Full Length Research Article

MARKOV QUEUE GAME WITH VIRTUAL REALITY STRATEGIES

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ABSTRACT
A non cooperative markov game with several unique characteristics was introduced. Some of these characteristics include: the existence of a single phase multi server queuing model and markovian transition matrix/matrices for each game, introduction of virtual situations (virtual reality) or dummies to improve the chances of winning or dominating the opponent, existence of payoff reduction factor as a result of introduction of the virtual realities/dummies etc. The game had two variants: variant 1 for games with customer/input based virtual reality strategies and variant 2 games without customer/input based virtual reality strategies. Based on some of these characteristics enumerated above, as well as other characteristics and assumptions, a mathematical model was formulated for studying the games. From the model, nash equilibria was proved to exist. The model could be effectively used in studying competitive queuing systems which involves crowd renting, crowd hiding and other virtual reality and dummy strategies.

Keywords: Markov games; Queuing; Virtual reality strategies; nash equilibrium, MSC CLASSIFICATION: 91A43

INTRODUCTION

Game theory is the study of the ways in which strategic interactions among rational players produce outcomes with respect to the preferences (or utilities) of those players, none of which might have been intended by any of them. Since the mathematical theory of games was invented by John von Neumann and Oskar Morgenstern (von Neumann & Morgenstern, 1947), a lot of work has been done to expound and advance the theory of games.

The work of Nash Jr (Nash 1950; Nash 1951) from 1950-1951 contributed much to our understanding of the mathematical analysis of non-cooperative games for pioneering analysis of equilibria in the theory of non-cooperative games. The concept of markov/stochastic game and markov strategies were discussed by Shapley (1953), Fink (1964), Fudenberg & Tyrole (1991). In these games there exists a set of states of nature S, a matrix game for every state s is a member of S and a probabilistic transition function that associates with every state s is a member of S and every combination of actions, one for each agent, a probability distribution on future states s' is a member of S.

The markov game concept in this work is remarkably different from those expounded by the above writers. The markov concept in this particular model corresponds to the markovian queuing model (Asmussen, 2003; Hamdy, 2004; Haribaskaran, 2006). Here the states represent the number of customers in the server and there is no matrix game for each state, rather, there exists transition matrix for each player and a corresponding payoff matrix. Introduction of virtual reality strategies change the transition matrices and the payoff matrices effectively changing the effective payoffs for the players.

A very peculiar characteristics of this model is the concept of virtual reality strategies or dummy strategies, and this work is an attempt to help formalize this concept in game theory. Hence, it is expected to bring a new dimension to game theory analysis with the expected introduction of numerous sophisticated mathematical models to expound the concept and apply them to our daily lives.

GAME CHARATERISTICS AND ASSUMPTIONS

The mathematical model was developed based on the following characteristics and assumptions:

1. A non zero sum game.
2. Each competitor (player) is perfectly aware of each other's strategy.
3. The game is a non-cooperative game.
4. The game is a simultaneous play game with finite number of players.
5. There is an (M/M/S) :(FCFS/N/infinity) queuing model behind every game.
6. A nash equilibrium point (NE) or saddle point exists, but the equilibrium point is dynamic.
7. There are two game variants: variant 1 for games with customer/input based virtual reality strategies and variant 2 games without customer/input based virtual reality strategies.
8. Each competitor aware of the markovian transition matrix/matrices underlying the game introduces dummies or virtual situations to improve his/her chances of winning or dominating the opponent.
9. Introduction of virtual realities results to a virtual/transitional probability matrix.

10. For variant 1 and 2 games, Introducing virtual reality strategies or dummies reduces the payoff but increases chances of a competitor dominating the opponent or winning the game.
11. For variant 1 games, the magnitude of payoff reduction is equal to the product of a factor z and the number of introduced virtual situations/dummies.
12. Depending on the magnitude of the z factor, the saddle point/NE may shift.
13. It is assumed that whatever virtual reality strategy adopted by a player does not affect the probable total payoff of his opponent.
14. In variant 2 games, virtual reality strategies include use of mirrors, entertainments etc to induce waiting.
15. Playing conditions may be perfectly the same for both parties. i.e. the transition matrix for the competitors are identical (equal) for the same strategy, or may be different which makes the game biased in favour of any of the competitors.
16. The game could be a 2-person game or an n-person game, but our analysis is based on 2-person game.

THE GAMES QUEUING MODEL

The game's queuing model corresponds to (M/M/S): (FCFS/N/infinity) model (Hamdy, 2004). Hence, the server capacity, N, is finite. Since the queue is markovian/poisson, arrivals are state dependent (Feinberg *et al.,* 2002; Asmussen, 2003; Hamdy, 2004; Haribaskaran, 2006) which means that it is dependent on the number of customers already in the facility. The markovian process could be reduced to a markovian chain with the homogenous markovian transition matrices shown in Figs 1 and 2.

DEVELOPMENT OF THE GAME'S MATHEMATICAL MODEL

The mathematical model as we had earlier noted, assumes that there is a markovian transition matrix corresponding to the game's payoff matrix, and a single phase multi channel queuing model behind every game. The transition matrix for the competitors, X and Y, is represented by: P_x for competitor X and P_y for competitor Y. Typical transition matrices for two competitors, X and Y, are shown in Figs 1 and 2.

Next State

Current State		0	1	... N
	0	$P_{1,1}$	$P_{1,2}$	$...P_{1,N}$
	1	0.0	$P_{2,2}$	$...P_{2,N}$
	.	.	.	.
	.	.	.	.
	.	.	.	.
	N	0.0	0.0	1.0

FIG. 1. TRANSITION MATRIX FOR COMPETITOR X, P_X

Next State

Current State		0	1	... N
	0	$P_{1,1}$	$P_{1,2}$	$...P_{1,N}$
	1	0.0	$P_{2,2}$	$...P_{2,N}$
	.	.	.	.
	.	.	.	.
	.	.	.	.
	N	0.0	0.0	1.0

FIG. 2. TRANSITION MATRIX FOR COMPETITOR Y, P_Y

For the transition matrices (Figs 1 and 2), we assumed that the probability of moving from a higher state to a lower state is zero (0). The states represent the number of customers/units in the facility and N represents the server capacity. A unified transition matrix P for the game produced by combining Px and Py is shown in Fig. 3.

	0	1	... N
0	$P_{1X,1X}$, $P_{1Y,1Y}$	$P_{1X,2X}$, $P_{1Y,2Y}$	… $P_{1X,NX}$,$P_{1Y,NY}$
1	$P_{2X,1X}$, $P_{2Y,1Y}$	$P_{2X,2X}$, $P_{2Y,2Y}$	… $P_{2X,NX}$,$P_{2Y,NY}$
.	.	.	.
.	.	.	.
.	.	.	.
N	$P_{NX,NX}$,$P_{NY,1Y}$	$P_{NX,2X}$, $P_{NY,2Y}$	… $P_{NX,NX}$,$P_{NY,NY}$

FIG. 3. UNIFIED TRANSITION MATRIX, P

Here P_{iXjX} corresponds to the probabilities in the transition matrix for competitor X. P_{iYjY} corresponds to the probabilities in the transition matrix for competitor Y. A typical payoff matrix P for the game is shown in Fig. 4.

	0	1	N
0	$G_{1X,1X}$, $G_{1Y,1Y}$	$G_{1X,2X}$, $G_{1Y,2Y}$	… $G_{1X,NX}$,$G_{1Y,NY}$
1	$G_{2X,1X}$, $G_{2Y,1Y}$	$G_{2X,2X}$, $G_{2Y,2Y}$	… $G_{2X,NX}$,$G_{2Y,NY}$
.	.	.	.
.	.	.	.
.	.	.	.
N	$G_{NX,1X}$,$G_{NY,1Y}$	$G_{NX,2X}$, $G_{NY,2Y}$	…$G_{NX,NX}$,$G_{NY,NY}$

FIG. 4. PAYOFF MATRIX, G

Here
$G_{1X,1X}$ = the payoff when player X moves from state 0 to 0, $G_{1Y,1Y}$ = the payoff when player Y moves from state 0 to 0, $G_{2X,1X}$ = the payoff when player X moves from state 1 to 0, $G_{2Y,1Y}$ = payoff when player Y moves from state 1 to 0 e.t.c. The effective payoff matrix EG is given by:

$$EG = P \cdot G \qquad \text{… (1)}$$

The dot operator in equation 1 carries out the operation of multiplying each payoff by its associated probability of occurrence. The transition and payoff matrices above represent the natural state of the game.

For variant 1 games, if virtual situations/dummies, n, are introduced, the elements in the transition matrix changes yielding a virtual transition/probability matrix, PT. The dummies represent individuals pretending to be part of the queue in order to attract more customers to the server.

If n strategy is introduced by each player, and the payoff reduction factor is z, then the transitional payoff matrix, GT, is given by:

$$GT = G - (n-s)z \qquad \text{...(2)}$$

Here, the operator, –, subtracts the quantity (n-s) z from the affected elements in the matrix G.
The operator, –, is such that:

$$GT_{IXJX} = G_{IXJX} - (n_X - s)z \; . \qquad \text{...(3)}$$
$$GT_{IYJY} = G_{IYJY} - (n_Y - s)z \; ... \qquad \text{... (4)}$$

HERE $i \leq n$

Combining GT_{ixjx} and GT_{iyjy} we obtain GT. The subscripts x and y denotes the quantities for player x and y respectively.

n= the value of the strategy i.e. number of dummies/virtual situations introduced
z= the payoff reduction factor; s= system state

Payoff reduction affects only rows in Fig. 4 affected by virtual reality, and the rows become biased in favour of the player/competitor that introduced the virtual reality strategy.

For variant 2 games, equation 2 is of the form:

$$GT = G - R \qquad \text{... (5)}$$

Here R is the payoff reduction.

$$GT_{IXJX} = G_{IXJX} - R_X \qquad \text{... (6)}$$
$$GT_{IYJY} = G_{IYJY} - R_Y \; .... \qquad \text{... (7)}$$

Hence, equation 1 changes to:

$$EG = PT \cdot GT \qquad \text{... (8)}$$

For variant 1 and 2 games, if virtual reality strategies are introduced by the players, then the grand payoff matrix, GG, is shown in Fig. 5.

$$\begin{array}{c|cccc|} & 0 & 1 & ... & N \\ 0 & GG_{1X,1X}, GG_{1Y,1Y} & GG_{1X,1X}, GG_{1Y,1Y} & ...GG_{1X,NX}, GG_{1Y,NY} \\ 1 & GG_{2X,1X}, GG_{2Y,1Y} & GG_{2X,2X}, GG_{2Y,2Y} & ...GG_{2X,NX}, GG_{2Y,NY} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ N & GG_{NX,1X}, GG_{NY,1Y} & GG_{NX,2X}, GG_{NY,2Y} & ...GG_{NX,NX}, GG_{NY,NY} \end{array}$$

FIG. 5. GRAND PAYOFF MATRIX, GG

Here
$GG_{iX,jX}$ = the cumulative payoff for player X in the effective payoff matrix, EG, when X uses strategy i-1. $GG_{iY,jY}$ = the cumulative payoff for player Y in the effective payoff matrix, EG, when Y uses strategy j-1.

$$GG_{IYJY} = \sum_{I=1}^{N} \sum_{J=1}^{N} EG_{IYJY} \qquad \text{...(9)}$$

$$GG_{IXJX} = \sum_{I=1}^{N} \sum_{J=1}^{N} EG_{IXJX} \qquad \text{...(10)}$$

N= maximum value of strategies

Theorem 1: Nash equilibrium exists in markov queue game with virtual reality strategies.

Theorem 2: Dynamic Nash equilibrium exists in markov queue game with virtual reality strategies.

PROOFS OF THEOREMS

Proof of Theorem 1:

From Nash's theorem (Nash, 1950), an equilibrium point must exist for the payoff matrix GG in Fig. 5, since there are finite number of pure strategies, and from our assumption, finite number of players.

Proof of Theorem 2:

Let the value of z in equation 3 be a at time t_0, hence:
GT_{an} = G – (n-s) a

Here n is finite and can have values: 0, 1, 2, 3,...N.

Thus, equation 8 changes to: $EG_{an} = PT \cdot GT_{an}$

The grand payoff matrix at z = a is GG_a. From Nash's theorem (Nash, 1950), an equilibrium point must exist for the payoff matrix GG_a, since there are finite number of pure strategies and finite number of players.

Hence, let k_a be the equilibrium point at z =a.

Let the value of z in equation 3 change to b at time t_1 while n remains unchanged, hence:

GT_{bn} = G – (n-s) b

Thus, equation 8 changes to: $EG_{bn} = PT \cdot GT_{bn}$ The grand payoff matrix at z = b is GG_b. Let k_b be the equilibrium point at z =b.

$$GG_a \neq GG_b$$

Hence:

$$K_a \neq K_b$$

Or $K_a = K_b$

Alternatively,

Let the value of R in equation 5 be a at time t_0, hence:
GT_{an} = G – a

Thus, equation 8 changes to: $EG_{an} = PT \cdot GT_{an}$

The grand payoff matrix at R = a is GG_a

From Nash's theorem (Nash, 1950), an equilibrium point must exist for the payoff matrix GG_a, since there are finite number of pure strategies and finite number of players.

Hence, let k_a be the equilibrium point at R =a.

Let the value of z in equation 3 change to b while n remains unchanged, hence:
$GT_{bn} = G - b$

Thus, equation 8 changes to: $EG_{bn} = PT \cdot GT_{bn}$

The grand payoff matrix at R = b is GG_b. Let k_b be the equilibrium point at R =b.

$GG_a \neq GG_b$
Hence:

$K_a \neq K_b$
Or $K_a = K_b$

EVALUATION OF A TYPICAL 3 STRATEGY, 2-PERSONS GAME

Typical transition matrices for two competitors are shown below:

$$P_X = \text{Current State} \begin{matrix} \\ 0 \\ 1 \\ 2 \end{matrix} \overset{\text{Next State}}{\begin{matrix} 0 & 1 & 2 \\ 0.20 & 0.60 & 0.20 \\ 0.00 & 0.30 & 0.60 \\ 0.00 & 0.00 & 1.00 \end{matrix}}$$

$$P_Y = \text{Current State} \begin{matrix} \\ 0 \\ 1 \\ 2 \end{matrix} \overset{\text{Next State}}{\begin{matrix} 0 & 1 & 2 \\ 0.20 & 0.60 & 0.20 \\ 0.00 & 0.30 & 0.60 \\ 0.00 & 0.00 & 1.00 \end{matrix}}$$

For the above transition matrices, we assumed that the probability of moving from a higher state to a lower state is zero (0).

If the game is biased in favour of X, P_x looks like this:

$$PB_X = \text{Current State} \begin{matrix} \\ 0 \\ 1 \\ 2 \end{matrix} \overset{\text{Next State}}{\begin{matrix} 0 & 1 & 2 \\ 0.20 & 0.60 & 0.20 \\ 0.00 & 0.30 & 0.70 \\ 0.00 & 0.00 & 1.00 \end{matrix}}$$

A unified transition matrix, P, of the game produced by combining Px and Py is shown below:

$$P_0 = \begin{matrix} \\ 0 \\ 1 \\ 2 \end{matrix} \begin{matrix} 0 & 1 & 2 \\ 0.20,0.20 & 0.60,0.60 & 0.20,0.20 \\ 0.00,0.00 & 0.40,0.40 & 0.60,0.60 \\ 0.00,0.00 & 0.00,0.00 & 1.00,1.00 \end{matrix}$$

A typical payoff matrix G_0 for the game is shown below:

$$G_0 = \begin{vmatrix} 0,0 & 3,3 & 6,6 \\ 0,0 & 0,0 & 3,3 \\ 0,0 & 0,0 & 0,0 \end{vmatrix}$$

The effective payoff matrix for the game, EG_0, is given by: $EG_0 = G_0 \cdot P_0$

$$EG_0 = \begin{vmatrix} 0.00,0.00 & 1.80,1.80 & 1.20,1.20 \\ 0.00,0.00 & 0.00,0.00 & 1.80,1.80 \\ 0.00,0.00 & 0.00,0.00 & 0.00,0.00 \end{vmatrix}$$

Possible total payoff for player X = 4.8
Possible total payoff for player Y = 4.8

The transition and payoff matrices above represent the natural state of the game.

For the payoff matrix P_0 above, if strategy 1 is used i.e. n=1 and z= 0.25, we have:

$$GT_1 = \begin{vmatrix} -0.25,-0.25 & 2.75,2.75 & 5.75,5.75 \\ 0.00,0.00 & 0.00,0.00 & 3.00,3.00 \\ 0.00,0.00 & 0.00,0.00 & 0.00,0.00 \end{vmatrix}$$

$$PT_1 = \begin{vmatrix} 0.10,0.10 & 0.60,0.00 & 0.30,0.00 \\ 0.00,0.60 & 0.40,0.40 & 0.60,0.00 \\ 0.00,0.30 & 0.00,0.60 & 1.00,1.00 \end{vmatrix}$$

The effective payoff matrix, EG_1, is given by: $EG_1 = GT_1 \cdot PT_1$

$$EG_1 = \begin{vmatrix} -0.03,-0.03 & 1.65,1.65 & 1.73,1.73 \\ 0.00,0.00 & 0.00,0.00 & 1.80,1.80 \\ 0.00,0.00 & 0.00,0.00 & 0.00,0.00 \end{vmatrix}$$

Total payoff for player X = 5.15
Total payoff for player Y = 5.15

If strategy 2 is used i.e. n=2 and z= 0.25, we have:

$$GT_2 = \begin{vmatrix} -0.50,-0.50 & 2.50,2.50 & 5.50,5.50 \\ 0.00,0.00 & -0.50,-0.50 & 2.75,2.75 \\ 0.00,0.00 & 0.00,0.00 & 0.00,0.00 \end{vmatrix}$$

$$PT_2 = \begin{vmatrix} 0.00,0.00 & 0.70,0.70 & 0.30,0.30 \\ 0.00,0.00 & 0.00,0.00 & 1.00,1.00 \\ 0.00,0.00 & 0.00,0.00 & 1.00,1.00 \end{vmatrix}$$

$EG_2 = GT_2 \cdot PT_2$

The effective payoff matrix, EG_2, is given by:

$$EG_2 = \begin{vmatrix} 0.00,0.00 & 1.75,1.75 & 1.65,1.65 \\ 0.00,0.00 & 0.00,0.00 & 2.75,2.75 \\ 0.00,0.00 & 0.00,0.00 & 0.00,0.00 \end{vmatrix}$$

Total payoff for player X = 6.15
Total payoff for player Y = 6.15

The grand payoff matrix, GG_1 for z = 0.25 is given by:

$$GG_1 = \begin{vmatrix} 4.80,4.80 & 4.80,5.15 & 4.80,6.15 \\ 5.15,4.80 & 5.15,5.15 & 5.15,6.15 \\ 6.15,4.80 & 6.15,5.15 & 6.15,6.15 \end{vmatrix}$$

The Nash equilibrium point corresponds to strategy 2, 2.

For the payoff matrix P_0 above, if strategy 1 is used i.e. n=1 and z= 0.5, we have:

$$GT_1 = \begin{vmatrix} -0.50,-0.50 & 2.50,2.50 & 5.50,5.50 \\ 0.00,0.00 & 0.00,0.00 & 3.00,3.00 \\ 0.00,0.00 & 0.00,0.00 & 0.00,0.00 \end{vmatrix}$$

$$PT_1 = \begin{vmatrix} 0.10,1.00 & 0.60,0.60 & 0.30,0.30 \\ 0.00,0.00 & 0.40,0.40 & 0.60,0.60 \\ 0.00,0.00 & 0.00,0.00 & 1.00,1.00 \end{vmatrix}$$

The effective payoff matrix, EG_1, is given by: $EG_1 = GT_1 \cdot PT_1$

$$EG_1 = \begin{vmatrix} -0.05,-0.05 & 1.50,1.50 & 1.65,1.65 \\ 0.00,0.00 & 0.00,0.00 & 1.80,1.80 \\ 0.00,0.00 & 0.00,0.00 & 0.00,0.00 \end{vmatrix}$$

Total payoff for player X = 4.90
Total payoff for player Y = 4.90

If strategy 2 is used i.e. n=2 and z= 0.5, we have:

$$GT_2 = \begin{vmatrix} -1.00,-1.00 & 2.00,2.00 & 5.00,5.00 \\ 0.00,0.00 & -0.50,-0.50 & 2.50,2.50 \\ 0.00,0.00 & 0.00,0.00 & 0.00,0.00 \end{vmatrix}$$

$$PT_2 = \begin{vmatrix} 0.00,0.00 & 0.70,0.70 & 0.30,0.30 \\ 0.00,0.00 & 0.00,0.00 & 1.00,1.00 \\ 0.00,0.00 & 0.00,0.00 & 1.00,1.00 \end{vmatrix}$$

$EG_2 = GT_2 \cdot PT_2$

The effective payoff matrix, EG_2, is given by:

$$EG_2 = \begin{vmatrix} 0.00,0.00 & 1.40,1.40 & 1.50,1.50 \\ 0.00,0.00 & 0.00,0.00 & 2.50,2.50 \\ 0.00,0.00 & 0.00,0.00 & 0.00,0.00 \end{vmatrix}$$

Total payoff for player X = 5.40
Total payoff for player Y = 5.40

The grand payoff matrix, GG_1 for z = 0.5 is given by:

$$GG_1 = \begin{vmatrix} 4.80,4.80 & 4.80,4.90 & 4.80,5.40 \\ 4.90,4.80 & 4.90,4.90 & 4.90,5.40 \\ 5.40,4.80 & 5.40,4.90 & 5.40,5.40 \end{vmatrix}$$

A look at the payoff matrix, GG_1, above shows that Nash equilibrium point corresponds to strategy 2, 2.

If strategy 1 is used i.e. n=1 and z= 1, we have:

$$GT_1 = \begin{vmatrix} -1.00,-1.00 & 2.00,2.00 & 5.00,5.00 \\ 0.00,0.00 & 0.00,0.00 & 3.00,3.00 \\ 0.00,0.00 & 0.00,0.00 & 0.00,0.00 \end{vmatrix}$$

$$PT_1 = \begin{vmatrix} 0.10,1.00 & 0.60,0.60 & 0.30,0.30 \\ 0.00,0.00 & 0.40,0.40 & 0.60,0.60 \\ 0.00,0.00 & 0.00,0.00 & 1.00,1.00 \end{vmatrix}$$

The effective payoff matrix, EG_1 is given by: $EG_1 = GT_1 \cdot PT_1$

$$EG_1 = \begin{vmatrix} -0.10,-0.10 & 1.20,1.20 & 1.50,1.50 \\ 0.00,0.00 & 0.00,0.00 & 1.80,1.80 \\ 0.00,0.00 & 0.00,0.00 & 0.00,0.00 \end{vmatrix}$$

Total payoff for player X = 4.40
Total payoff for player Y = 4.40

If strategy 2 is used i.e. n=2 and z= 1, we have:

$$PT_2 = \begin{vmatrix} 0.00,0.00 & 0.70,0.70 & 0.30,0.30 \\ 0.00,0.00 & 0.00,0.00 & 1.00,1.00 \\ 0.00,0.00 & 0.00,0.00 & 1.00,1.00 \end{vmatrix}$$

$$GT_2 = \begin{vmatrix} -2.00,-2.00 & 2.00,2.00 & 5.00,5.00 \\ 0.00,0.00 & -2.00,-2.00 & 2.00,2.00 \\ 0.00,0.00 & 0.00,0.00 & 0.00,0.00 \end{vmatrix}$$

The effective payoff matrix EG_2 , is given by: $EG_2 = GT_2 \cdot PT_2$

$$EG_2 = \begin{vmatrix} 0.00,0.00 & 0.70,0.70 & 1.20,1.20 \\ 0.00,0.00 & 0.00,0.00 & 2.00,2.00 \\ 0.00,0.00 & 0.00,0.00 & 0.00,0.00 \end{vmatrix}$$

Total payoff for player X = 3.90
Total payoff for player Y = 3.90

The grand payoff matrix, GG_2 for z = 1 is given by:

$$GG_2 = \begin{vmatrix} 4.80,4.80 & 4.80,4.40 & 4.80,3.90 \\ 4.40,4.80 & 4.40,4.40 & 4.40,3.90 \\ 3.90,4.80 & 3.90,4.40 & 3.90,3.90 \end{vmatrix}$$

The Nash equilibrium/saddle point corresponds to strategy 0, 0.

CONCLUSION

The mathematical model developed for the game in this work could be used to analyze games where queues and Markov processes are involved, and dummies are used as game strategies to increase payoffs and improve chances of winning or dominating the opponent. The virtual reality strategies in variant 1 and variant 2 games could be combined in a single game to develop variant 3 games. The mathematical foundation for modeling variant 3 games has been laid in this work.

The model as already stated, could be effectively used in studying games which involves crowd renting, crowd hiding and other virtual reality strategies such as competitive human loading systems, political campaign and electioneering strategies in many countries, and in other systems yet to be identified.

Further more the concept of virtualization/virtual reality, payoff reduction and Nash equilibrium/saddle point shift introduced in this work could be applied to business, economics, management, psychology and in other systems yet to be identified. The mathematical foundations for such systems would be the subject of a future publication.

REFERENCES

Asmussen, S. (2003). *Applied Probability and Queues*. Springer Publications, New York.

Clemen, R. J. & Reilly, T. (1996). *Making Hard Decisions: An Introduction to Decision Analysis*, 2nd edition, Duxbury, Pacific Groove, CA.

Feinberg, E. A. & Shwartz, A. (2002), *Handbook of Markov Decision Processes*, Kluwer.

Fink, A. M. (1964). Equilibrium points of stochastic noncooperative games. *Journal of Science Hiroshima University Series A* 28, 89-93.

Fudenberg, D. & Tirole, J. (1991). *Game Theory*. MIT Press, Cambridge, MA.

Hamdy, A. T. (2004). *Operations Research*. Prentice Hall of India Ltd, New Delhi.

Haribaskaran, G. (2006). *Probability, Queuing Theory and Reliability Engineering*, 2nd Edition. Laxmi Publications.

Nash, J. (1950). Equilibrium Points in *n*-Person Games. *National Academy of Sciences of the USA* 36:48-49.

Nash, J. (1951). Non-cooperative Games. *Annals of Mathematics Journal* 54:286-295.

Shapley, L., (1953). Stochastic games. Proceeding of National Academy of Sciences of the USA 39, 1095–1100.

von Neumann, J. & Morgenstern, O. (1947). *The Theory of Games and Economic Behavior*. Princeton: Princeton University Press, 2nd edition.