Full Length Research Article

STOCHASTIC INTEREST RATES MODEL IN COMPOUNDING

*GALADIMA, D. J.[1], CHOJI, D. N.[2] & ADEJO ,B. O.[3]

[1]School of Science Technology, Federal College of Chemical and Leather Technology, PMB 1034, Zaria. Nigeria.
[2] Department of Mathematics, University of Jos, Nigeria.
[3]Department of Mathematical Sciences, Kogi State University, Lokoja, Nigeria.
*galadimadauda@yahoo.com

ABSTRACT
Interest rates considerations in cash flows are fundamental concepts in finance, real estate, insurance, accounting and other areas of business administration. The assumption that future rates are fixed and known with certainty at the beginning of an investment, is a restrictive and theoretical assumption that is not obtainable in real situations. A more realistic approach would be, to report the expected future value and its variance for a given return process. This paper derives formulae for the mean and variance of future values for a single cash flow and sequences of cash flows when returns processes are randomly and independently distributed. Numerical examples are given to illustrate the magnitude of the change from the fixed rate of return process to stochastic (random) rate of return processes.

***Keywords*:** Stochastic, Cash flow, Rate of returns, future value

INTRODUCTION
Compounding of cash flows are fundamental concepts in financial mathematics, real estate, insurance, actuarial science and business administration (McCutcheon & Scott, 1989).

Most textbooks written in these areas include how to find the future value of a current cash flow and the present value of a future flow. An investor is basically interested in the future value of his investment for T periods at an interest rate or rate of return of *r* per annum (Bowers *et al*., 1986). The traditional compounding formula as discussed by Galadima *et al.* (2007) for unit finance is given as $(1+r)^T$.

The assumption that future interest rates are known with certainty at the time of investment is not practicable. The validity of this restrictive assumption has raised criticism in the real world. The argument is that, only zero coupon bonds are fully hedged and has fixed and known-in-advance rates of return.

Most investments have future reinvestment rates that may vary randomly with time. This suggests that we ought to consider the problem in a broader context of randomly varying interest rates in compounding cash flows when returns are randomly distributed overtime. This paper develops the formulae for the future value of a single cash flow as well as sequences of flows under the assumption that, interest rates are identically and independently distributed. Examples and applications are included to illustrate the methods of compounding.

FUTURE VALUE OF A SINGLE CASH FLOW
Let It be the interest rate or rate of return for period t, (t = 1, 2… T) on a one period pure discount bond X_0 invested at time t = 0, will accumulate after T periods to

$$FV(X_0) = X_0\left[(1+I_1)(1+I_2)\ldots(1+I_T)\right]$$

Taking natural logarithm,

$$\ln(FV(X_0) = \ln\left(X_0\left[(1+I_1)(1+I_2)\ldots(1+I_T)\right]\right)$$
$$\ln(FV(X_0) = \ln(X_0) + \ln(1+I_1) + \ln(1+I_2) + \ldots + (1+I_T)$$

$$\ln(FV(X_0) = \ln(X_0) + \sum_{t=1}^{T}\ln(1+I_t)$$

From which we obtain

$$FV(X_0) = X_0\exp\left(\sum_{t=1}^{T} r_t\right) \qquad \ldots \qquad (1)$$

where $r_t = \ln(1+I_t)$ is the continuously compounded returns for the period t. The textbook approach is a special case of equation (1) and may be obtained by setting r_t = r (a known constant) for all the periods. Generalising the compounding formula with future rates as random variables, the actual value of I_t at some future date t, can not be known as at the time of the initial investment. Now denote by M(y,i) the moment generating function (Mgf) of the random variable y evaluated at the real number i. By definition (Giaccotto, 1989), M(y,i) equals E(exp(iy)).

Let $y = \sum_{t=1}^{T} r_t,$ and i =1

it is clear that the expected future value of an investment X_0 is the mgf of the sum of the interest rates evaluated at the point +1:

$$E\big(FV(X_0)\big) = X_0\left(M\left(\sum_{t=1}^{T} r_t, +1\right)\right)$$

The second moment of the expected future value $FV(X_0)$ is the mgf evaluated at +2:

$$E\big(FV^2\big(X_0^2\big)\big) = X_0^2\left(M\left(\sum_{t=1}^{T} r_t, +2\right)\right) \qquad \ldots \qquad (2)$$

from which we obtain the variance of the future value of X_0 as;

$$Var\big(FV\big(X_0\big)\big) = X_o^2\left\{M\sum_{t=1}^{T} r_t, +2 - \left[M\left(\sum_{t=1}^{T} r_t, +1\right)\right]^2\right\} \qquad \ldots \qquad (3)$$

Next, we assume that r_t is a normally distributed random variable with mean μ and variance σ2 and r_t is independent of r_s for t ≠s. The mgf for normally distributed random variables has a closed form solution given by:

$$m(y,i) = \exp\left(i\mu + i^2\left(\frac{\sigma^2}{2}\right)\right)$$

where μ is the mean of y and σ^2 is its variance. The normality of r_t implies that the sum $\sum_{t=1}^{T} r_t$ is $N(T\mu, T\sigma^2)$.

Thus the stochastic formula for compounding a single cash flow is given by

$$E\big(FV\big(X_O\big)\big) = X_O \exp\left(T\mu + T\frac{\sigma^2}{2}\right) \qquad \ldots \qquad (4)$$

and the corresponding variance around the mean is given by

$$Var\big(FV\big(X_O\big)\big) = X_O^2\left\{\exp\big(2T\mu + T\sigma^2\big)\right\}\left\{\exp\big(T\sigma^2\big) - 1\right\} \qquad \ldots \qquad (5)$$

We observe that the expected future value is positively related to both μ and σ^2. Hence an investment is likely to be worth more, after T periods as either the mean return or the variability of returns increases (Kotiah, 1991). The variance of the future value is a positive function of μ and σ^2, but the coefficient of variation depends on σ^2 only. Rewrite Equation (5) as

$$Var\big(FV\big(X_O\big)\big) = \left\{E(FV(X_0))\right\}^2\left\{\exp\big(T\sigma^2\big) - 1\right\} \qquad \ldots \qquad (6)$$

Both Francis (1995) and Murray & Larry (1999) defined coefficient of variation (CV) as the standard deviation divided by the mean, that is,

$$CV\left(FV\left(X_O\right)\right) = \left\{\exp\left(T\sigma^2\right)\right\}^{\frac{1}{2}} \quad \ldots \quad (7)$$

We observe that if σ^2 is zero, Equation (4) corresponds to the formula found in textbooks for (continuous) compounding with fixed and known reinvestment rates. In this case Equation (5) reduces to zero, implying that the future value of Xo is known for sure at the time of the initial investment Galadima (2005).

The main difference between the traditional formula for the future value i.e. $FV(X_0) = X_0 \exp(T\mu)$ and Equation (4) is the presence of the term $T\dfrac{\sigma^2}{2}$ in the exponent. This reveals that the net effect of going from fixed to random interest rates is to increase the effective compounding rate by half the variance of the return process. Hence the variance around the mean may be used to measure the degree of confidence about the possible future value of an investment.

FUTURE VALUE FOR A SEQUENCE OF CASH FLOWS

Consider a sequence of cash flows, $X_1, X_2,...X_T$. Consistent with the traditional treatment, the flows are assumed to occur at the end of each period. Now, define the vector of (T) real numbers

$A_t = 0....0,1...1$ where there are t 0s and T–t 1s, also define the vector of random interest rates ***I*** = (r_1, r_2... r_T), and let $a_t = A_t I$. The future value of X_1, X_2... X_T is given by

$$FV\left(X_1, X_2,..., X_T\right) = \sum_{t=1}^{T} X_t \exp\left(a_t\right) \quad \ldots \quad (8)$$

It is clear that the expected future value of Equation (8) is the sum of the mgf of each a_t evaluated at the point +1:

$$E(FV\left(X_1, X_2,..., X_T\right) = \sum_{t=1}^{T} X_t M\left(a_t, +1\right) \quad \ldots \quad (9)$$

and the second moment of Equation (8) is

$$E(FV^2\left(X_1, X_2,..., X_T\right)$$

$$= \sum_{t=1}^{T} X_t M(a_t, +2) + 2\sum_{t=1}^{T-1}\sum_{j=i+1}^{T} X_i X_J M\left(a_t + a_j, +1\right) \quad \ldots \quad (10)$$

To compute these moments, we obtain the mean and variance of at as

$$E\left(a_t\right) = \left(T - t\right)\mu \quad \ldots \quad (11)$$

$$\operatorname{var}\left(a_t\right) = \left(T - t\right)\sigma^2 \quad \ldots \quad (12)$$

hence the formula for compounding a sequence of cash flows X1, X2… XT is given by

$$E\left(FV\left(X_1, X_2,..., X_T\right)\right) = \sum_{t=1}^{T} X_t \exp\left[\mu\left(T - t\right) + \frac{\sigma^2}{2}\left(T - t\right)\right] \quad \ldots \quad (13)$$

For a sequence of constant payments at equal intervals, (i.e. an annuity) the future value factor of annuity (FVA) of the sequence obtained using geometric summation reduces to,

$$E\left\{FVA\left(X\right)\right\} = X\left[\frac{\exp\left\{\left(\mu + \frac{\sigma^2}{2}\right)T\right\} - 1}{\exp\left(\mu + \frac{\sigma^2}{2}\right) - 1}\right] \quad \ldots \quad (14)$$

Equation (14) corresponds to the traditional textbook formula for future value of annuity

$$FVA(X) = X\left\{\frac{\exp(T\mu)-1}{\exp(\mu)-1}\right\} \quad \ldots \quad (15)$$

Hence, the compounding rate is increased by $\frac{\sigma^2}{2}$; therefore the traditional formula could be used with an effective interest rate of $\mu + \frac{\sigma^2}{2}$.

To compute the variance of the future value, the variance of the sum $a_i + a_j$, using the fact that $a_i + a_j = 2a_j + (a_j - a_i)$, we have

$$\text{Var}(a_i + a_j) = 4\text{Var}(a_i) + \text{Var}(a_j - a_i) + 4\text{cov}(a_i, a_j - a_i) \quad \ldots \quad (16)$$

The covariance between a_i and a_i-a_j is zero, that is;

$$4\text{cov}(a_i, a_j - a_i) = 0$$

$$Var(a_i + a_j) = 4(T - j)\sigma^2 + (j - i)\sigma^2 \quad \ldots \quad (17)$$

Therefore, $\text{var}(FV(X_1, X_2, \ldots, X_T))$

$$= E\left[FV^2(X_1, X_2, \ldots, X_T)\right] - \left\{E\left[FV(X_1, X_2, \ldots, X_T)\right]\right\}^2 \quad \ldots \quad (18)$$

where $E\left[FV(X_1, X_2, \ldots, X_T)\right]$ is given in (13) and

$$E\left[FV^2(X_1, X_2, \ldots, X_T)\right] = \sum_{i=1}^{T} X_i^2 \exp\left[2\mu(T-t) + 2\sigma^2((T-t))\right]$$
$$+2\sum_{i=1}^{T-1}\sum_{j=i+1}^{T} X_i X_j \exp\left[\mu(2T-j-i) + \left(\frac{\sigma^2}{2}\right)(4T-3j-i)\right] \quad \ldots \quad (19)$$

To illustrate the formulae for compounding cash flows when rates of returns are random variables, we have computed the mean and standard deviation (given right below the corresponding mean) for the future value of unit finance, both as a single cash flow and as an annuity for T periods. The results are displayed in Tables I and II. The values of μ used are 12%, 16% and 20% respectively. For each value of μ the moment have been computed four times; once with $\sigma = 0$, a second time with $\sigma = \frac{\mu}{4}$, a third time with $\sigma = \frac{\mu}{2}$ and lastly with $\sigma = \mu$.

These are displayed respectively below each μ value (The μ values are written in bold type and the corresponding σ values are not in bold). Note that when the standard deviation is zero (the first element in each cell) we have the standard case of constant and known returns and the mean value given in any standard mathematics of finance textbook. For $\sigma > 0$, future values are not known with certainty. The degree of uncertainty is reflected by the non-zero standard deviation in Table 1 and Table 2.

CONCLUSION

An area of application that could be handled with the present approach of random rates of return includes contribution to a pension fund, cooperative society's contributions or an Individual Retirement Account (IRA). If the rates of return are assumed fixed and known in advance, the future value of a sequence of contributions to an IRA will be known with certainty. Choosing between one fund and another in such a situation is trivial because there is no risk; obviously investors will prefer the asset with the highest possible return. Many advertisements by banks and investment firms routinely report the future value of an IRA as if such a value could be achieved with certainty under all investment possibilities. A more realistic approach would be, to report the expected future value and its variance for a given return process. An individual could then choose an IRA portfolio that maximizes the expected future value subject to a given level of risk. The formulae developed above could be used to estimate both moments for a given sequence of contributions and the return process. A number of numerical examples are given to illustrate the magnitude of the change from fixed to random returns.

TABLE 1. FUTURE VALUE OF A SINGLE PAYMENT IN T PERIODS

μ	12%				16%				20%			
T \ σ	**0%**	**3%**	**6%**	**12%**	**0%**	**4%**	**8%**	**16%**	**0%**	**5%**	**10%**	**20%**
1	**1.127**	**1.128**	**1.130**	**1.136**	**1.174**	**1.174**	**1.177**	**1.189**	**1.221**	**1.223**	**1.228**	**1.246**
	0.00	0.03	0.07	0.14	0.00	0.05	0.09	0.19	0.00	0.06	0.12	0.25
2	**1.271**	**1.272**	**1.276**	**1.290**	**1.377**	**1.379**	**1.386**	**1.413**	**1.492**	**1.496**	**1.507**	**1.553**
	0.00	0.05	0.11	0.22	0.00	0.08	0.16	0.32	0.00	0.11	0.21	0.45
3	**1.433**	**1.435**	**1.441**	**1.465**	**1.616**	**1.620**	**1.632**	**1.679**	**1.822**	**1.829**	**1.850**	**1.935**
	0.00	**0.07**	**0.15**	**0.31**	0.00	**0.11**	**0.23**	**0.47**	0.00	**0.16**	**0.32**	**0.69**
4	**1.616**	**1.619**	**1.628**	**1.663**	**1.896**	**1.903**	**1.921**	**1.996**	**2.226**	**2.237**	**2.270**	**2.411**
	0.00	0.10	0.20	0.41	0.00	0.15	0.31	0.66	0.00	0.22	0.46	1.00
5	**1.822**	**1.826**	**1.839**	**1.889**	**2.226**	**2.234**	**2.261**	**2.373**	**2.718**	**2.735**	**2.787**	**3.004**
	0.00	0.12	0.25	0.52	0.00	0.20	0.41	0.88	0.00	0.31	0.63	1.41
6	**2.054**	**2.060**	**2.077**	**2.145**	**2.612**	**2.624**	**2.662**	**2.820**	**3.320**	**3.345**	**3.421**	**3.743**
	0.00	0.15	0.31	0.64	0.00	0.26	0.53	1.15	0.00	0.41	0.85	1.95
7	**2.316**	**2.324**	**2.346**	**2.436**	**3.065**	**3.082**	**3.134**	**3.352**	**4.055**	**4.091**	**4.200**	**4.665**
	0.00	0.18	0.37	0.79	0.00	0.33	0.67	1.49	0.00	0.54	1.13	2.65
8	**2.612**	**2.621**	**2.650**	**2.767**	**3.597**	**3.620**	**3.690**	**3.984**	**4.953**	**5.003**	**5.155**	**5.812**
	0.00	0.22	0.45	0.97	0.00	0.41	0.85	1.90	0.00	0.71	1.49	3.57
9	**2.945**	**2.957**	**2.993**	**3.142**	**4.221**	**4.251**	**4.344**	**4.736**	**6.050**	**6.118**	**6.328**	**7.243**
	0.00	0.27	0.54	1.17	0.00	0.51	1.06	2.41	0.00	0.92	1.94	4.77
10	**3.320**	**3.335**	**3.380**	**3.568**	**4.953**	**4.993**	**5.114**	**5.629**	**7.389**	**7.482**	**7.768**	**9.025**
	0.00	0.32	0.65	1.40	0.00	0.63	1.31	3.04	0.00	1.19	2.52	6.33

μ = mean value of the return process
σ = standard deviation of the return process
Each cell contains the expected future value FV and, right below, its standard deviation

TABLE 2. FUTURE VALUE OF A UNIT ANNUITY FOR T PERIODS

μ	12%				16%				20%			
T \ σ	**0%**	**3%**	**6%**	**12%**	**0%**	**4%**	**8%**	**16%**	**0%**	**5%**	**10%**	**20%**
1	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**
	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	**2.13**	**2.13**	**2.13**	**2.14**	**2.17**	**2.17**	**2.18**	**2.19**	**2.22**	**2.22**	**2.23**	**2.25**
	0.0	0.0	0.1	0.2	0.0	0.0	0.9	0.2	0.0	0.0	0.1	0.4
3	**3.40**	**3.40**	**3.41**	**3.43**	**3.55**	**3.55**	**3.56**	**3.60**	**3.71**	**3.72**	**3.73**	**3.80**
	0.0	0.1	0.1	0.2	0.0	0.1	0.2	0.3	0.0	0.2	0.3	0.7
4	**4.83**	**4.84**	**4.85**	**4.89**	**5.17**	**5.17**	**5.19**	**5.28**	**5.54**	**5.55**	**5.58**	**5.73**
	0.0	0.1	0.2	0.4	0.0	0.2	0.5	0.7	0.0	0.4	1.0	2.4
5	**6.45**	**6.45**	**6.47**	**6.55**	**7.06**	**7.08**	**7.12**	**7.28**	**7.76**	**7.78**	**7.85**	**8.14**
	0.0	0.2	0.4	0.5	0.0	0.3	0.9	1.3	0.0	0.7	1.8	3.1
6	**8.27**	**8.28**	**8.31**	**8.44**	**9.29**	**9.31**	**9.38**	**9.65**	**10.48**	**10.52**	**10.64**	**11.15**
	0.0	0.3	0.5	0.6	0.0	0.4	1.2	3.6	0.0	1.3	3.3	5.5
7	**10.32**	**10.34**	**10.39**	**10.59**	**11.90**	**11.94**	**12.04**	**12.47**	**13.80**	**13.86**	**14.06**	**14.89**
	0.0	0.4	0.8	1.3	0.0	0.6	1.8	5.1	0.0	2.3	5.0	8.1
8	**12.64**	**12.66**	**12.74**	**13.02**	**14.97**	**15.02**	**15.17**	**15.82**	**17.85**	**17.96**	**18.26**	**19.56**
	0.0	0.5	1.0	2.0	0.0	1.6	2.3	6.5	0.0	4.0	6.5	14.6
9	**15.25**	**15.29**	**15.38**	**15.79**	**18.56**	**18.64**	**18.86**	**19.81**	**22.81**	**22.96**	**23.42**	**25.37**
	0.0	0.6	1.3	2.7	0.0	1.4	4.4	9.1	0.0	5.1	11.7	21.0
10	**18.20**	**18.24**	**18.38**	**18.93**	**22.78**	**22.89**	**23.21**	**24.54**	**28.86**	**29.08**	**29.75**	**32.61**
	0.0	0.7	1.6	3.2	0.0	2.9	6.6	10.4	0.0	8.5	19.8	28.4

μ=mean value of the return process
σ=standard deviation of the return process
Each cell contains the expected FV and, right below, its standard deviation.

REFERENCES

Bowers, N. L. Gerber, H.U. Hickman, J .C. Jones, D. A. & Nesbitt, C. J. (1986). Actuarial mathematics; The *Society of Actuaries. Chicago,Illinois, USA.*

Francis, A. (1995). *Business mathematics and statistics*; 4th edition. D P Publications. Aldine House/ Aldine Place.

Galadima, D. J. (2005). Application of Difference Equation in Cash Flow Analysis. Unpuplished M.Sc. Thesis. University of Jos, Nigeria.

Galadima, D. J; Choji, D. N; OKON, U. E. (2007). Mathematical Annuity Models: Application in Cash flow Analysis. *Science World Journal* 2 (3): 7-10

Giaccotto, C. (1989). Compounding and Discounting with Stochastic Interest Rates.*Journal. of Business Finance and Accounting* 16(5): 745-769.

Kotiah, T. C. T. (1991). Difference and Differential Equations in the Mathematics of Finance. *International Journal of Mathematics, Education, Science and Technology.* 22:5:783-789.

McCutcheon, J. J. & Scott, W. F. (1989). *An Introduction to the Mathematics of Finance*; Heinemann; Oxford.

Murray, R. S., Larry, J. S. (1999). *Statistics*, Third Edition; McGraw-Hill.

Pandy, I. M. (1999). *Financial Management*, 8th edn. Vikas Publishing House, PVT, Ltd.