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ON EXISTENCE OF CONTROL FOR A CLASS OF UNCERTAIN DYNAMICAL SYSTEMS

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ABSTRACT
In this paper we prove the existence of control for input bounded uncertain dynamical system, modeled on Euclidean spaces of dimensions n and m. We apply the Conjugate Gradient Method (C.G.M) in generating algorithms to compute control signals for the class of problem under consideration.

Keywords: Control, Uncertain Dynamical Systems, Conjugate Gradient Method.

INTRODUCTION

We consider system with input bounded uncertainties. Our model is defined on Euclidean spaces of dimension n and m

Problem Formulation and Basic Assumptions

Let R^n and R^m be real Euclidean spaces of dimension n and m. Let $B(R^n, R^m)$ and $B(R^m, R^n)$ be spaces of bounded linear operators mapping R^n onto R^m and R^m onto R^n, respectively.

Denote by $\|.\|$ norm of vectors and operators.

Let $L_2[(0,T), R^n]$ and $L_2[(0,T), R^m]$ be spaces of $R^n - valued$ and $R^m - valued$ square integrable function on $[0,T]$. We consider a linear system subject to bounded input disturbance defined by:

$$\dot{x}(t) = Ax(t) + B_1u(t) + B_1v(t) \quad \dots (1)$$
$$x(0) = x_0, t \in [0,T]$$

where A and B_1 are matrix operators such that $A \in R^{n\times n}$ and $B_1 \in R^{m\times m}$ respectively. $x(t)$ is the state vector in R^n, $u(t)$ and $v(t)$ are the control vector and disturbance vector respectively such that:

$$u(t) \in U \subset R^m : U = \{\|u\| \le m_1, m_1 \in (0,\infty)\} \quad \dots (2)$$

$$v(t) \in V \subset R^m : V = \{\|v\| \le m_2, m_2 \in (0,\infty)\}$$

$U \cap V = \phi$ for each $t \in [0,T]$ a.e.

We shall be interested in computing a stable control, which renders the system sufficiently stable in the presence of some disturbances.

We associate the following cost functional with (1) which is defined by:

$$\text{Minimize } I(x,u) = \int_0^T \{x(t)^T Qx(t) + u(t)^T Pu(t)\}dt \dots (3)$$

where $Q \in R^{n\times n}$, non-negative definite symmetric matrix.

P and P^T are $m \times m$ symmetric and positive definite matrices.

In what now follows, we make the following assumptions

Assumptions:
C1: The matrix function A (.), B (.), Q (.) and P (.) are constant matrices

C2: We assume that the system (1) describes a zero-sum differential game, hence there exists a saddle point

C3: Control vector $u(t)$ and disturbance vector $v(t)$ are minimiser and maximiser of (3) respectively. The vectors are generated by strategies such that:

$$\gamma(.) : R^n \to R^m$$

$$\alpha(.) : R^n \to R^m$$

and if $\overline{u}(t)$ and $\overline{v}(t)$ denote optimal strategy for minimiser and maximiser respectively then

$$\gamma(\bar{x}(t)) = \bar{u}(t)$$

$$\alpha(\bar{x}(t)) = \bar{v}(t)$$

Where $\bar{x}(t):[0,T] \to R^n$ is a solution of (1)

C4: The linear equation defined by:

$$\dot{x}(t) = Ax(t), t \in [0,T]$$
$$x(0) = x_0 \qquad \text{... (4)}$$

where A is assumed to be stable, generate a fundamental matrix $S(t)$ on R^n. Also

$u(.)$ and $v(.)$ are measurable functions such that;
$u(.), v(.) \in L[0,T,R^m]$.

Necessary Condition for Existence of Stable Control

In addition to assumption C1 – C4, we assume that the system is controllable. Let the inner product of n-dimensional vectors x and y in $L[(0,T),R^n]$ be denoted by $< x, y >_1$ and be defined by:

$$< x, y >_1 = \int_0^T x(t)y(t)dt \qquad \text{... (5)}$$

Similarly, define the inner product for control $u \in R^m$ such that we have
$< u_{1,} u_2 >_2$

The cost functional $I(x,u)$ is therefore written as:

$$I(x,u) = < x(t), Qx(t) >_1 + < u(t), Pu(t) >_2 \qquad \text{... (6)}$$

Given the assumptions C4, we write (1) as an integral equation of the form:

$$x(t) = S(t)x_0 + \int_0^t S(t-r)\{B_1 u(r) + B_1 v(r)\}dr \qquad \text{... (7)}$$

Using the assumption C2 we set:

$$v(t) = \beta u(t), \beta > 0. \qquad \text{... (8)}$$

Equation 7 becomes

$$x(t) = S(t)x_0 + \int_0^t S(t-r)\tilde{H}u(r)dr \qquad \text{... (9)}$$

where

$$\tilde{H} = (I+\beta)B_1 \quad \text{... (10)} \qquad \text{... (10)}$$

Next we define a linear integral operator on $L_2[(0,T),R^n]$ as follows:

$$Lu = \int_0^t K(t,s)u(s)ds \qquad \text{... (11)}$$

where

$$L \in B[L_2(0,T), R^m), L_2(0,T), R^n)]$$

and

$$K(t,s) = S(t-s)\tilde{H} \qquad \text{... (12)}$$

$K(t,s)$ is linear and continuous in the domain $0 \le t \le s \le T$, hence it is bounded.

Define

$$r(t) = s(t)x_0, r(t) \in R^n \qquad \text{... (13)}$$

Now the cost functional is expressed as

$$I(u) = < r + Lu, Qr + QLu > + < u, Pu > \qquad \text{... (14)}$$

Let L^T be the adjoint operator of L then,

$L^T \in B[L_2(0,T,R^n), L_2(0,T,R^m)]$ and

$< z, Lw > = < L^T z, w >, z \in L_2\{0,T,R^m]$ and $w \in L_2[0,T,R^m]$ holds.

Equation (14) in an expanded form becomes:

$$I(u) = <r,Qr> + 2 < L^T Qr, u> + < L^T QL, u> + <Pu, u> \qquad \text{... (15)}$$

where

$$L^T Qr(t) = \int_0^T L^T(t,s)Qr(t)dt \qquad \text{... (16)}$$

$$L^T QLu(t) = \int_0^T \Lambda(s,t)u(\tau)d\tau \qquad \text{... (17)}$$

and

$$\Lambda(s,t) = \int_0^T K^T(t,s)QK(t,s)d\tau, \max(t,s), 0 \le t \le s \le T \qquad \text{...(18)}$$

Define anther operator by:

$$R = L^T QL + P \qquad \text{... (19)}$$

Then (eqn 15) is expressed as:

$$I(u) = < Ru, u >_2 + 2 < L^T Qr, u > + < r(t), Qr(t) >_1 \qquad \text{... (20)}$$

R is clearly self-adjoint and positive definite.

Now from the relation $\|u\| \le m_1$ and $\|v\| \le m_2$ we have the following relation:

$$m_1 - u^2(t) \ge 0$$
$$m_2 - v^2(t) \ge 0 \qquad \text{... (21)}$$

Let

$$\Phi_1(u) = \begin{bmatrix} m_1 - u_1^2(t) \\ . \\ . \\ . \\ m_1 - u_m^2(t) \end{bmatrix}$$

... (22)

$$\Phi_2(v) = \begin{bmatrix} m_2 - v_1^2(t) \\ . \\ .. \\ . \\ m_2 - v_m^2(t) \end{bmatrix}$$

Clearly

$\Phi_1(u) \ge 0, \Phi_2(v) \ge 0$ hence,

$$\Phi_1(u) + \Phi_2(v) \ge 0 \qquad \text{... (23)}$$

By virtue of (8)

$$\Phi(u) = \begin{bmatrix} M-(1+\beta)u_1^2(t) \\ . \\ . \\ . \\ M-(1+\beta)u_m^2(t) \end{bmatrix} \qquad \text{... (24)}$$

where

$\Phi(u) = \Phi_1(u) + \Phi_2(v)$ and $M = m_1 + m_2$

Functional (20) is concave while functional (23) is convex. Now according to Hausdorff (1972)

$I(u)$ is minimized by $\bar{u}$ provided there exist a non-negative $m-$ dimensional vector function:

$\lambda(t) = (\lambda_1, ...\lambda_m) \in R^m$ such that:

$$J(u,\lambda) = I(u) - <\lambda, \Phi(u)>_2 \qquad \text{... (25)}$$

satisfies the saddle point inequality defined by:

$$J(u,\bar{\lambda}) \ge J(\bar{u},\bar{\lambda}) \ge J(\bar{u},\lambda). \qquad \text{... (26)}$$

The expression (26) shows that $J(u,\lambda)$ is maximized by $\lambda \ge 0.$ In what now follows, we write (20) as:

$$J(u,\lambda) =< Ru,u > +2 < L^T Qr(t),u > + < r(t),Qr(t) > - < \lambda, \Phi(u) > . \qquad \text{... (27)}$$

Let $D(h)J(u,\lambda)$ denote the Frechet derivatives of J at u

then

$$D(h)J(u,\lambda) = 2 < Ru,h > +2 < L^T Qr(t),h > - < \frac{\partial \Phi}{\partial u}\lambda, h > \qquad \text{... (28)}$$

where

$$\frac{\partial \Phi}{\partial u} = \frac{\partial \Phi_i}{\partial u_j} = -2(1+\beta)\begin{bmatrix} u_1.....0 \\ . \\ . \\ . \\ 0.....u_m \end{bmatrix}$$

Setting $D(h)J(\lambda,u) = 0$ for arbitrary h then;

$$Ru + L^T Qr(t) - \frac{1}{2}\frac{\partial \Phi}{\partial u}\lambda = 0 \qquad \text{... (29)}$$

Set $L^T Qr(t) = f(t)$ and from (16), (17) and (18)

$$Ru = \int \Lambda(s,\tau)u(\tau)d\tau + Pu(s), \tau, s \in [0,T] \qquad \text{... (30)}$$

Using Haudorff (1972), we apply the generalized Kunh-Tucker condition to (27) to get the following relations:

$$\lambda(t) \ge 0 \qquad \text{... (31)}$$
$$< \lambda(t), (M-(1+\beta)u^2(t) >= 0 \qquad \text{... (32)}$$
$$\int_0^T \Lambda(s,\tau)u(\tau)d\tau + Pu(s) - (1+\beta)\lambda u(t) = -f(t) \qquad \text{... (33)}$$

where

$$L^T Qr(t) = f(t) \qquad \text{... (34)}$$

From (31) and (32) we found that if $\lambda(t) = 0$ then $\|u(t)\| \le M/(1+\beta)$ now chose:

$$m = (\frac{m_1}{1+\beta}, \frac{m_2}{1+\beta})$$

Then $\|u(t)\| \le m$ and $\|v(t)\| \le m$ also if $x(t) > 0,$ then,

$$u(t) = \sqrt{\frac{M}{(1+\beta)}} \le m \qquad \text{... (35)}$$

Set

$$w(t) = Pu(t) - (1+\beta)\lambda(t)u(t)$$

in (33) let the relationship between $u(t)$ and $w(t)$ be denoted by $\Omega(w(t))$ then we can express (33) as

$$w(t) + \int_0^T \Lambda(t,s)\Omega(w(s))ds = -f(t) \qquad \text{... (36)}$$

The integral equation in (36) is of Hammerstein type. We wish to apply a version of Shauder-Tychnov fixed point theorem stated below for subsequent development of this paper.

Theorem (1): Let K be a closed and convex subset of a Banach space X. Let $T : K \to K$ be a compact mapping, then T has a fixed point in $K.$

We now prove the following theorem using theorem (1)

Theorem (2): The integral equation defined by (33) has a solution in $L_2[0,T]$ provided

(i) $f(t) \in L_2[0,T]$

(ii) The function $\Lambda(.,.) : (0,T) \times (0,T) \to R$ is continuous for all $0 \le t \le s \le T$ and square integrable.

(iii) The function $\Omega : (0,T) \to R$ is such that $\Omega(w(t))$ is continuous and bounded for all $0 \le t \le T.$

Proof: Define an operator $\hat{T}$ mapping $L_2[0,T]$ into $L_2[0,T]$ by

$$[\hat{T}w](t) = \int_0^T \Lambda(t,s)\Omega(w(s))ds. \qquad \text{... (37)}$$

Then w is a solution of (33) if it is a fixed point of (37). Now, we first show that $[\hat{T}w] \in L_2[0,T], \quad 0 \le t \le T.$

To see this, consider

$$\int_0^T \left[\int_0^T \Lambda(t,s)\Omega(w(s))ds \right]^2 dt$$
$$\le \int_0^T \left[\int_0^T |\Lambda(t,s)\|\Omega(w(s))| ds \right]^2 dt$$
$$\le \int_0^T \left[\left(\int_0^T |\Lambda(t,s)|^2 ds \right)^{\frac{1}{2}} \left(\int_0^T |\Omega(w(s)|^2 ds \right)^{\frac{1}{2}} \right]^2 dt$$

(By Holder's inequality)

$$= \|\Omega(w(s)\|_{L_2[0,T]} \int_0^T \int_0^T |\Lambda(t,s)|^2 dsdt$$

Set

$$\int_0^T \int_0^T |\Lambda(t,s)|^2 dsdt = \alpha^2$$

Then

$$\|\Omega(w(s))\|_{L_2[0,T]} \int_0^T \int_0^T |\Lambda(t,s)|^2 dsdt$$

$$= \alpha^2 \|\Omega(w(s))\|^2_{L_2[0,T]} < \infty$$

This shows that $[\hat{T}w](t) \in L_2[0,T] \quad 0 \le t \le T.$

Now $[\hat{T}w] \in L_2$ [0, T] implies $\|\hat{T}w\| < \infty$ and this implies the existence of $\hat{M} > 0,$ such that:

$$\|\hat{T}w\|_{L_2[0,T]} \le \hat{M}, \hat{M} < \infty$$

Define K such that:

K=$\{w \in L_2[0,T], \|\hat{T}w\| \le \tilde{M}\}$

Clearly K is closed and convex. Let H be a subset of K such that

H=$\{\hat{T}w : w \in K\}$

We show that H is relatively compact, by showing that

(i) H is uniformly bounded.
(ii) H is equicontnuous

H is uniformly bounded

Let $\tau \in [0,T]$ then for $\hat{T}w \in H,$

$$\|\hat{T}w\| = \left(\int_0^T \left(\int_0^T \Lambda(t,s)\Omega(w(s))ds \right)^2 \right)^{\frac{1}{2}} dt \le \sup\|\hat{T}w(\tau)\| \le M_{\tau\in[0,T]}$$

This implies that $\|\hat{T}w\| \le M$ hence H is uniformly bounded.

H is equicontnuous

Since $\Lambda(.,.)$ is continuous on a compact domain [0, T] then it is uniformly continuous. Therefore given $\tau_1, \tau_2 \in [0,T]$ and $\varepsilon_1 > 0, \exists$ a $\delta > 0$ such that

$\|\Lambda(\tau_1, s) - \Lambda(\tau_2, s)\| \le \varepsilon_1$ whenever $|\tau_1 - \tau_2| < \delta$.

Now for $\hat{T}w \in H$

$$\|(\hat{T}w)(\tau_1) - (\hat{T}w)(\tau_2)\|$$

$$= \left| \int_0^T (\Lambda(\tau_1, s) - \Lambda(\tau_2, s))\Omega(w(s))ds \right|$$

$$\le \int_0^T |\Lambda(\tau_1, s) - \Lambda(\tau_2, s)\|\Omega(w(s))|ds$$

$$\le \int_0^T \varepsilon_1 |\Omega(w(s))|ds$$

Whenever $|\tau_1 - \tau_2| < \delta$

Since $\Omega(w(s))$ is bounded then there exist M_1 >0 such that $|\Omega(w(s))| < M_1$ hence

$$\int_0^T \varepsilon_1 |\Omega(w(s))|ds \le \varepsilon_1 M_1 \int_0^T ds = \varepsilon_1 M_1 T$$

Whenever $|\tau_1 - \tau_2| < \delta$.

Now since ε_1 is arbitrary, given $\varepsilon > 0$, choose $\varepsilon_1 < \dfrac{\varepsilon}{M_1 T}$, then

$$\|(\hat{T}w)(\tau_1) - (\hat{T}w)(\tau_2)\| < \varepsilon$$

whenever $|\tau_1 - \tau_2| < \delta$. This implies that H is uniformly continuous hence equicontnuous. Therefore by Arzela Ascoli theorem, K is relatively compact.

We now check the continuity of $[\hat{T}w]$.

$$\|\hat{T}w_1 - \hat{T}w_2\|^2 = \int_0^T \left\| \left[\int_0^T \Lambda(t,s)(\Omega(w_1(s) - \Omega(w_2(s))ds \right] \right\|$$

$$\le \int_0^T \left[\left(\int_0^T |\Lambda(t,s)|^2 ds \right)^{\frac{1}{2}} \left(\int_0^T |\Omega(w_1(s) - \Omega(w_2(s)|^2 \right)^{\frac{1}{2}} ds \right]^2$$

Therefore

Choose $\varepsilon > \alpha\|\Omega(w_1(s)) - \Omega(w_2(s))\|$, then $\|\hat{T}w_1 - \hat{T}w_2\| < \varepsilon$ whenever $|w_1 - w_2| < \delta$ therefore $\hat{T}$ is continuous and so by Shauder's theorem $\hat{T}$ has a fixed point.

Remark (1): The existence of fixed point for integral equation (36) shows that we can construct a solution on iterative basis which will eventually converge to a fixed point of (36) This fixed point need not be unique however if $\Lambda(t,s)$ in (36) is strictly positive then (36) has a unique solution (Arika, 1976). Equation (36) is necessary for existence of stable control $u(t)$.

Numerical Technique

In this section we show that there exist computational techniques for computing stable control under uncertainty. We recall the conjugate gradient method of Hestenes & Stiefel (1952), which was initially developed for solution of the abstract algebraic equation of the form:

$$Min \;\; F(x) = T_o + < a, x >_H + \frac{1}{2} < x, Ax >_H \;\; \ldots \;\; (37)$$

Where H is a Hilbert space and A is a positive definite operator. The technique admits the following simple steps.

Step 1: Guess an initial point x_0 and compute the gradient at x_0 and denote the gradient by g_0 such that:

$$g_0 = a + Ax \,/\, x = x_0$$

Step 2: Set $p_0 = -g_0$

Step 3: Update the descent sequence as follows:

$$x_{i+1} = x_i + \alpha_i p_i, i \in [0,1,2,\ldots\ldots\ldots..]$$

$$\alpha_i = \frac{< g_i, g_i >}{< p_{i,} Ap_i >}$$

$$g_{i+1} = g_i + \alpha_i Ap_i$$

$$p_{i+1} = -g_{i+1} + \gamma_i p_i$$

$$\gamma_i = \frac{< g_{i+1}, g_{i+1} >}{< g_i , g_i >}$$

Step 4: If there exist a $g_i = 0$ for a particular i, then terminate the procedure else set $i = i + 1$ and go to step 3

The conjugate gradient algorithm had been applied and found to work elegantly on a number of problems such as problem which involve the determination of minimum time paths for climb phase of a V/STOL aircraft (Bryson & Mehra, 1969). Worthy of mention is the achievement made in 1983, when the extended conjugate gradient method, was developed to handle problems in control theory (Ibiejugba, & Onumanyi (1984). Also the convergence estimate of the technique in the upper direction had been established in the work of Ibiejugba & Onumanyi (1984) and recently, the lower bound convergent estimate was attempted in Ibiejugba & Abiola (1985). In this work, a little investigation is made on the algorithm to allow it being applied to the class of problem defined in (1)-(3)

Now recall equation (20) and define it by:

$$I(u) =< Ru, u > +2 < L^T Qr(t), u(t) > + < r(t), Qr(t) > . \quad \ldots (38)$$

It is known that R is self-adjoint and positive definite, where

$$R = L^T QL + P \quad \ldots (39)$$

and

$$L^T QLu = \int_0^T \Lambda(s,\tau)u(\tau)dt \quad \ldots (40)$$

$$\Lambda(s,t) = \int_0^T K^T (t,s)QK(t,s)dt \quad \ldots (41)$$

We can now observe that by explicit determination of the operator $L^T QL$, it is an easy matter to apply the gradient technique to the problem under investigation. Suppose $g(u)$ is the gradient of $I(u)$, we then state the technique in the form applicable to (38)

The Algorithm:

Step 1: Guess $u = 0$, arbitrarily and compute $g_0(u)$ at u_0

Step2: Set $p_0(u) = -g_0(u)$

Step3: update descent sequence as

$$u_{i+1} = u_i + \alpha_i(u)p_i(u)$$

$$\alpha_i(u) = \frac{< g_i(u), g_i(u) >}{< p_i(u), Rp_i(u) >}$$

$$g_{i+1}(u) = g_i(u) + \alpha_i(u)Rp_i(u)$$

$$p_{i+1}(u) = -g_{i+1}(u) + \gamma_i(u)Rp_i$$

$$\gamma_i(u) = \frac{< g_{i+1}(u), g_{i+1}(u) >}{< g_i(u), g_i(u) >}; i \in [0,1,2,......]$$

Remark (2): The convergence of the above algorithm can be established in a manner similar to the one described in Ibiejugba & Abiola (1985).

We summarize the findings in this section by the following proposition.

Proposition (1). The above algorithm generates a sequence $\{u_N\}$ of controls which finally converges to a stable control u^* for problem (1)-(3) given the assumption C1 – C4 holds and with the explicit determination of operator R in (39) which is defined as:

$$R = L^T QL + P \quad \ldots (42)$$

and

$$L^T QLu = \int_0^T \Lambda(s,t)u(\tau)dt \quad \ldots (43)$$

$$\Lambda(s,t) = \int_0^T K^T (t,s)QK(t,s)dt \quad \ldots (44)$$

Conclusion: We have proved the existence of control for a class of uncertain dynamical systems where the uncertainty enters into the system as input bounded uncertainty. In the subsequent paper we are going to apply the results in this paper into the determination of controls for water pollution problem.

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