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Exponential Time Differencing With Runge-Kutta Time Stepping for Convectively Dominated Financial Problems

Désiré Yannick Tangman *
Faculty of Science,
University of Mauritius,
Email: *y.tangman@uom.ac.mu*

Ashvin Gopaul
Faculty of Science,
University of Mauritius
Email: *a.gopaul@uom.ac.mu*

Muddun Bhuruth
Faculty of Science,
University of Mauritius
Email*: mbhuruth@uom.ac.mu*

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Abstract

We describe how to use Exponential Time Differencing with Runge-Kutta time stepping (ETDRK) for convectively dominated financial problems. For European options with low volatility, we illustrate how the use of ETDRK with flux limiters gives non-oscillatory prices and sensitivity parameters. We also show how accurate Asian option prices can be efficiently obtained by combining high resolution methods with ETDRK to solve the resulting two-dimensional convection dominated Black-Scholes type PDE.

Keywords: Exponential time integrators, Runge-Kutta, Option pricing, TVD discretisations.

**For correspondences and reprints*

1. INTRODUCTION

It is well known in the financial community (Zvan et al. 1998) that spurious oscillations occur if a central approximation is used for the spatial discretisation for convectively dominated financial problems. This is due to the slow decay of longer wavelengths components (Tavella & Randall 2000). Therefore special TVD discretisations involving high-order non-linear flux limiters have been employed by Zvan et al. (1998) in order to obtain non-oscillatory solutions. However, this procedure results in non-linear equations which are computationally expensive to solve using Newton iterations.

We describe here a resourceful scheme based on ETDRK to avoid the non-linear problem. ETDRK methods have been recently introduced by Cox & Matthews (2002) and more recently by Kassam & Trefethen (2005) to solve PDEs that combine high order linear terms with low order nonlinear terms. Most of the test cases considered by these authors involve periodic or constant boundary conditions. Here we show how to incorporate time dependent boundaries in order to apply such schemes in the financial settings. Indeed, using exponential time integrators with Runge-Kutta time stepping allows us to treat the stiff linear part through the exponential operator while the Runge-Kutta time stepping is used to integrate the non-linear advection discretisation terms explicitly. We then show how to price Asian options using ETDRK.

We use in this paper, the two-dimensional PDE as in Wilmott et al. (1995) and Hugger (2004) to price Asian options under the Black-Scholes (BS) model. An Asian option is a path-dependent security since its payoff depends on the average a of the underlying asset price S over a given period of time. However no diffusion terms exist in the average spatial dimension making the pricing equation convectively dominated. Hugger (2004) has tried to prevent the use of non-linear discretisations by adding artificial diffusion but his procedure does not give accurate prices. This means that the high resolution methods proposed in Zvan et al. (1998) and Oosterlee et al. (2004) are essential.

This paper is structured as follows. In section 2, we describe the PDE approach for pricing European options and fixed strike arithmetic Asian options. Next in section 3, we show how to use of ETDRK in combination with various special spatial discretisation for hyperbolic problems such as Kappa schemes with flux limiters, ENO and WENO discretisations. We also include numerical experiments which confirm the efficiency of the proposed methodology. Finally our conclusion is found in section 4.

2. OPTIONS PRICING IN THE BLACK-SCHOLES FRAMEWORK

We consider a financial market with a single asset with price S which follows the geometric Brownian motion

$$dS = (r-\delta)S\,dt + \sigma S\,dW_t,$$

where r is the interest rate, δ the amount of dividend, σ the volatility and dW_t is the standard Weiner process at time t. The value $V(S,\tau)$ of a European option on the asset, solves the initial boundary value problem of the Black Scholes

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta)S\frac{\partial V}{\partial S} - rV, \quad 0 \le S < \infty, \quad 0 \le \tau \le T, \tag{1}$$

where $\tau = T - t$ is the time from expiry and T is the expiry time. It is the feature of each option that characterises the condition at expiry and the boundary conditions at the ends of our computational domain. We now describe these conditions for the well-posed formulation of European and Asian option pricing problems.

European options can only be exercised at maturity and admit an analytical (Black & Scholes1973). So they serve as a guideline to evaluate the effectiveness of our schemes in cases when (1) is convection dominated. We distinguish between a call and a put option which give its holder the right to buy or sell at a fixed strike price E. For a call option, boundary and initial conditions are given by

$$\begin{aligned} V(S,0) &= \max(S-E,0), \\ V(0,\tau) &= 0, \\ V(S,\tau) &= Se^{-\delta\tau} - Ee^{-r\tau}. \end{aligned} \tag{2}$$

Asian options are strongly path dependent. The payoff

$$V(S,a,0) = \max\left(\frac{a}{T} - E, 0\right), \tag{3}$$

of the underlying asset price. It is because of this extra independent variable that it does not satisfy the one dimensional Black-Scholes PDE (1), but rather a two dimensional PDE of the form (Hugger 2004,Wilmott et al. 1995)

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta)S\frac{\partial V}{\partial S} - rV + S\frac{\partial V}{\partial a}. \tag{4}$$

In the case $S = 0$, the option value is just the discounted payoff

$$V(S,a,\tau) = e^{-r\tau}\max\left(\frac{a}{T} - E, 0\right). \tag{5}$$

Also, a is a non-decreasing function of $-\tau$ and for $a \ge ET$, we are guaranteed a positive payoff. For the case $\delta = 0$, the Feynman-Kac theorem (Barraquand &

Pudet 1996, Tavella & Randall 2000) states that under the risk neutral process $\hat{S}_u$ where

$$\mathrm{E}_{T-\tau}\left[\hat{S}_u\right] = S_{T-\tau}, \quad u \le T-\tau, \quad \text{and}$$
$$\mathrm{E}_{T-\tau}\left[\hat{S}_u\right] = S_{T-\tau} e^{r(u-T+\tau)}, \quad u > T-\tau,$$

the option value is

$$\begin{aligned} V(S,a,\tau) &= \mathrm{E}_{T-\tau}\left[e^{-r\tau}\left(\frac{a}{T} - E\right)\right], \\ &= \frac{e^{-r\tau}}{T}\left[\int_0^{T-\tau} \mathrm{E}_{T-\tau}\left[\hat{S}(\xi)\right] d\xi + \int_{T-\tau}^{T} \mathrm{E}_{T-\tau}\left[\hat{S}(\xi)\right] d\xi\right] - e^{-r\tau}E, \\ &= \frac{e^{-r\tau}}{T}\left[\int_0^{T-\tau} S(\xi)\, d\xi + \int_{T-\tau}^{T} e^{r(\xi-T+\tau)} S_{T-\tau}\, d\xi\right] - e^{-r\tau}E, \\ &= e^{-r\tau}\left(\frac{a}{T} - E\right) + \frac{S}{rT}\left(1 - e^{-r\tau}\right). \end{aligned} \tag{6}$$

The extension to dividend yield is straightforward. First we notice that $w(S,a,\tau) = e^{\delta\tau} V(S,a,\tau)$ will satisfy (4) with r replaced by $r-\delta$ in the coefficient of the reaction term (Wilmott et al.1995). So using (6), the Asian option value with dividend will be

$$V(S,a,\tau) = \left(\frac{a}{T} - E\right)e^{-r\tau} + \frac{S}{(r-\delta)T}\left(e^{-\delta\tau} - e^{-r\tau}\right), \quad \text{for} \quad a \ge ET. \tag{7}$$

Similarly for $\sigma = 0$, it can be shown that the option value will be

$$V(S,a,\tau) = \max\left(\left(\frac{a}{T} - E\right)e^{-r\tau} + \frac{S}{(r-\delta)T}\left(e^{-\delta\tau} - e^{-r\tau}\right), 0\right), \tag{8}$$

and this can be regarded as a boundary condition for large asset price $\hat{S}_{\max}$. With the boundary conditions (3), (5), (7) and (8) defined over the whole surface edge of our two-dimensional computational domain, we can compute the option value for $a = 0$. The boundary value formulation for the fixed strike Asian option has been shown to be well-posed by Hugger (2003). We notice that (4) differs from (1) by the addition of a first order derivative in the average spatial dimension. Also, no diffusion exists in a so that with central differencing, the slow decay of longer wavelengths components causes oscillations (Tavella & Randall 2000). As a result, more sophisticated spatial discretisations are necessary.

3. EXPONENTIAL TIME DIFFERENCING SCHEMES

To describe ETD schemes for the pricing of different options, we first consider a European call option without dividend. For a finite difference discretisation of the spatial derivatives in

(1), we need to truncate the semi-infinite S-domain $(0,\infty)$ to a bounded domain $\Omega_S = (0, S_{\max})$. We therefore consider a computational grid $\Omega_{\Delta S} \subset \Omega_S$ defined by

$$\Omega_{\Delta S} = \{S_i \in \Re_+ : S_i = i\Delta S,\ i = 0,1,\ldots,m,\ \Delta S = S_{\max}/m\},$$

and define the central second order approximations to the first and second order spatial derivatives with respect to S by the difference matrices

$$D_S^1 = \frac{1}{2\Delta S}\operatorname{tridiag}[-1,0,1]; \quad D_S^2 = \frac{1}{(\Delta S)^2}\operatorname{tridiag}[1,-2,1];$$

respectively. Then, by discretising the spatial operator in (1), we obtain the matrix

$$A = \frac{1}{2}\sigma^2 S^2 D_S^2 + (r-\delta)S\, D_S^1 - rI_S,$$

where $I_S \in \Re^{m-1\times m-1}$ is the identity matrix for S. Thus PDE (1) becomes the ODE system

$$V'(\tau) = AV(\tau) + b(\tau), \quad 0 \le \tau \le T, \tag{10}$$
$$V(0) = [\max(S_1 - E, 0), \ldots, \max(S_{m-1} - E, 0)]^{\mathrm{T}},$$

where

$$b(\tau) = \left[0,0,\ldots,0,\gamma_{m-1}(S_m - Ee^{-r\tau})\right]^{\mathrm{T}}. \tag{11}$$

To derive an exponential time difference scheme, we multiply (10) by the integrating factor $e^{-A\tau}$ and integrate over the time interval $0 \le \tau \le T$. This gives

$$V(T) = e^{AT}V(0) + e^{AT}\int_0^T e^{-A\tau} b(\tau)\, d\tau. \tag{12}$$

The special structure of the vector $b(\tau)$ allows a closed form expression for the integral term in (12). For this, we write $b(\tau) = \gamma_{m-1}\left(S_m - Ee^{-r\tau}\right)e_{m-1}$ where e_{m-1} is the last vector in the canonical basis of $\Re^{m-1}$. It is then easy to prove the following result.

LEMMA 3.1. *Let*

$$g = e^{AT}\int_0^T e^{-A\tau} b(\tau)\, d\tau.$$

Then

$$g = \gamma_{m-1}\left(S_m A^{-1}\left(e^{AT} - I\right) - E\left(A + rI\right)^{-1}\left(e^{AT} - e^{-rT}I\right)\right)e_{m-1}. \tag{13}$$

Since the matrices A and $(A+rI)$ are tridiagonal, we observe that the implementation of the formula (13) can be efficiently carried out. The term $\left(e^{AT}-e^{-rT}I\right)e_{m-1}$ simply equals the $(m-1)$ th column of the matrix $\left(e^{AT}-e^{-rT}I\right)$ and this term is easily computed once the matrix e^{AT} is available.

3.1 Dominant Convection

When the Black-Scholes PDE (1) is convection-dominated, the central spatial discretisation (9) will cause oscillations in the calculation of the option's price as well as in the greeks delta and gamma. To restore the positivity of the schemes, we use an approximation similar to that used by Zvan, Forsyth & Vetzal (1998) for the convective term. This takes the form

$$\frac{\partial V}{\partial S}=\frac{V_{i+\frac{1}{2}}-V_{i-\frac{1}{2}}}{\Delta S},$$

which is the difference between the in-flux $V_{i+\frac{1}{2}}$ and the out-flux $V_{i-\frac{1}{2}}$. We now explain the different ways to approximate these fluxes. For the computation of $V_{i+\frac{1}{2}}$, upwinding schemes generally set

$$\left[V_{down},V_{up},V_{2up}\right]=\begin{cases}\left[V_i,V_{i+1},V_{i+2}\right], & \textit{if } (r-\delta)\geq 0;\\ \left[V_{i+1},V_i,V_{i-1}\right], & \textit{otherwise}\end{cases}$$

so that

$$V_{i+\frac{1}{2}}=V_{up},$$

for the first order upwind scheme. The κ schemes (van Leer 1977); see also (Wesseling 2000, p.149) provide a framework for formulating upwind biased schemes when convection is dominant. A higher order term is added to the convective flux as follows:

$$V_{i+\frac{1}{2}}=\frac{1}{2}\left(V_i+V_{i+1}\right)+\frac{\kappa-1}{4}\left(V_{2up}-2V_{up}+V_{down}\right).$$

For typical financial parameters, we usually have $r\geq\delta$ and the flux difference will be

$$\Delta S\frac{\partial V}{\partial S}\approx\left(V_{i+1}-V_i\right)-\frac{\kappa}{2}\left(V_{i+1}-V_i\right)+\left(\frac{1+\kappa}{4}\right)\left(V_i-V_{i-1}\right)-\left(\frac{1-\kappa}{4}\right)\left(V_{i+2}-V_{i+1}\right).$$

The case $\kappa=1/3$ gives a third order upwind biased scheme (Anderson et al. 1986) and $\kappa=1/2$ gives the QUICK scheme of Leonard (1979). These schemes are still linear and can be implemented as previously as a one-time step algorithm. The only difference is to incorporate the ghost cell boundaries in the boundary vector g.

However the κ schemes are not monotone and produce spurious oscillations at places that have a discontinuity or a strong gradient. Thus, to make the schemes TVD, we apply flux limiters to the high-order terms as follows:

$$\Delta S \frac{\partial V}{\partial S} \approx \left(V_{i+1} - V_i\right) - \frac{\kappa}{2}\phi\left(q_{i+\frac{1}{2}}\right)\left(V_{i+1} - V_i\right) + \left(\frac{1+\kappa}{4}\right)\phi\left(q_{i-\frac{1}{2}}\right)\left(V_i - V_{i-1}\right) - \left(\frac{1-\kappa}{4}\right)\phi\left(q_{i+\frac{3}{2}}\right)\left(V_{i+2} - V_{i+1}\right),$$

where

$$q_{i+\frac{1}{2}} = \frac{V_{i+2} - V_{i+1}}{V_{i+1} - V_i},$$

and

$$\phi(q) = \frac{q + |q|}{1 + |q|},$$

is the van Leer limiter in van Leer (1974). For $\kappa = 1$, we obtain the scheme proposed by Zvan, Forsyth & Vetzal (1998) to solve the convectively dominated Black-Scholes PDE.

Another way to obtain nonoscillatory solutions for problems with high drift term and low volatility is to use a WENO scheme (Liu et al. 1994, Jiang & Shu 1996). For example a fifth-order WENO scheme computes the upwinding cell-reconstructions over the 5-point stencil $S_r(i) = (S_{i-1}, \ldots, S_{i+3})$ for $r \geq \delta$ as

$$V_{i+\frac{1}{2}} = \sum_{r=0}^{2} \omega_r V^r_{i+\frac{1}{2}},$$

where the quantities

$$V^0_{i+\frac{1}{2}} = \frac{1}{6}\left(11V_{i+1} - 7V_{i+2} + 2V_{i+3}\right),$$

$$V^1_{i+\frac{1}{2}} = \frac{1}{6}\left(2V_i + 5V_{i+1} - V_{i+2}\right),$$

$$V^2_{i+\frac{1}{2}} = \frac{1}{6}\left(-V_{i-1} + 5V_i + 2V_{i+1}\right),$$

are third order accurate. To avoid oscillations, the reconstruction containing the discontinuity is discounted automatically by weights

$$\omega_r = \frac{\alpha_r}{\sum_{i=0}^{2} \alpha_i} \quad \text{where} \quad \alpha_i = \frac{d_i}{\left(\beta_i + \varepsilon\right)^2},$$

with $d_0 = 0.1, d_1 = 0.6, d_2 = 0.3, \varepsilon = 10^{-6}$ and

$$\beta_0 = \frac{1}{12}\left[13\left(V_{i+1} - 2V_{i+2} + V_{i+3}\right)^2 + 3\left(3V_{i+1} - 4V_{i+2} + V_{i+3}\right)^2\right],$$

$$\beta_1 = \frac{1}{12}\left[13\left(V_i - 2V_{i+1} + V_{i+2}\right)^2 + 3\left(V_i - V_{i+2}\right)^2\right],$$

$$\beta_2 = \frac{1}{12}\left[13\left(V_{i-1} - 2V_i + V_{i+1}\right)^2 + 3\left(V_{i-1} - 4V_i + 3V_{i+1}\right)^2\right],$$

represent the smoothness indicators.

It is the way the weights are computed that introduces non-linearity into the discretisation. The Newton iteration for solving the flux limiter scheme with Crank-Nicolson timestepping has been considered in Zvan et al. (1998) and multigrid techniques for fifth-order WENO with Blended Backward Differentiation Formula (BDF2) were studied in Oosterlee et al. (2004). However such techniques are computationally costly. Here we show how flux limiting can be efficiently implemented with ETD and a combination of Runge-Kutta (RK) steps. For a non-linear scheme, the semi-discrete system (10) becomes

$$V^1(\tau) = \mathrm{B}V(\tau) + b(\tau) + N(V(\tau)), \qquad 0 \le \tau \le T, \tag{14}$$

where $N(V(\tau))$ represents the extra nonlinear term due to the flux limiter. Here the matrix B comprises of only diffusion and reaction terms that have been discretised by central difference approximations. Now, integrating (14) over a time step $\Delta\tau$ gives the exact equation

$$e^{-B\tau_{k+1}}V(\tau_{k+1}) = e^{-\mathrm{B}\tau_k}V(\tau_k) + \int_{\tau_k}^{\tau_{k+1}} e^{-\mathrm{B}\varsigma}\left[b(\varsigma) + N(V(\varsigma))\right]d\varsigma.$$

It is the way we approximate the integral term that characterizes the order of an ETDRK scheme. A second order ETD scheme with Runge Kutta time stepping (ETDRK2) (Cox & Matthews 2002) can be formulated as follows. First we use a constant approximation $N(V(\varsigma)) \approx N(V(\tau_k))$ over the interval $\tau_k \le \tau \le \tau_{k+1}$ to obtain

$$a(\tau_k) = e^{B\Delta\tau}V(\tau_k) + g + \Delta\tau\,\varphi_1(B\Delta\tau)N(V(\tau_k)), \tag{15}$$

where $\varphi_1(z) = (e^z - I)/z$ and the update

$$g = \gamma_{m-1}\left(S_m B^{-1}\left(e^{B\Delta\tau} - 1\right) - e^{-r\tau_k}E\left(B + rI\right)^{-1}\left(e^{B\Delta\tau} - e^{-r\Delta\tau}I\right)\right)e_{m-1,}$$

incorporates the boundaries for RK-timestepping. We then use the approximation

$$N(V(\varsigma)) \approx N(V(\tau_k)) + \varsigma(N(a(\tau_k)) - N(V(\tau_k))),$$

to obtain

$$V^{k+1} = a(\tau_k) + \Delta\tau\varphi_2(B\Delta\tau)(N(a(\tau_k)) - N(V(\tau_k))), \qquad (16)$$

where $\varphi_2(z) = (\varphi_1(z) - I)/z$. For an efficient implementation, the matrix vector multiplications in g can be precomputed once at the start since they do not depend on τ_k. Then only vector manipulations remain to be done at each time step. We numerically illustrate the superior performance of the ETDRK2 scheme described by equations (15) and (16) for a European call option with parameters $r = 0.15, o = 0.01, E = 15$ and $T = 1$ and from Table 1, it is the Van Leer flux limiter scheme with $\kappa = 1$ that produces numerical solutions with errors of least magnitude. For central schemes $(\kappa = 1)$ without limiter, we can see oscillations in delta and gamma in Figure 1 while its TVD version computes accurately the option price and the hedging parameters. For this test example we use MATLAB® 6.1 on a computer with 256 MB RAM and 2.8 GHZ, and found that ETDRK2 is about 60 times faster than the Newton iteration used in Zvan et al. (1998) and runs in 0.5470 seconds with an error of lesser magnitude.

	κ-Schemes			Upwind	WENO5
	$\kappa = 1$	$\kappa = 1$	$\kappa = 1$		
No limiter	0.0303	0.0082	0.0100	0.1247	0.0055
Van Leer Limiter	0.0023	0.0301	0.0257	-	-

Table 1. Infinity norm error for various schemes.

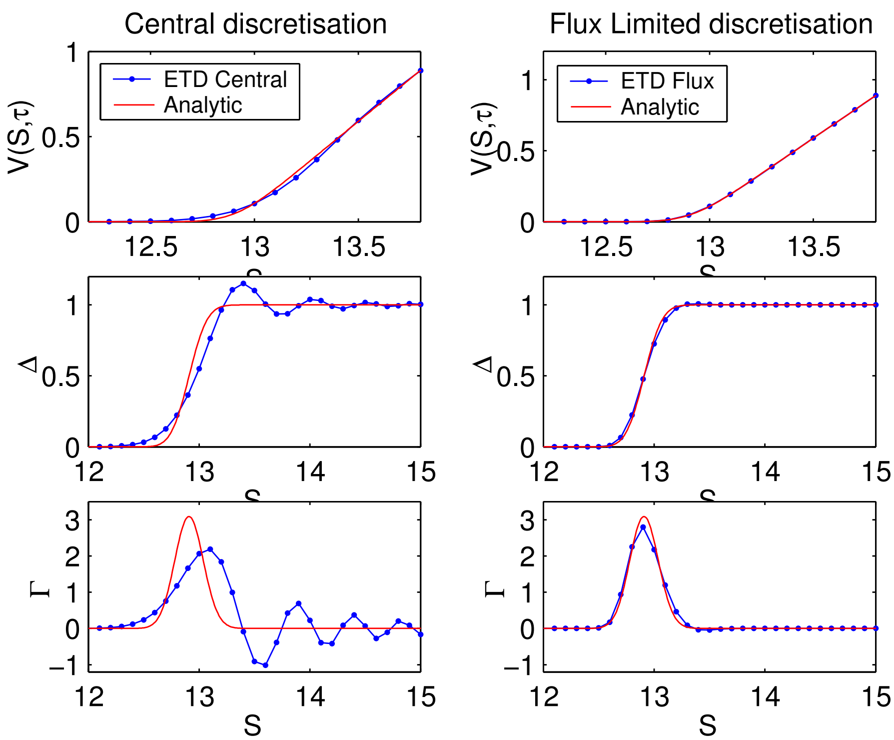

Figure 1. Accuracy of central and flux discretisation with the ETDRK scheme.

For an Asian option, semi-discretisation of (4) gives

$$V(\tau_{k+1}) = e^{\mathrm{B}\Delta T} V(\tau_k) + e^{\mathrm{B}_{\tau_{k+1}}} \int_{\tau_k}^{\tau_{k+1}} e^{-\mathrm{B}\varsigma} \left[b(\varsigma) + N_S(V(\varsigma)) + N_a(V(\varsigma)) \right] d\varsigma, \quad (17)$$

where $N_S(V(\varsigma))$ is the advection in the S spatial direction and $N_a(V(\varsigma))$ is the advection in a. Similar formulas as (15) and (16) can be easily derived for the ETDRK2 version of (17). However when the van Leer flux limiter is used to discretise the convective term, a difficulty is encountered since there exists no boundary condition at $a = 0$. Hugger (2004) proposed to extend the computational domain in the negative a direction and impose the analytic solution for $\sigma = 0$ as boundary. We propose here a better approach which computes a second order upwind approximations for $\partial V / \partial a$ only at $a = 0$. Such a procedure does not result in extra computations due to a grid extension. Note that the way of formulating (17) to separate the linear and non-linear part, has also separated S and a. In fact, the matrix B which consists of diffusion and reaction terms has the same size $m-1$ as for the one dimensional problem (14). In comparison to the method used in Zvan et al. (1998), the discretisation matrix will have rank $(m-1)^2$ and will be truly two dimensional if both S and a are discretised simultaneously with the same number of grid nodes. We solve the two dimensional PDE with the van Leer flux limiter and we use ETDRK2 for the time stepping. The numerical results for $E = 100$ and $r = 0.1$ are given in Table 2 for a wide range of parameters. They clearly show that the new scheme is faster and more accurate than if the Crank-Nicolson scheme was employed with a central spatial discretisation (CNCentral). Also, we observed in Figure 2 that there is no oscillation in the computation of both the delta and gamma when using ETDRK2. This is not the case for CNCentral.

σ	T	ETDRK2	Bounds	ZFV	CNCentral
	0.25	1.8834	1.851	1.793	2.0320
0.1	0.5	3.1136	3.104	3.052	2.9006
	1	5.2990	5.255	5.261	4.8458
	0.25	2.9289	2.932	2.928	2.8547
0.2	0.5	4.5053	4.505	4.511	4.3234
	1	7.0477	7.042	7.042	6.8328
	0.25	5.1564	5.168	5.175	5.0315
0.4	0.5	7.5642	7.572	7.574	7.4194
	1	11.1211	11.121	11.115	10.961
CPU		0.9690			29.346

Table 2. Results for Asian fixed strike call option (The bounds are from the paper by Rogers & Shi (1995) and the prices under the column heading ZFV are from Zvan, Forsyth & Vetzal (1998)).

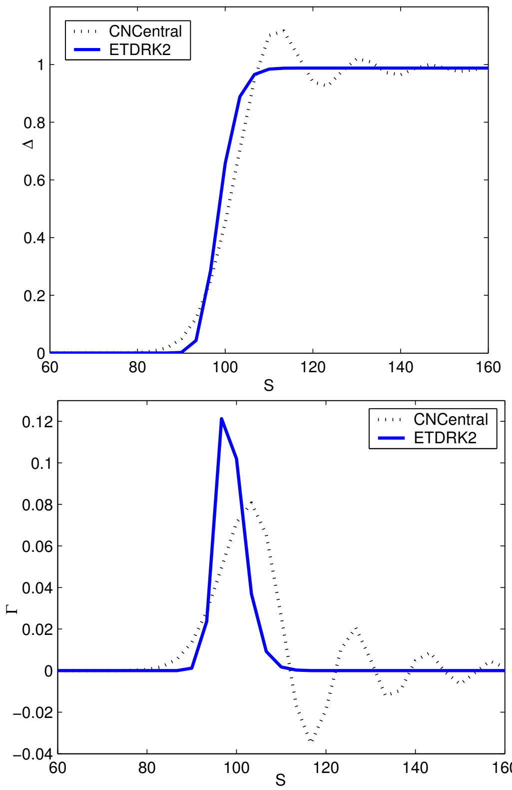

Figure 2. Delta and Gamma for a fixed strike Asian call using Crank-Nicolson and ETDRK2.

4. CONCLUSIONS

The new approach of using ETD with Runge-Kutta time stepping schemes for option pricing proves to be very effective. This is because ETD solves the stiff linear part exactly through the integrating factor $e^{-A\Delta\tau}$ so that explicit Runge-Kutta can be applied for the nonlinear part. This gives an algorithm that is faster than the Newton's non-linear iterative solver since no linear or nonlinear solvers are required at each time step. The framework developed in this paper is robust enough to price with reliable accuracies convectively dominated European and Asian pricing problems.

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