0^\infty(1+\lambda t)^{-1/(\lambda r)}\varphi(t)\,dt$
Classical Sumudu	$\frac{1}{r}$	$-1,\ \lambda\to 0$	$\mathbf{S}\{\varphi(t)\}(r)=\frac{1}{r}\int_0^\infty e^{-t/r}\varphi(t)\,dt$
Degenerate Elzaki	r	-1	$\mathbf{E}_\lambda\{\varphi(t)\}(r)=r\int_0^\infty(1+\lambda t)^{-1/(\lambda r)}\varphi(t)\,dt$
Classical Elzaki	r	$-1,\ \lambda\to 0$	$\mathbf{E}\{\varphi(t)\}(r)=r\int_0^\infty e^{-t/r}\varphi(t)\,dt$

which corresponds to the classical generalized integral transform of $\sin(at)$. Similarly,

$$\lim_{\lambda\to 0}\mathtt{H}_\lambda\{\cos_\lambda(at)\}(r)=\mathtt{H}\{\cos(at)\}(r)=u(r)\frac{r^\vartheta}{r^{2\vartheta}+a^2}$$

These results are consistent with the classical Laplace transforms $L\{\sin(at)\}=\frac{a}{r^2+a^2}$ and $L\{\cos(at)\}=\frac{r}{r^2+a^2}$ when $u(r)=1$ and $\vartheta=1$.

Kakichev et al. [29] investigated a method for constructing generalized integral convolutions. Yakubovich et al. [30] discussed about the hypergeometric method for convolutions and integral transforms. Xuan [31] focused on the Fourier sine and cosine transforms using generalized convolution with a weight function. Giang et al. [32] studied applications and convolutions for Fourier transforms with geometric variables. We develop a convolution operation for the suggested integral transform based on the ideas from these studies.

Lemma 3.15. *The kernel $K_\lambda(t,r)=(1+\lambda t)^{-r^\vartheta/\lambda}$ defined in Equation (3.2) satisfies*

$$K_\lambda(t_1,r)K_\lambda(t_2,r)=K_\lambda(t_1\oplus_\lambda t_2,r) \tag{3.22}$$

where $t_1\oplus_\lambda t_2=t_1+t_2+\lambda t_1t_2$ for all $t_1,t_2\geq 0$.

Using the definition of the kernel in Equation (3.2):

$$\begin{aligned}K_\lambda(t_1,r)K_\lambda(t_2,r)&=(1+\lambda t_1)^{-r^\vartheta/\lambda}(1+\lambda t_2)^{-r^\vartheta/\lambda}\\&=[(1+\lambda t_1)(1+\lambda t_2)]^{-r^\vartheta/\lambda}\\&=[1+\lambda(t_1+t_2+\lambda t_1t_2)]^{-r^\vartheta/\lambda}\\&=[1+\lambda(t_1\oplus_\lambda t_2)]^{-r^\vartheta/\lambda}\\&=K_\lambda(t_1\oplus_\lambda t_2,r).\end{aligned}$$

The operation $\oplus_\lambda$ is commutative and associative, with identity element 0.

Define the convolution

$$(\varphi *_\lambda \varrho)(t) = \int_0^t \varphi(\tau)\varrho\left(\frac{t-\tau}{1+\lambda\tau}\right)\frac{d\tau}{1+\lambda\tau}, \tag{3.23}$$

provided the integral converges absolutely.

Theorem 3.16. *Let φ, ϱ be functions such that all integrals converge absolutely. Then*

$$\mathtt{H}_\lambda\{\varphi *_\lambda \varrho\}(r) = \frac{\mathtt{H}_\lambda\{\varphi\}(r)\mathtt{H}_\lambda\{\varrho\}(r)}{u(r)}. \tag{3.24}$$

Proof. Define the transforms

$$\Phi_\lambda(r) = \frac{\mathtt{H}_\lambda\{\varphi\}(r)}{u(r)} = \int_0^\infty K_\lambda(\tau,r)\varphi(\tau)\,d\tau,$$

and

$$R_\lambda(r) = \frac{\mathtt{H}_\lambda\{\varrho\}(r)}{u(r)} = \int_0^\infty K_\lambda(\sigma,r)\varrho(\sigma)\,d\sigma.$$

Using the pseudo-multiplication formula (3.22), we compute

$$\begin{aligned}\Phi_\lambda(r)R_\lambda(r) &= \int_0^\infty\int_0^\infty K_\lambda(\tau,r)K_\lambda(\sigma,r)\varphi(\tau)\varrho(\sigma)\\ &\quad d\tau\,d\sigma\\ &= \int_0^\infty\int_0^\infty K_\lambda(\tau\oplus_\lambda\sigma,r)\\ &\quad\times\varphi(\tau)\varrho(\sigma)\,d\tau\,d\sigma.\end{aligned}$$

For fixed τ, make the change of variables $t = \tau\oplus_\lambda\sigma = \tau+\sigma+\lambda\tau\sigma$, which gives

$$\sigma = \frac{t-\tau}{1+\lambda\tau}, \quad d\sigma = \frac{dt}{1+\lambda\tau}, \quad t\in(\tau,\infty).$$

Substituting, we obtain

$$\begin{aligned}\Phi_\lambda(r)R_\lambda(r) &= \int_0^\infty\int_\tau^\infty K_\lambda(t,r)\varphi(\tau)\\ &\quad\times\varrho\left(\frac{t-\tau}{1+\lambda\tau}\right)\tfrac{dt}{1+\lambda\tau}\,d\tau.\end{aligned}$$

The domain of integration is $\{(\tau,t): 0\le\tau<t<\infty\}$. By Fubini's theorem, we interchange the order of integration:

$$\begin{aligned}\Phi_\lambda(r)R_\lambda(r) &= \int_0^\infty K_\lambda(t,r)\Big[\int_0^t\varphi(\tau)\varrho\left(\frac{t-\tau}{1+\lambda\tau}\right)\\ &\quad\frac{d\tau}{1+\lambda\tau}\Big]\,dt\\ &= \int_0^\infty K_\lambda(t,r)(\varphi *_\lambda\varrho)(t)\,dt.\end{aligned}$$

The final integral is precisely $\mathtt{H}_\lambda\{\varphi *_\lambda \varrho\}(r)/u(r)$. Therefore,

$$\frac{\mathtt{H}_\lambda\{\varphi\}(r)\mathtt{H}_\lambda\{\varrho\}(r)}{u(r)^2} = \frac{\mathtt{H}_\lambda\{\varphi *_\lambda\varrho\}(r)}{u(r)},$$

which yields

$$\mathtt{H}_\lambda\{\varphi *_\lambda \varrho\}(r) = \frac{\mathtt{H}_\lambda\{\varphi\}(r)\mathtt{H}_\lambda\{\varrho\}(r)}{u(r)}.$$

This completes the proof.

Remark 3.17. In the limit $\lambda\to0$, the λ-deformed sum reduces to ordinary addition:

$$t_1\oplus_\lambda t_2 \longrightarrow t_1+t_2.$$

Consequently, the λ-convolution (3.23) reduces to the classical convolution

$$(\varphi*\varrho)(t) = \int_0^t \varphi(\tau)\varrho(t-\tau)\,d\tau.$$

Theorem 3.18. ***(Convolution Theorem in the Limiting Case)*** *Let $\mathtt{H}_\lambda\{\varphi\}(r) = \Phi_\lambda(r)$ and $\mathtt{H}_\lambda\{\varrho\}(r) = R_\lambda(r)$. Then, in the limiting case $\lambda\to0$, the standard convolution theorem holds:*

$$\lim_{\lambda\to0}\mathtt{H}_\lambda\{\varphi*\varrho\}(r) = \frac{1}{u(r)}\left[\lim_{\lambda\to0}\Phi_\lambda(r)\right]\left[\lim_{\lambda\to0}R_\lambda(r)\right],$$

where $(\varphi*\varrho)(t) = \int_0^t\varphi(t-\tau)\varrho(\tau)\,d\tau$.

Proof. Using Definition (3.2), we have

$$\mathtt{H}_\lambda\{\varphi*\varrho\}(r) = u(r)\int_0^\infty(1+\lambda t)^{-r^\vartheta/\lambda}\left(\int_0^t\varphi(t-\tau)\varrho(\tau)\,d\tau\right)dt.$$

Interchanging the order of integration and substituting $\eta = t-\tau$, we get

$$\mathtt{H}_\lambda\{\varphi*\varrho\}(r) = u(r)\int_0^\infty\int_0^\infty(1+\lambda(\eta+\tau))^{-r^\vartheta/\lambda}\varphi(\eta)\varrho(\tau)\,d\eta\,d\tau.$$

Now, taking the limit $\lambda\to0$:

$$\lim_{\lambda\to0}\mathtt{H}_\lambda\{\varphi*\varrho\}(r) = \lim_{\lambda\to0}u(r)\int_0^\infty\int_0^\infty(1+\lambda(\eta+\tau))^{-r^\vartheta/\lambda}\varphi(\eta)\varrho(\tau)\,d\eta\,d\tau.$$

Using the standard limit, $\lim_{\lambda\to0}(1+\lambda x)^{-1/\lambda} = e^{-x}$, we obtain

$$\lim_{\lambda\to0}(1+\lambda(\eta+\tau))^{-r^\vartheta/\lambda} = e^{-r^\vartheta(\eta+\tau)} = e^{-r^\vartheta\eta}e^{-r^\vartheta\tau}.$$

Therefore,

$$\begin{aligned}\lim_{\lambda\to0}\mathtt{H}_\lambda\{\varphi*\varrho\}(r) &= u(r)\int_0^\infty\int_0^\infty e^{-r^\vartheta\eta}e^{-r^\vartheta\tau}\varphi(\eta)\varrho(\tau)\,d\eta\,d\tau\\ &= u(r)\left[\int_0^\infty e^{-r^\vartheta\eta}\varphi(\eta)\,d\eta\right]\left[\int_0^\infty e^{-r^\vartheta\tau}\varrho(\tau)\,d\tau\right].\end{aligned}$$

Since $\lim_{\lambda\to0}\mathtt{H}_\lambda\{\varphi\}(r) = u(r)\int_0^\infty e^{-r^\vartheta t}\varphi(t)\,dt$, we have

$$\int_0^\infty e^{-r^\vartheta t}\varphi(t)\,dt = \frac{1}{u(r)}\lim_{\lambda\to0}\Phi_\lambda(r).$$

Similarly,

$$\int_0^\infty e^{-r^\vartheta t}\varrho(t)\,dt = \frac{1}{u(r)}\lim_{\lambda\to0}R_\lambda(r).$$

Therefore,

$$\begin{aligned}&\lim_{\lambda\to0}\mathtt{H}_\lambda\{\varphi*\varrho\}(r)\\ &= u(r)\left[\frac{1}{u(r)}\lim_{\lambda\to0}\Phi_\lambda(r)\right]\left[\frac{1}{u(r)}\lim_{\lambda\to0}R_\lambda(r)\right]\\ &= \frac{1}{u(r)}\left[\lim_{\lambda\to0}\Phi_\lambda(r)\right]\left[\lim_{\lambda\to0}R_\lambda(r)\right].\end{aligned}$$

This completes the proof.

From Definition 3.1, we observe that:

$$\mathtt{H}_\lambda\{\varphi(at)\}(r) = u(r)\int_0^\infty (1+\lambda t)^{-r^\vartheta/\lambda}\varphi(at)\,dt.$$

Upon setting $\varsigma = at \implies t = \varsigma/a,\ dt = d\varsigma/a$, we obtain:

$$\mathtt{H}_\lambda\{\varphi(at)\}(r) = u(r)\int_0^\infty (1+\lambda\varsigma/a)^{-r^\vartheta/\lambda}\varphi(\varsigma)\frac{1}{a}\,d\varsigma.$$

Thus,

$$\mathtt{H}_\lambda\{\varphi(at)\}(r) = \frac{u(r)}{a}\int_0^\infty \left(1+\frac{\lambda}{a}\varsigma\right)^{-r^\vartheta/\lambda}\varphi(\varsigma)\,d\varsigma.$$

Special Functions

Definition 3.19. **(p-k-Mittag-Leffler Function):** For $p,k\in\mathbb{R}^+$, $n,q\in\mathbb{N}$, and $\gamma,\beta,\alpha\in\mathbb{C}$ with $\Re(\alpha)>0$, $\Re(\beta)>0$, and $\Re(\gamma)>0$, the generalized p-k-Mittag-Leffler function [27] is expressed as:

$$(3.25)\quad {}_pE^{\gamma,q}_{k,\alpha,\beta}(t) = \sum_{m=0}^\infty \frac{{}_p(\gamma)_{mq,k}}{{}_p\Gamma_k(\alpha m+\beta)}\frac{t^m}{\Gamma(m+1)},$$

where ${}_p(\gamma)_{mq,k}$ is the p-k Pochhammer symbol [27].

Theorem 3.20. *The degenerate generalized integral transform of Eq. (3.25) is given by:*

$$\mathtt{H}_\lambda\left\{t^{\beta-1}{}_pE^{\gamma,q}_{k,\alpha,\beta}(at^\alpha)\right\}(r) = u(r)\sum_{m=0}^\infty\left[\frac{{}_p(\gamma)_{mq,k}\lambda^{\alpha m+\beta}\Gamma(\alpha m+\beta)}{{}_p\Gamma_k(\alpha m+\beta)\Gamma(m+1)r^{\vartheta(\alpha m+\beta)}}\times\Gamma_{\lambda/r^\vartheta}(\alpha m+\beta)\right].$$

provided the series converges absolutely.

Proof. Using Definition 3.19, we have

$$\mathtt{H}_\lambda\left\{t^{\beta-1}{}_pE^{\gamma,q}_{k,\alpha,\beta}(at^\alpha)\right\}(r) = u(r)\int_0^\infty (1+\lambda t)^{-r^\vartheta/\lambda}\sum_{m=0}^\infty\left[\frac{{}_p(\gamma)_{mq,k}a^m}{{}_p\Gamma_k(\alpha m+\beta)}\times\frac{t^{\alpha m+\beta-1}}{\Gamma(m+1)}\right]dt.$$

Interchanging the summation and integration, we get:

$$(3.26)\quad I = \sum_{m=0}^\infty \frac{{}_p(\gamma)_{mq,k}}{{}_p\Gamma_k(\alpha m+\beta)\Gamma(m+1)}u(r)\times\int_0^\infty (1+\lambda t)^{-r^\vartheta/\lambda}t^{\alpha m+\beta-1}\,dt.$$

Using Theorem 3.12, we have:

$$(3.27)\quad I = u(r)\sum_{m=0}^\infty \times\frac{{}_p(\gamma)_{mq,k}\lambda^{\alpha m+\beta}\Gamma(\alpha m+\beta)}{{}_p\Gamma_k(\alpha m+\beta)\Gamma(m+1)r^{\vartheta(\alpha m+\beta)}}\Gamma_{\lambda/r^\vartheta}(\alpha m+\beta).$$

This completes the proof.

Definition 3.21. **(k-Gauss Hypergeometric Function):** The k-Gauss hypergeometric function [28] is defined as

$$(3.28)\quad {}_2F_{1,k}(a,b;c;t) = \sum_{m=0}^\infty \frac{(a)_{m,k}(b)_{m,k}}{(c)_{m,k}}\frac{t^m}{\Gamma(m+1)}.$$

Theorem 3.22. *The degenerate generalized integral transform of the k-Gauss hypergeometric function is given by*

$$\mathtt{H}_\lambda\left\{{}_2F_{1,k}(a,b;c;t)\right\}(r) = u(r)\frac{\lambda\Gamma(1+r^\vartheta/\lambda)}{r^\vartheta\Gamma(r^\vartheta/\lambda)}\times\sum_{m=0}^\infty\frac{(a)_{m,k}(b)_{m,k}}{(c)_{m,k}}\frac{1}{\prod_{j=1}^{m+1}(r^\vartheta-j\lambda)}.$$

provided $r^\vartheta > (m+1)\lambda$.

Proof. The proof of Theorem 3.22 follows in a similar manner to that of Theorem 3.20. Thus, the details are omitted.

Definition 3.23. **(The Fox-Wright Function):** The Fox-Wright function [28] is given by

$${}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\dots,(a_p,A_p)\\(b_1,B_1),\dots,(b_q,B_q)\end{matrix};t\right] = \sum_{m=0}^\infty\frac{\prod_{i=1}^p\Gamma(a_i+A_im)}{\prod_{j=1}^q\Gamma(b_j+B_jm)}\frac{t^m}{\Gamma(m+1)}.$$

Theorem 3.24. *The degenerate generalized integral transform of the Fox-Wright function is given by*

$$\mathtt{H}_\lambda\left\{{}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\dots,(a_p,A_p)\\(b_1,B_1),\dots,(b_q,B_q)\end{matrix};t\right]\right\}(r) = u(r)\frac{\lambda\Gamma(1+r^\vartheta/\lambda)}{r^\vartheta\Gamma(r^\vartheta/\lambda)}\times\sum_{m=0}^\infty\frac{\prod_{i=1}^p\Gamma(a_i+A_im)}{\prod_{j=1}^q\Gamma(b_j+B_jm)}\frac{1}{\prod_{j=1}^{m+1}(r^\vartheta-j\lambda)}.$$

provided the series converges absolutely.

Proof. The proof of Theorem 3.24 follows in a similar manner to that of Theorem 3.20. Thus, the details are omitted. All the transform pairs derived in this section reduce to their classical Laplace counterparts when we set $u(r)=1$, $\vartheta=1$, and take the limit $\lambda\to 0$. This demonstrates the unifying nature of the degenerate generalized integral transform, which seamlessly bridges classical and degenerate transforms.

4. Application

In order to solve initial value problems with variable coefficients, this section presents methods derived from the degenerate generalized integral transform.

Example 4.1. Consider the following first-order variable coefficient ordinary differential equation:

$$(1+\lambda t)\varphi'(t) - 2\varphi(t) = 0, \qquad \varphi(0) = 1, \tag{4.1}$$

where $\lambda > 0$ is a parameter.

Now we apply the degenerate generalized integral transform $\mathtt{H}_\lambda$ to both sides of Equation (4.1):

$$\mathtt{H}_\lambda\{(1+\lambda t)\varphi'(t)\} - 2\mathtt{H}_\lambda\{\varphi(t)\} = 0. \tag{4.2}$$

In fact,

$$(1+\lambda t)\varphi'(t) = \frac{d}{dt}[(1+\lambda t)\varphi(t)] - \lambda\varphi(t). \tag{4.3}$$

Applying the transform to both sides of (4.3):

$$\begin{aligned}&\mathtt{H}_\lambda\{(1+\lambda t)\varphi'(t)\}\\ &= \mathtt{H}_\lambda\left\{\frac{d}{dt}\big[(1+\lambda t)\varphi(t)\big]\right\} - \lambda\mathtt{H}_\lambda\{\varphi(t)\}.\end{aligned} \tag{4.4}$$

From the derivative property of $\mathtt{H}_\lambda$, for any function $\varrho(t)$:

$$\mathtt{H}_\lambda\{\varrho'(t)\} = r^\vartheta\mathtt{H}_\lambda\{(1+\lambda t)^{-1}\varrho(t)\} - u(r)\varrho(0).$$

Let $\varrho(t) = (1+\lambda t)\varphi(t)$. Then:

$$\varrho(0) = (1+\lambda\cdot 0)\varphi(0) = \varphi(0) = 1,$$

and

$$(1+\lambda t)^{-1}\varrho(t) = (1+\lambda t)^{-1}(1+\lambda t)\varphi(t) = \varphi(t).$$

Thus,

$$\mathtt{H}_\lambda\left\{\frac{d}{dt}[(1+\lambda t)\varphi(t)]\right\} = r^\vartheta\mathtt{H}_\lambda\{\varphi(t)\} - u(r). \tag{4.5}$$

Substituting Equation (4.5) into Equation (4.4):

$$\begin{aligned}&\mathtt{H}_\lambda\{(1+\lambda t)\varphi'(t)\}\\ &= \big[r^\vartheta\mathtt{H}_\lambda\{\varphi(t)\} - u(r)\big] - \lambda\mathtt{H}_\lambda\{\varphi(t)\}\\ &= (r^\vartheta - \lambda)\mathtt{H}_\lambda\{\varphi(t)\} - u(r).\end{aligned} \tag{4.6}$$

Substituting Equation (4.6) into Equation (4.2):

$$\begin{aligned}&[(r^\vartheta - \lambda)\mathtt{H}_\lambda\{\varphi(t)\} - u(r)] - 2\mathtt{H}_\lambda\{\varphi(t)\} = 0\\ &\implies (r^\vartheta - \lambda - 2)\mathtt{H}_\lambda\{\varphi(t)\} = u(r).\end{aligned} \tag{4.7}$$

Therefore,

$$\mathtt{H}_\lambda\{\varphi(t)\} = \frac{u(r)}{r^\vartheta - \lambda - 2}. \tag{4.8}$$

From the known transform pair for the degenerate exponential function:

$$\mathtt{H}_\lambda\{(1+\lambda t)^{a/\lambda}\} = \frac{u(r)}{r^\vartheta - a - \lambda}. \tag{4.9}$$

Comparing (4.8) with (4.9), we identify $a = 2$:

$$\mathtt{H}_\lambda\{(1+\lambda t)^{2/\lambda}\} = \frac{u(r)}{r^\vartheta - 2 - \lambda} = \frac{u(r)}{r^\vartheta - \lambda - 2}. \tag{4.10}$$

Therefore, the solution after taking the inverse transform is

$$\varphi(t) = (1+\lambda t)^{2/\lambda}.$$

For $\lambda \to 0$, the differential equation becomes:

$$\varphi'(t) - 2\varphi(t) = 0, \qquad \varphi(0) = 1, \tag{4.11}$$

and the solution is

$$\varphi(t) = e^{2t} = \lim_{\lambda\to 0}(1+\lambda t)^{2/\lambda}.$$

Example 4.2. Consider the following second-order variable coefficient problem:

$$(1+\lambda t)^2\varphi''(t) + 2\lambda(1+\lambda t)\varphi'(t) + (1+\lambda^2)\varphi(t) = 0, \quad \varphi(0) = 1, \quad \varphi'(0) = 0. \tag{4.12}$$

The solution is given by:

$$\begin{aligned}\varphi(t) = (1+\lambda t)^{-\frac{1}{2}}\Bigg[&\cos\left(\frac{\sqrt{4+3\lambda^2}}{2\lambda}\ln(1+\lambda t)\right)\\ &+ \frac{\lambda}{\sqrt{4+3\lambda^2}}\sin\left(\frac{\sqrt{4+3\lambda^2}}{2\lambda}\ln(1+\lambda t)\right)\Bigg].\end{aligned}$$

For $\lambda \to 0$, we recover the classical solution $\varphi(t) = \cos t$.

Figure 1 displays the graphs of the classical exponential function $f(t) = e^{2t}$ and the degenerate exponential function $f(t) = (1+\lambda t)^{2/\lambda}$ for different values of the parameter λ. As $\lambda \to 0$, the degenerate exponential function converges to the classical exponential function e^{2t}, confirming the consistency of the degenerate framework with classical results. Figure 2 illustrates the 3D solution profiles of the second-order variable coefficient problem. Figure 3 shows the graphs of $f(t) = (1+\lambda t)^{2/\lambda}$ for various values of λ.

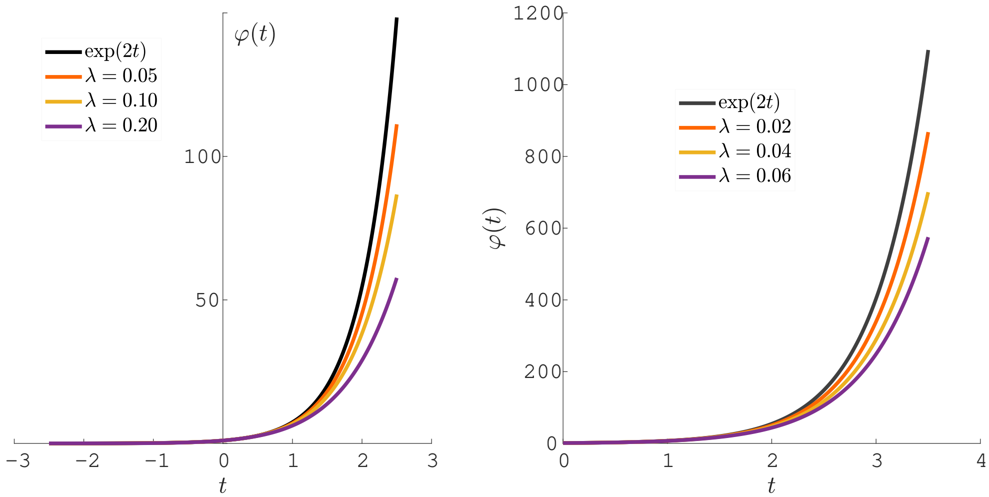

FIGURE 1. Graphs of $f(t) = e^{2t}$ and $f(t) = (1+\lambda t)^{2/\lambda}$ for Example 1 at different values of λ

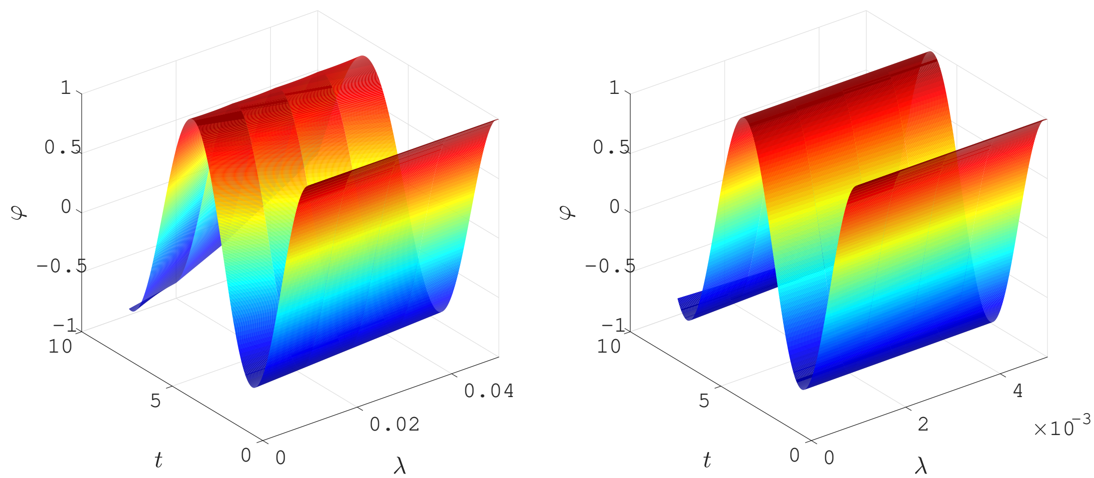

FIGURE 2. 3D solution profiles of the second-order variable coefficient problem for different values of λ for Example 2

5. Conclusion

He, J.-H., et al. [17] introduced the generalized integral transform of the form:

$$\mathtt{H}\{\varphi(t)\}(r) = \mathtt{H}(r) = u(r)\int_0^\infty e^{-r^\vartheta t}\varphi(t)\,dt,$$

(5.1) where $r \in \mathbb{C},\ \vartheta \in \mathbb{N},\ u(r) \neq 0.$

Motivated by this study, in the present paper, we have introduced the degenerate generalized integral transform by the improper integral found in Equation (3.2):

$$\mathtt{H}_\lambda\{\varphi(t)\}(r) = u(r)\int_0^\infty (1+\lambda t)^{-r^\vartheta/\lambda}\varphi(t)\,dt,$$

where $\lambda \in (0,\infty)$, $u(r) \neq 0$, $\vartheta \in \mathbb{Z}$, and $r \in \mathbb{C}$. This transform unifies classical transforms (Laplace, Sumudu, Elzaki, etc.) with their degenerate forms, and reduces to the classical generalized integral transform [17] in the limit $\lambda \to 0$. We have established fundamental properties of

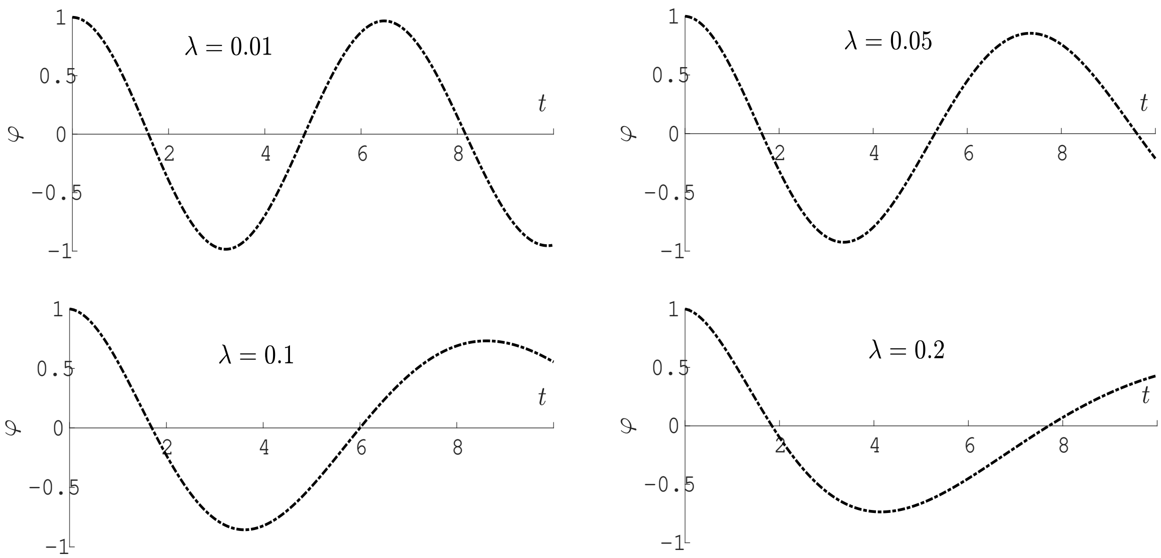

FIGURE 3. Graphs of $f(t) = (1 + \lambda t)^{2/\lambda}$ for various values of λ for Example 2

the proposed transform, including: existence theorem, linearity, differentiation formulas, and convolution theorem We have derived degenerate generalized transforms of various functions, including: power functions, degenerate exponential functions, degenerate sine and cosine functions, degenerate hyperbolic sine and cosine functions. We have also obtained a relation between the degenerate generalized integral transform and the degenerate gamma function. In addition, we derived transform pairs for some special functions, including the p-k-Mittag-Leffler function, the k-Gauss hypergeometric function, and the Fox-Wright function. Applications to variable coefficient ordinary differential equations were demonstrated through two illustrative examples. Future research directions include: Extension of the transform to higher-dimensional problems and application to fractional differential equations and fractional kinetic equations.

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