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PROPAGATION OF WAVES IN AN ELASTIC CYLINDER WITH VOIDS

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ABSTRACT

A study of axial waves propagating in an infinitely long, homogeneous and isotropic elastic circular cylinder containing a distribution of small pores (voids) is presented. Assuming the cylinder to be of uniform cross-section and its surface to be traction- free, the frequency equation is obtained and analysed in two limiting cases. It is found that if the cylinder has a small radius and the frequency is low the waves travel more slowly than in the absence of voids. When both the radius and the frequency are large the waves behave like surface waves in a half-space with voids.

Keywords: Elastic waves, voids, cylinder, frequency.

INTRODUCTION

The theory of elastic materials with voids is one of the recent extensions of the classical theory of elasticity. It is concerned with elastic materials consisting of a distribution of small pores (voids) which are assumed to contain nothing of mechanical or energetic significance. The non-linear version of this theory was proposed by Nunziato & Cowin (1979), and the linear version was developed by Cowin &Nunziato (1983).

In the latter version, the void volume is included as an additional kinematic variable, and in the limiting case of the vanishing of this volume, the theory reduces to the classical linear theory of elasticity. Accordingly, a new feature of this theory, over other theories on porous materials, is that it permits a porous body to enlarge or reduce the overall volume the body occupies in the absence of body forces. The theory is intended to find applications in the treatment of the mechanics of granular materials and manufactured porous bodies for which the classical theory proves inadequate.

The object of the present paper is to discuss the propagation of waves in an infinitely long, homogeneous and isotropic elastic circular cylinder with voids, by making use of the field equations obtained by Cowin & Nunziato (1983). We assume that the axis of the cylinder coincides with the z-axis, and that an axial wave is travelling in the z direction with a given frequency. We obtain the frequency equation and show that in the absence of voids it reduces to the classical Pochhammer frequency equation. As limiting cases we consider small frequency waves in a cylinder of small radius, and high frequency waves in a cylinder of large radius. In the first case we obtain the expression for the phase speed and find that because of the influence of the voids wave propagation is slower than its classical counterpart. Thus for a typical (hypothetical) material model (Puri & Cowin,1985) the decrease in speed is found to be 3.76 % . Further we confirm that in the absence of voids the expression for the bar velocity is recovered. In the second case the frequency equation reduces to that of surface waves in a half-space with voids.

PROBLEM FORMULATION

In the context of the theory presented by Cowin & Nunziato (1983), the field equations for a homogeneous and isotropic material, in the absence of body forces, are given in the notation of Cartesian tensors as follows :

$$\mu\nabla^2 u_i + (\lambda+\mu)u_{k,ki} + \beta\phi_{,i} = \rho\frac{\partial^2 u_i}{\partial t^2} \tag{1}$$

$$\alpha\nabla^2\phi - \xi\phi - \omega\frac{\partial\phi}{\partial t} - \beta u_{k,k} = \rho k\frac{\partial^2\phi}{\partial t^2} \tag{2}$$

The relation between the stress tensor τ_{ij} and ϕ is given by

$$\tau_{ij} = \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \beta\delta_{ij}\phi \tag{3}$$

In these equations, u_i is the displacement vector, ϕ is the so-called volume fraction field, λ, μ are the usual Lamé constants, ρ is the mass density, $\alpha, \beta, \xi, \omega$ and k are the new material constants characterizing the presence of voids, δ_{ij} is the Kronecker delta and t is time. In the absence of voids, we have $\phi = 0$; equation (1) then reduces to the Navier equation of classical elasticity.
The classical technique of Helmholtz decomposition of the displacement vector was adopted by Chandrasekharaiah (1987), who studied the propagation of Rayleigh-Lamb waves in an elastic plate with voids. We follow his approach and retain some of his solutions and notation. Hence on setting

$$\mathbf{u} = \nabla p + \nabla\wedge\mathbf{q}, \tag{4}$$

equation (1) reduces to

$$\left(\nabla^2 - \frac{1}{c_L{}^2}\frac{\partial^2}{\partial t^2}\right)p = -\frac{\beta}{\rho c_L{}^2}\phi \tag{5}$$

$$\left(\nabla^2 - \frac{1}{c_S{}^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{q} = 0 \tag{6}$$

Elimination of ϕ from equations (5) and (2) then yields

$$\left[\left\{\nabla^2 - \frac{1}{\alpha^*}\left(1 + \omega^*\frac{\partial}{\partial t} + k^*\frac{\partial^2}{\partial t^2}\right)\right\}\left\{\nabla^2 - \frac{1}{c_L{}^2}\frac{\partial^2}{\partial t^2}\right\} + \beta^*\nabla^2\right]p = 0 \tag{7}$$

where

$$c_L{}^2 = \frac{\lambda + 2\mu}{\rho},\ c_S{}^2 = \frac{\mu}{\rho},\ \alpha^* = \frac{\alpha}{\xi},\ \omega^* = \frac{\omega}{\xi},\ k^* = \frac{\rho k}{\xi},\ \beta^* = \frac{\beta^2}{\rho\alpha c_L{}^2} \tag{8}$$

Equations (5)-(7) now serve as field equations and are to hold in the region $-\infty < z < \infty$ and $0 \le r < a$, where a is the radius of the cylinder. In addition there has to be satisfied on the boundary, the condition that there should be no traction. For the problem considered here this amounts to (Cowin and Nunziato (1983)) :

$$\tau_{ij}\, n_j = 0,\ \phi_{,j}\, n_j = 0 \tag{9}$$

where n_j is the unit outward normal to the boundary.

SOLUTION OF THE FIELD EQUATIONS

We shall work in cylindrical polar co-ordinates (r, θ, z) and since we shall consider only axially-symmetric displacements, the field variables will be independent of θ. Hence $\mathbf{u} = [u(r,z,t), 0, w(r,z,t)]$ while for q it is only the θ component which is relevant, and we denote it by q. Further we presume that a wave-like disturbance has been established outside our region of interest (i.e. in $z \to -\infty$). Accordingly we shall seek steady-state solutions to equations (5)-(7) representing a right-moving wave in the form

$$(p, q) = (P, Q)\exp[i(\kappa z - \varpi t)] \tag{10}$$

where P and Q are functions of r only, and where κ and ϖ are, respectively, the wave number and frequency of the wave. Substitution of p from (10) into (7) yields the following fourth-order differential equation

$$r^3 \frac{d^4 P}{dr^4} + 2r^2 \frac{d^3 P}{dr^3} + \left[Dr^3 - r\right]\frac{d^2 P}{dr^2} + \left[Dr^2 + 1\right]\frac{dP}{dr} + Fr^3 P = 0 \quad (11)$$

where

$$D = \left(\beta^* - \frac{1}{\alpha^*}\right) + \varpi^2\left(\frac{k^*}{\alpha^*} + \frac{1}{{c_L}^2}\right) + \frac{i\omega^* \varpi}{\alpha^*} - 2\kappa^2$$

$$F = -\kappa^2 (D - \kappa^2) + \frac{\varpi^2}{\alpha^* {c_L}^2}(k^* \varpi^2 + i\omega^* \varpi - 1)$$

Equation (11) has general solution

$$P(r) = A_1 J_0(m_1 r) + A_2 J_0(m_2 r) + A_3 Y_0(m_1 r) + A_4 Y_0(m_2 r)$$

where $A_1, \cdots, A_4$ are arbitrary constants, J_0 and Y_0 are zero-order Bessel functions of the first and second kind respectively, and m_1, m_2 are the (complex) roots of the bi-quadratic equation

$$m^4 - Dm^2 + F = 0$$

The substitution for q from (10) into (6) gives

$$q = [A_5 J_1(m_3 r) + A_6 Y_1(m_3 r)]\exp[i(\kappa z - \varpi t)]$$

where

$${m_3}^2 = \varpi^2 / {c_S}^2 - \kappa^2$$

and A_5, A_6 are arbitrary constants. To maintain boundedness of the solutions for $r = 0$, we must have $A_3 = A_4 = A_6 = 0$, so that after relabelling the constants we obtain

$$P(r) = AJ_0(m_1 r) + BJ_0(m_2 r)$$

$$q = CJ_1(m_3 r)\exp[i(\kappa z - \varpi t)]$$

It follows from equation (4) that the displacement components are

$$u = [-Am_1 J_1(m_1 r) - Bm_2 J_1(m_2 r) - i\kappa CJ_1(m_3 r)]\exp[i(\kappa z - \varpi t)] \quad (12)$$

$$w = [i\kappa AJ_0(m_1 r) + i\kappa BJ_0(m_2 r) + m_3 CJ_0(m_3 r)]\exp[i(\kappa z - \varpi t)] \quad (13)$$

while from equation (5) we have the volume- fraction field given by

$$\phi = \frac{\rho c_L{}^2}{\beta}[n_1 AJ_0(m_1 r) + n_2 BJ_0(m_2 r)]\exp[i(\kappa z - \varpi t)] \quad (14)$$

where $n_{1,2} = m_{1,2}{}^2 + \kappa^2 - \varpi^2 / c_L{}^2$

If the surface of the cylinder is to be stress-free the boundary conditions (9) reduce to

$$(\lambda + \mu)\frac{\partial u}{\partial r} + \lambda\left(\frac{u}{r} + \frac{\partial w}{\partial z}\right) + \beta\phi = 0; \quad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = 0; \quad \frac{\partial \phi}{\partial r} = 0 \text{ at } \quad r = a$$

With the aid of equations (12)-(14) these conditions yield the following homogeneous system of equations :

$$\begin{bmatrix} (\kappa^2 - m_3{}^2)aJ_{01} + 2m_1 J_{11} & (\kappa^2 - m_3{}^2)aJ_{02} + 2m_2 J_{12} & 2i\kappa(J_{13} - am_3 J_{03}) \\ 2i\kappa m_1 J_{11} & 2i\kappa m_2 J_{12} & (m_3{}^2 - \kappa^2)J_{13} \\ m_1 n_1 J_{11} & m_2 n_2 J_{12} & 0 \end{bmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0$$

where $J_{kl} = J_k(m_l a)$

THE FREQUENCY EQUATION

The vanishing of the determinant of the above matrix yields the frequency equation. Before expanding the determinant, however, we make the following substitutions :

$$\Gamma = \kappa a, \quad M_i = m_i a, \quad \Omega = \varpi a / c_s, \quad V = \Omega c_s / \Gamma, \; {R_{1,2}}^2 = 1 + n_{1,2} {c_L}^2 / \varpi^2 \qquad (15)$$

so that

$$ {M_{1,2}}^2 = \Gamma^2 \left(\frac{V^2 {R_{1,2}}^2}{{c_L}^2} - 1 \right), \quad {M_3}^2 = \Gamma^2 \left(\frac{V^2}{{c_S}^2} - 1 \right) \qquad (16)$$

We note that only the real part of the complex phase-speed V is physically relevant. Further, we find from the bi-quadratic equation above that $R_{1,2}$ satisfy the equation

$$\Omega^* R^4 + (1 - N^* - \Omega^*) R^2 - 1 = 0 \qquad (17)$$

where

$$N^* = \frac{\alpha^* \beta^* a^2}{a^2 - i\omega^* \Omega c_s a - k^* \Omega^2 {c_s}^2}, \quad \Omega^* = \frac{\alpha^* \Omega^2 {c_s}^2}{{c_L}^2 [a^2 - i\omega^* \Omega c_s a - k^* \Omega^2 {c_s}^2]}$$

Furthermore, in the absence of voids, we may take $R_1 = 1, \; R_2 = 0$ With the aid of the above substitutions we obtain the required frequency equation as

$$(2\Gamma^2 - \Omega^2)^2 [M_2 ({R_2}^2 - 1) J_{01} J_{12} J_{13} - M_1 ({R_1}^2 - 1) J_{02} J_{11} J_{13}]$$

$$= 2 M_1 M_2 J_{11} J_{12} ({R_1}^2 - {R_2}^2) [2\Gamma^2 M_3 J_{03} - \Omega^2 J_{13}] \qquad (18)$$

Clearly the waves are dispersive. Further, in the absence of voids this equation reduces to

$$(\Gamma^2 - {M_3}^2) J_{01} J_{13} + 4\Gamma^2 M_1 M_3 J_{11} J_{03} - 2 M_1 (\Gamma^2 + {M_3}^2) J_{11} J_{13} = 0$$

which is the classical Pochhammer equation (see Achenbach (1984) eq (6.131), where his k, p, q are our Γ, M_1, M_3 respectively.)

Explicit analytic solutions of equation (18) are not possible although numerical solutions should not be difficult to obtain if the values of the material constants are known. We therefore analyse it in two limiting cases.

First, when the radius of the cylinder is very small (so that $|\Gamma| << 1, \Omega \to 0$) we have on using the asymptotic expansions of Bessel functions for small arguments, a first approximation of equation (18) given by

$$(2\Gamma^2 - \Omega^2)^2[M_2^{\ 2}(R_2^{\ 2} - 1) - M_1^{\ 2}(R_1^{\ 2} - 1)] = M_1^{\ 2}M_2^{\ 2}(R_1^{\ 2} - R_2^{\ 2})[4\Gamma^2 - \Omega^2] \quad (19)$$

Substituting for W and $M_{1,2}$ from equations (15) and (16) with

$$R_1^{\ 2} = (1 - N)^{-1}, \ \ R_2 \to \infty \ ; \ N = \alpha^*\beta^* = \beta^2 / \xi(\lambda + 2\mu)$$

the phase-speed V is given by

$$V = \frac{c_S[4c_S^{\ 2} - 3(1 - N)c_L^{\ 2}]^{1/2}}{[c_S^{\ 2} - (1 - N)c_L^{\ 2}]^{1/2}} \quad (20)$$

In the absence of voids $(N = 0)$, equation (20) reduces to

$$V = \frac{c_S[3c_L^{\ 2} - 4c_S^{\ 2}]^{1/2}}{[c_L^{\ 2} - c_S^{\ 2}]^{1/2}} = \sqrt{\frac{E}{\rho}} = V_{bar}$$

where E is Young’s modulus and V_{bar} is the classical bar velocity. The constant N is known to satisfy the inequality (Puri & Cowin,1985)

$$0 \le N < 1$$

so that we easily deduce that waves of small frequency propagating in a small-radius cylinder with voids are slower than in one without voids. To determine the reduction in speed we use a hypothetical material proposed by Puri & Cowin (1985) for which the relevant parameters are

$$c_L = 3873 \ m/s, \ \ c_S = 1937 \ m/s, \ \ N = 0.2778$$

These give the phase-speeds $V_{voids} = 3044\ m/s,\ V_{bar} = 3163\ m/s$ *, i.e., a 3.76 % reduction in speed of the waves in the presence of voids.*

Secondly, when the cylinder is of very large radius and the frequency is high, equation (18) reduces, on using the Bessel asymptotic expansions for large arguments, to

$$(2\Gamma^2 - \Omega^2)^2[M_2(R_2^{\ 2} - 1) - M_1(R_1^{\ 2} - 1)] = 4M_1M_2M_3(R_1^{\ 2} - R_2^{\ 2})\Gamma^2 \qquad (21)$$

It is easily shown that equation (21) is the frequency equation for surface waves in a half-space with voids, the analysis of which is given in Chandrasekharaiah (1987).

Furthermore, in the absence of voids the equation yields

$$(2\Gamma^2 - \Omega^2)^2 + 4M_2M_3\Gamma^2 = 0$$

which is precisely the classical Rayleigh wave equation.

CONCLUSION

In this article we have shown that propagation of axial waves in an infinitely long elastic circular cylinder is affected by the presence of voids. Indeed, there is a reduction in the phase-speed of low-frequency waves. To quantify this decrease, use has been made of a hypothetical material and it is found that there is a 3.76 % drop in speed. This is in marked contrast with the case of the thin elastic plate with voids, (Chandrasekharaiah,1987), where the corresponding reduction for the same material is 6.64 %.

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