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Interval Forecast for Smooth Transition Autoregressive Model

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Abstract

In this paper, we propose a simple method for constructing interval forecast for smooth transition autoregressive (STAR) model. This interval forecast is based on bootstrapping the residual error of the estimated STAR model for each forecast horizon and computing various Akaike information criterion (AIC) function. This new interval forecast suggest definite and better coverage to the future sample path than the conventional method of using a multiple of standard error of the forecast distribution using bootstrap method. Simulation studies are used to illustrate the proposed method.

Key words: AIC; bootstrap; Interval forecast; STAR

1. Introduction

One of the major reasons for time series modelling is to forecast future values. Forecasting future values of time series data may take the form of point forecast or an interval forecast. For non-linear time series model, the construction of point and interval forecast is problematic. It is known that in linear autoregressive moving average (ARMA(p, q)) model, forecast interval can be constructed theoretically using a weight function derived from the estimated model, see Box and Jenkins (1976). In non-linear time series models, a theoretical forecast interval is not easy to construct. Chatfield (1993) presented a road map for constructing interval forecast for stationary and non-stationary models and Christofferson (1998) gave a general method for evaluating interval forecast while other work on interval forecast was done using Sieve bootstrap method, see Bühlmann (1997), Alonso *et.al* (2002, 2003) for examples. These authors only achieve good prediction interval results for linear ARMA models whereas their prediction intervals failed for non-linear models. The simple reason for this is that linear ARMA model can be represented as an infinite linear process while non-linear model cannot be represented so. Giordano *et.al* (2007) used a neural network sieve bootstrap to construct interval forecast for STAR model. This was also made possible by using a class of neural network to approximate the original non-linear model. Their approach is rather complicated since the neural network requires training. At the moment, there is no standard method of obtaining interval forecast for non-linear time series model. In this study we propose a simple forecast interval for non-linear time series model with particular attention given to logistic smooth transition autoregressive (LSTAR) model. Our method makes use of bootstrapping the residuals generated from the estimated STAR model and computing various AIC function, due to Akaike (1969).

The rest of this work is organized as follows. In Section 2, we describe briefly the basic representation of STAR model and point forecast. In Section 3, we present the proposed methodology for constructing interval forecast for STAR model. Section 4 is on simulation experiments and results to illustrate the proposed method. In Section 5, the results of the simulation studies are presented. Section 6 concludes the paper.

2. *Representation of the basic STAR model and point forecast*

We first consider the autoregressive (AR(p)) model of order p before considering the basic STAR model. An AR(p) model of order p, for some positive integer p ≥ 1, can be written as:

$$y_t = \phi' w_t + \varepsilon_t \qquad (2.1)$$

where $w_t = (1, y_{t-1}, y_{t-2}, \ldots, y_{t-p})'$ and $\phi = (\phi_0, \phi_1, \ldots, \phi_p)'$ are real parameters such that the zeroes of $\phi(z) = 1 - \alpha_1 z - \cdots - \alpha_p z^p$ lie outside the unit disk and ε_t is normally distributed with mean zero and variance σ_ε^2. The smooth transition autoregressive (STAR) model for a univariate time series y_t which is observed at $t = 1,2,\ldots,n$ is given by:

$$y_t = \phi_1' w_t + \phi_2' w_t G(s_t; \gamma, c) + \varepsilon_t \tag{2.2}$$

where $\phi_i = (\phi_{i,0}, \ldots, \phi_{i,p})'$, $i = 1,2$, $\gamma > 0$, w_t and ε_t are as defined in (2.1). $G(s_t; \gamma, c)$ is called the transition function and lies between 0 and 1.The transition function could either be of the logistic type or the exponential type. The variable s_t is called the transition variable which could be assumed to be a lagged endogenous variable that is $s_t = y_{t-d}$ for certain integer $d > 0$ or linear trend $(s_t = t)$, which gives rise to smoothly changing parameters.

If the transition function is of the Logistic type, we have

$$G(\gamma, c; s_t) = \frac{1}{(1 + \exp\{-\gamma(s_t - c)\})} \tag{2.4}$$

After estimation of the parameters of any of the model given by (2.1) and (2.2) and the model found to be adequate, then it can be used for forecasting. We illustrate how to generate point forecast using a general model of order 1,

$$y_t = F(y_{t-1}; \phi) + \varepsilon_t, \tag{2.5}$$

for some linear or non-linear function of $F(y_{t-1}; \phi)$. The optimal point forecast of future values of the time series are given by their conditional expectations. See Box and Jenkins (1976), Franses and Van Dijk (2000) for exposition. The h-step-ahead forecast of the future values y_{t+h} at time t is given by:

$$\hat{y}_{t+h} = E[y_{t+h} / \Omega_t] \tag{2.6}$$

where Ω_t denotes the past history of the time series up to and including the observation at time t. An optimal 1-step-ahead forecast, using the fact that $E[\varepsilon_{t+1} / \Omega_t] = 0$ is obtained as:

$$\hat{y}_{t+1} = E[y_{t+1} / \Omega_t] = F(y_t; \phi) \tag{2.7}$$

This optimal 1-step-ahead forecast for $F(y_t;\phi)$ is the same for linear and non-linear model. For h-step-ahead greater than or equal 2 forecasts can be generated recursively without difficult for linear model, while this is no longer easy to handle for non-linear model. In fact, an optimal 2-step-ahead forecast using (2.6) and (2.7) for non-linear model is given by:

$$\hat{y}_{t+2} = E[y_{t+2}/\Omega_t] = E[F(y_{t+1};\phi)/\Omega_t] \qquad (2.8)$$

The linear conditional expectation operator E cannot be interchanged with the non-linear operator F; hence (2.8) can be expressed using the relationship that exists between 1- and-2 -steps ahead forecasts which is now written as:

$$\hat{y}_{t+2} = E[F(F(y_t;\phi)+\varepsilon_{t+1};\phi)/\Omega_t] = E[F(\hat{y}_{t+1}+\varepsilon_{t+1};\phi)/\Omega_t] \qquad (2.9)$$

The exact method of forecast from (2.9), is

$$\hat{y}_{t+2} = \int_{-\infty}^{\infty} F(y_{t+1};\phi)\,d\phi(\varepsilon)\,d\varepsilon \qquad (2.10)$$

(2.10) requires numerical integration and the dimension of integration increases as the forecast horizon increases. As a result of this, this exact method is usually replaced by simulation method, given by:

$$\hat{y}_{t+2} = (1/M)\sum_{i=1}^{M} F(y_{t+1}+\varepsilon_i;\phi) \qquad (2.11)$$

(2.11) is called the Monte-Carlo method if an assumed error distribution is used or the bootstrap method if the error generated from the fitted method such as the one in (2.2) is used.

3 Methodology for Constructing Interval Forecast for STAR Model

Given a time series $\{y_t\}, t = 1,2,...,n$; Franses and Van Dijk (2000) listed three methods of constructing interval forecast for STAR models. These are:

3.1 An interval symmetric around the mean, it is defined by:

$$S_\alpha = (\hat{y}_{t+h} - w, \quad \hat{y}_{t+h} + w),$$

where w is chosen such that $P(y_{t+h} \varepsilon S_\alpha / \Omega_t) = 1-\alpha$, $\hat{y}_{t+h}$ is forecast made at time origin t for a specific horizon h, and y_{t+h} is the expected future value.

For linear models, w is given by $w = z_{\alpha/2}(FMSE(h))^{1/2}$, while for non-linear models, $w = M(sde)$ where sde is the standard error of the forecast distribution at each forecast horizon, $M > 0$ is an integer. Thus, we must generate the forecast before we can construct the forecast interval as suggested by Hyndman (1995).

3.2 An interval between $\alpha/2$ and $(1-\alpha/2)$ quantiles of the forecast distribution denoted

by $q_{\alpha/2}$ and $q_{1-\alpha/2}$ respectively. This interval is given by:

$$Q_\alpha = (q_{\alpha/2}, q_{1-\alpha/2})$$

3.3 The highest density region (HDR) that is,

$$HDR_\alpha = \{y_{t+h} / g(y_{t+h} / \Omega_t) \geq g_\alpha\}$$

where g_α is such that $P(y_{t+h} \varepsilon HDR_\alpha / \Omega_t) = 1-\alpha$

Hyndman (1996), asserted that the three methods of constructing forecast region using (3.1)-(3.3) has been found to be the same when the forecast distribution is normal. That for non-normal forecast distributions; the regions are all different and recommend that highest-density forecast regions be used. For linear models, w is given by $w = z_{\alpha/2}(FMSE(h))^{1/2}$, where $FMSE(h)$ can be obtained theoretically. For non-linear models, $w = M(sde)$ where sde is the standard error given by $sde = (FMSE(h))^{1/2}$ which cannot be obtained theoretically and $M \geq 2$ (an integer) is used to construct interval forecast. For non-linear model w is set equal to a multiple of standard error of the forecast distribution. This method of interval forecast in non-linear model does not suggest an appropriate or optimal interval forecast.

Now, for any forecast made there exists a forecast error or prediction error e_{t+h} given by:

$$e_{t+h} = y_{t+h} - \hat{y}_{t+h} \qquad (3.4)$$

where y_{t+h} is the future values and $\hat{y}_{t+h}$ is the forecast made at time t. Franses and Van Dijk (2000) stated that it is desirable to choose the forecast $\hat{y}_{t+h}$ that minimizes the forecast mean squared error (FMSE) given by:

$$FMSE(h) \equiv E(e_{t+h}^2) = E[(y_{t+h} - \hat{y}_{t+h})^2] \qquad (3.5)$$

It has been showed in Box-Jenkins (1976) that the forecast that minimizes (3.5) is the conditional expectation of y_{t+h} at time t, that is,

$$\hat{y}_{t+h} = E[y_{t+h} / \Omega_t].$$

Assuming normality, a $100(1-\alpha)\%$ forecasting interval for $\hat{y}_{t+h}$ in ARMA(p, q) is given by the following interval:

$$\hat{y}_{t+h} - Z_{\alpha/2}.(FMSE(h))^{1/2} \text{ and } \hat{y}_{t+h} + Z_{\alpha/2}.(FMSE(h))^{1/2} \qquad (3.6)$$

where $FMSE(h)$ can be obtained theoretically and it is given by:

$$FMSE(h) = \sigma_\varepsilon^2 \sum_{j=0}^{h-1} \psi_j^2 \qquad (3.7)$$

where $\psi_0 = 1$, ψ_j for $j = 1,2,...,h-1$ are the forecast weights which can be computed recursively from infinite representation of ARMA(p, q) model and σ_ε^2 is the residual error variance estimated from a fitted ARMA(p, q) model.

The AIC function may be given by any of the following:

$$AIC(k) = n\log(\hat{\sigma}_\varepsilon^2) + 2k \qquad (3.8)$$

$$AIC(k) = -n\log(\hat{\sigma}_\varepsilon^2) + 2k \qquad (3.9)$$

The expressions for AIC in (3.8) and (3.9) can either be positive or negative. These values are used indiscriminately in the literature. In order to obtain positive values for AIC, Ekhosuehi (2010) adopted the following expression:

$$AIC(k) = \begin{cases} n\log(\hat{\sigma}_\varepsilon^2) + 2k, & \log(\hat{\sigma}_\varepsilon^2) \geq 0 \\ -n\log(\hat{\sigma}_\varepsilon^2) + 2k, & \log(\hat{\sigma}_\varepsilon^2) < 0 \end{cases} \qquad (3.10)$$

where k denote the number of parameters in the model, n is the sample size and $\hat{\sigma}_\varepsilon^2 = n^{-1}\sum_{t=1}^{n} \hat{\varepsilon}_t^2$, where $\hat{\varepsilon}_t$ is the residuals generated from the fitted model. Ekhosuehi and Omosigho (2012), show that the sampling distribution of the AIC function represented by (3.8) and (3.9) has the normal distribution, while the one given by (3.10) has a Chi-square distribution. The specification given by (3.10) is that it permits positive values of AIC at all times. In a simulation experiment, Ekhosuehi and

Omosigho (2010) established a close relationship between $FMSE(h)$ given in (3.7) and the AIC given in (3.10) namely:

$$(FMSE(h))^{1/2} = \frac{AIC(k)}{\sqrt{n}} \tag{3.11}$$

Our method for obtaining w is based on the bootstrap method to estimate the FMSE for LSTAR model. This method is achieved by constructing B number of AIC function from the re-sampled error sequence, and then computes the standard error. The idea is similar to the Sieve bootstrap for times series proposed by Bühlmann (1997) and Sieve bootstrap interval by Alonso *et.al* (2002, 2003). The AIC function that is used in this regard is the one defined by (3.8) or (3.9) since they yield the same variance when their bootstrap is conducted.

The procedure for constructing the AIC prediction interval is given by the following steps:

Step1. Given a time series observations $\{y_t\}$, use JMULTI (a time series software package) to model an appropriate LSTAR model.

Step2. Generate the residuals $\hat{\varepsilon}_t$ from the fitted LSTAR model for $t = p+1, \cdots, n$

Step4. Compute the empirical distribution function of the standardized residuals,

$$\hat{F}_{\varepsilon^*}(x) = (n-p)^{-1}\sum_{t=p+1}^{n} 1_{\{\varepsilon^* \le x\}}, \quad \text{where} \quad \varepsilon_t^* = \frac{\tilde{\varepsilon}_t}{\hat{\sigma}}, \tilde{\varepsilon}_t = \hat{\varepsilon}_t - (n-p)^{-1}\sum_{t=p+1}^{n}\hat{\varepsilon}_t$$

and

$$\sigma^2 = (n-p)^{-1}\sum_{t=p+1}^{n}\hat{\varepsilon}_t^2$$

Step5. Draw a resample ε_t^* of i.i.d observations from the standardized residuals to obtain

forecast values of length h using the bootstrap method and also compute the AIC

function for each bootstrap replication.

Step6. Compute the standard error of the bootstrap AIC. Where the standard error is given as:

$$s\hat{e}_{B(AIC)} = \left(\frac{1}{(B-1)} \sum_{b=1}^{B} (AIC^{*b} - AI\overline{C}*)^2 \right)^{1/2} \tag{3.12}$$

where $AI\overline{C}* = \frac{1}{B} \sum_{b=1}^{B} AIC^{*b}$

Step7. Construct the interval forecast using $S_\alpha = (\hat{y}_{t+h} - w, \quad \hat{y}_{t+h} + w)$ where $w = 3(s\hat{e}_{B(AIC)})$

4 Simulation Experiment

We consider the following data generating process (DGP) using the logistic smooth transition autoregressive (LSTAR) model. Model (I) is an LSTAR(1) model while Model (II) is an LSTAR(2) model.

Model (I) $y_t = 0.8y_{t-1} - 0.8y_{t-1}(1 + \exp(-10y_{t-1}))^{-1} + \varepsilon_t$

Model (II) $y_t = 1.8y_{t-1} - 1.06y_{t-2} + (0.02 - 0.9y_{t-1} + 0.795y_{t-2})G(y_{t-1}) + \varepsilon_t$

where $G(y_{t-1}) = (1 + \exp\{-20(y_{t-1} - 0.02)\})^{-1}$.

The sequence of error, ε_t, which is normally distributed with mean 0 and variance 1, was generated using the random number generator in MATLAB 7.5.0. We generated $(300 + n)$ sample sizes using Models (I) and (II). Only the last n observations are kept, while the first 300 are discarded to minimize initialization effect. Among the n generated artificial time series observations, the first $(n - h0)$ observations were used for modeling while the remaining $h0$ observations are kept for out-of-sample performance. The modeling cycle which involves testing for non-linearity, model specification and evaluation such as in Luukonnen *et al*. (1988a), Luukonnen *et al*. (1988b), Teräsvirta (1994), Eitrheim and Teräsvirta (1996), Van Dijk *et al*. (2002) were executed using a Time series software package called JMULTI. After estimation of the model, it is then used for forecasting and for constructing forecast intervals. The forecasting method employed is the bootstrap method using the idea of Efron and Tibshirani (1998), Martinez and Martinez (2002). This stage of computing point forecast and constructing forecast interval in non-linear LSTAR model was written and executed in MATLAB7.5.0, since JMULTI does not provide forecast option in their package. The forecasting horizon (h) is taken from $h = 1$ to $h0 = 10$, so as to agree with the number of observations kept for the out-of-sample performance.

This process was replicated many times. We however, present few results in the next section to illustrate this new interval forecast.

Figures 1 and 2 shows the upper and lower forecast limits using the AIC bootstrap with a multiple of the standard error computed from the forecast mean square error using Model (I) for sample size n = 100 and 200 respectively, while Figures 3 and 4 shows the upper and lower forecast limits using the AIC bootstrap with a multiple of the standard error computed from the forecast mean square error using Model (II) for sample size n = 100 and 200 respectively. All the graphs also contain the actual observations kept for out-of-sample performance and the point forecast made. The multiple of the standard error of the forecast distribution used for constructing the interval forecast for the bootstrap method ranges from 2 to 5.

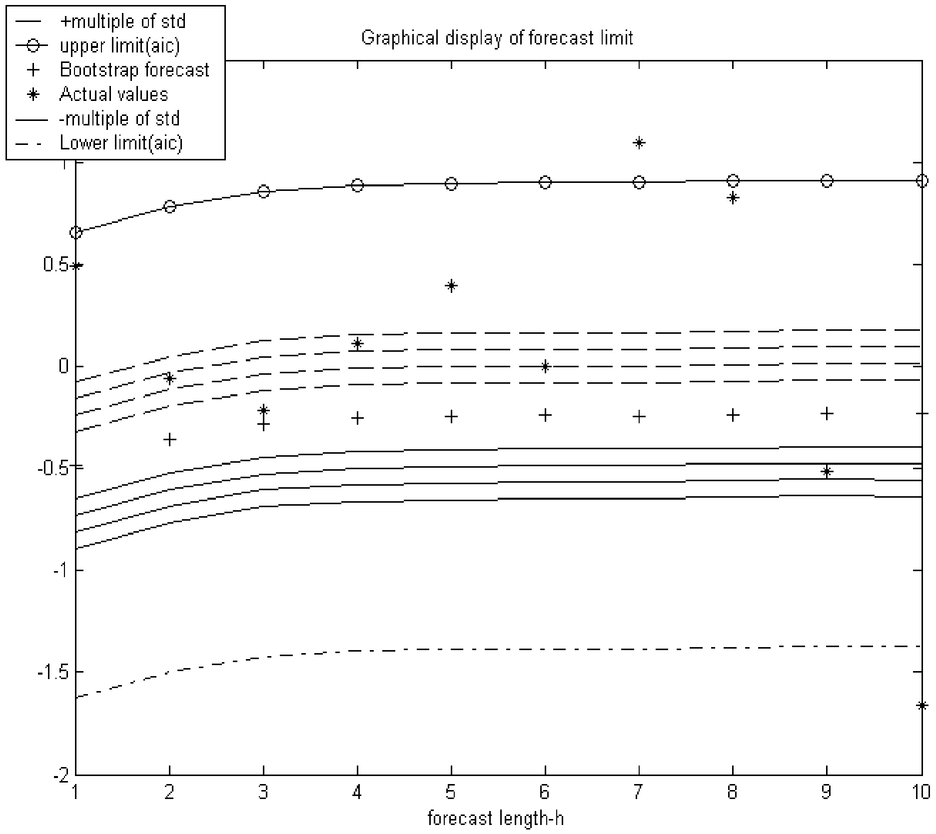

Figure 1: Represent forecast interval using plus or minus a multiple of standard deviation as forecast interval and the proposed AIC forecast interval using thrice bootstrap standard error for model (I) for sample size $n - h0 = 100$, $h0 = 10$.

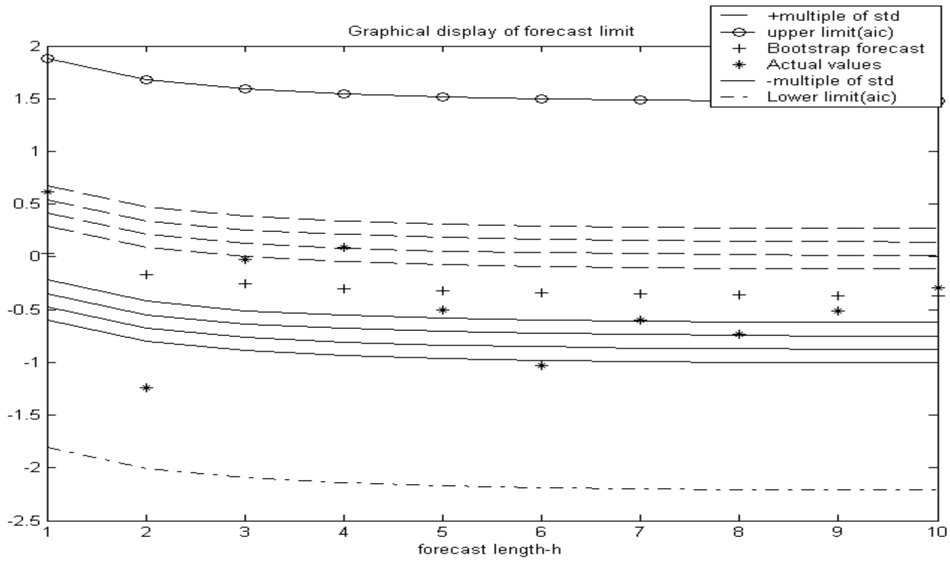

Figure 2: Represent forecast interval using plus or minus a multiple of standard deviation as forecast interval and the proposed AIC forecast interval using thrice bootstrap standard error for model (I) for sample size $n\text{-}h0 = 200$, $h0 = 10$.

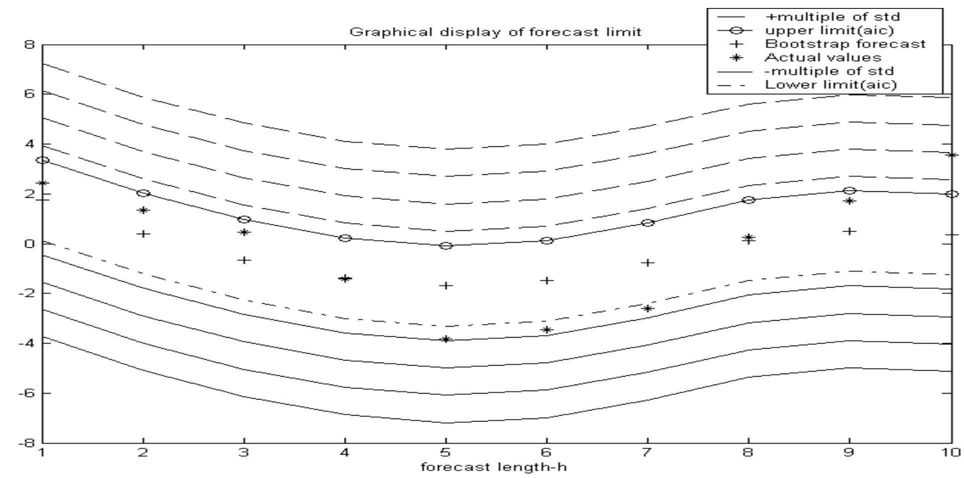

Figure 3: Represent forecast interval using plus or minus a multiple of standard deviation as forecast interval and the proposed AIC forecast interval using thrice bootstrap standard error for model (II) for sample size $n\text{-}h0 = 100$, $h0 = 10$.

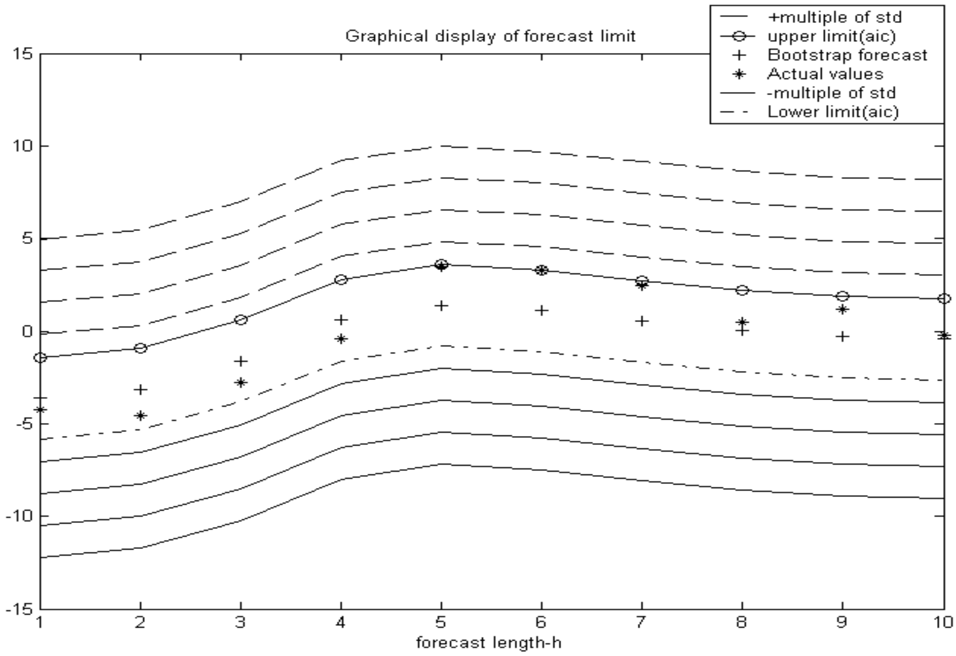

Figure 4: Represent forecast interval using plus or minus a multiple of standard deviation as forecast interval and the proposed AIC forecast interval using thrice bootstrap standard error for Model (II) for sample size $n - h0 = 200$, $h0 = 10$.

5 Discussion of Results

The interval forecast in Figures 1 and 2 are constructed using plus or minus a multiple of standard deviation for the forecast distribution when bootstrap method is applied and the AIC bootstrap forecast interval for sample sizes n = 100 and 200 respectively using Model (I). Similarly, Figures 3 and 4 are constructed using plus or minus a multiple of standard deviation for the forecast distribution when bootstrap method is applied and the AIC bootstrap forecast interval for sample sizes n = 100 and 200 respectively using Model (II). In Figures 1 and 2, it is observed that the largest interval of the multiple of standard deviation, which is 5 times standard deviation gives 70% and 80% coverage to the expected future values respectively while the bootstrap AIC forecast interval gives a 90% and 100% coverage to the expected future values respectively. In Figure 3 and 4, the smallest interval of the multiple of standard deviation, which 2 times standard deviation gives 100% coverage to the expected future values respectively while the bootstrap AIC forecast interval gives 70% and 100% coverage to the expected future values.

6 Conclusion

In this paper, a new method of constructing interval forecast for non-linear LSTAR model using the bootstrap AIC function is presented. This new interval forecast is constructed using the bootstrap standard error of AIC obtained by re-sampling from the generated residuals of the estimated LSTAR model. Simulation results show that the interval predicted by this method gives a better coverage compared to the conventional method of using a multiple of the standard deviation computed from the forecast distribution which does not indicate or suggest an appropriate forecast interval.

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