A Mathematical Model for Stock Price Forecasting

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Abstract

Many mathematical models of stochastic dynamical systems were based on the assumption that the drift and volatility coefficients were linear function of the solution. In this work, we arrive at the drift and the volatility by observing the dynamics of change in the selected stocks in a sufficiently small interval Δt*. We assumed that only one change occurs within* $\Delta t = t_{i+1} - t_i$*. During this time, a stock may gain one unit [+1], remain stable [0], or loss one unit [-1]. The likelihood of each change occurring were noted and the expectation (the drift) and the covariance (the volatility) of the change were computed leading to the formulation of the system of linear stochastic differential equations. To fit data to the model, changes in the prices of the stocks were studied for an average of 30 times. A simple checklist was used to determine the likelihood of each event of a loss, again or stable occurring. The drift and the volatility coefficients for the SDE were determined and the multi-dimensional Euler-Maruyama scheme for system of stochastic differential equations was used to simulated prices of the stocks for* $1 < t < 30$*. The simulated prices was compared to the observed price and we observed that the simulated prices is sufficiently close to the observed price and there for suitable for forecasting the price of the stocks for a short time interval.*

Key words: stock price forecasting, system of linear stochastic differential equations, model, volatility coefficients, simulate, forecasting

Introduction

Investment management was considered over the years as an act rather than a science. However, advancement in technology leading to globalization and the consequent expansion in the universe of investable assets have made it necessary to explore mathematically based approach to forecasting the price of assets rather than mere speculation. Furthermore, the large size of the markets today makes it imperative to adopt safe and repeatable methodologies in arriving at the choice of which asset to invest in or not. This justifies the need for a more scientific approach to investment management than the mere traditional arts based on speculation. The science of designing (or engineering) of contracts and portfolios of contracts that result in predetermined cash flows contingent to different events is referred to as financial engineering [3]. In other words, financial engineering employs scientific techniques using mathematical tools to give quantitative management of risk and uncertainty.

The application of financial engineering principles in investment management has changed the landscape of financial modeling so that computationally intensive methods such as Monte Carlo simulations and simulation of sample paths of stochastic differential equations are now widely used to represent variation in stock prices. Powerful specialized mathematical languages and vast statistical software libraries have also been developed over the years. The ability to program sequences of statistical operations within a single programming language has been a big step forward [3].

The focus of this work is to formulate the stochastic model for the dynamics of change in the prices some selected stocks and to use same model to simulate prices of the selected stocks.

Review

The binomial Asset Pricing model looked at the price of assets as going up or down by a factor u or d respectively. This model however does not take cognizance of the possibility of the asset price remaining stable within the time interval considered [4]. Edward Allen [1] proposed a method of formulating stochastic models based on observation of the relevant parameters rather than the assumption that the drift and the volatility coefficient are linear functions of the solutions. The model formulated here an extension of the model proposed by Allen in [1]

The mathematical development of present-day economic and finance theory began in Lausanne, Switzerland at the end of the nineteenth century, with the development of the mathematical equilibrium theory by Leon Walras and Wilfredo Pareto [9]. At the beginning of the twentieth century, Louis Bachelier in Paris and Filip Lundberg in Uppsala (Sweden) developed sophisticated mathematical tools to describe uncertain price and risk processes. Since then, observations of prices of stocks, has been modeled as a stochastic process. A stochastic process with state space S is a collection of random variables $\{X_t, \ t \in T\}$ defined on the same probability space (Ω, f, P) evolving over time. The set T is called its parameter set. If $T = \{0, 1, 2, \ldots\}$, the process is said to be of a discrete parameter set. If T is not countable, the process is said to have a continuous parameter set, for example, $T = R+ = [0, 1)$ and $T = [a, b] \in R$. The index t represents time, and X_t as the "state" or the "position" of the process at time t. The state space is R in most usual examples, and then the process is said real-valued.

Variation in stock prices and other phenomenon involving uncertainties when modeled as a stochastic process, leads to

Stochastic Differential Equations (SDE). For example, in finance and insurance, the concept of cooperate defaults, operational failures, insured accidents, uncertainties in foreign exchange and prices of stocks can be naturally modeled by Stochastic Differential equations (SDEs) [7] Stochastic differential equations with jumps was used for modeling credit events like defaults and credit rating changes Jarion [6] acknowledge that stochastic differential equations could be applied to modeling of short time rate typically set by the central banks. Models for the dynamics of financial quantities specified by SDEs have become increasing popular in the recent past. Models of this king can be found [8], [4], [11], [12] to mention a few.

Stochastic Differential Equations

Differential equations are used generally to describe the evolution of a system over time. Stochastic differential equation arises when a differential equation is subject to some random perturbation called the White noise. If x(t) is a differential equation defined for t $\geq$ 0, u(x,t) is a function of x and t and the following relation is satisfied for all t, 0$\leq$ t $\leq$T

$$\frac{dx(t)}{dt} = x'(t) = u(x(t),t), \quad x(0) = x_0 \qquad (1)$$

$x(t)$ is said to be the solution of the ordinary differential equation (1) with the initial condition $x(0) = x_0$. Equation (1) can be written as $dx(t) = u(x(t),t)\,dt$ and by the continuity of $x'(t)$, we can write:

$$x(t) = x(0) + \int_0^t U(x(s),s)ds \qquad (2)$$

The differential equation (1) above becomes a stochastic differential equation if we subject it to a random perturbation by a White noise which is considered to be derivative of a Brownian motion i.e. $\xi(t) = \frac{dB(t)}{dt} = B'(t)$. It should be noted that the White noise does not exist as usual function of t since a Brownian motion is now where differentiable. If the intensity of the noise is denoted by $\sigma(t)$ at a point x at time t, then we can write

$$\int_0^T \sigma(t)(x(t),t)\xi(t) = \int_0^T \sigma(t)(x(t),t)dB'(t)d(t)$$

$$= \int_0^T \sigma(t)(x(t),t)dB(t) \qquad (3)$$

where the integral in (3) above is an Ito integral. Therefore, we can say that stochastic differential equation arises when the coefficients of ordinary differential equation are perturbed by white Noise.

Itó Integral

The stochastic calculus of Itô originated with the investigation of conditions under which the drift and the diffusion coefficient of the diffusion of Markov process could be used to characterize this process [12]. Following the Einstein's explanation of Brownian motion in the first decade of the 19th century, there were rigorous efforts by researchers to formulate the dynamics of the motion in terms of stochastic differential equation.

The resulting equation was written in the form

$$dX_t = a(t, X_t) + b(t, X_t)dW_t \qquad (4)$$

This symbolic differential form can be written in integral form as

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_t \quad (5)$$

Since the Brownian motion B_t is a derivative of the Wiener process W_t, the second integral in equation (5) above cannot be interpreted as the Riemann or Lebesgue integral because Brownian motion is nowhere differentiable. Furthermore, because the continuous sample path of Brownian motion is not of a bounded variation in any interval, the integral cannot also be interpreted as Riemann-Stieljes integral [11]. Therefore, the integral of the form $\int_a^b X(t)dB(t)$ where $X(t)$ is adapted to the filtration $\mathcal{F}_t = \sigma\{B(s); s \leq t\}$ is called a stochastic or Itó integral. This type of integral arises naturally as a solution of stochastic differential equations or martingale.

3.1 Approach to the Model Formulation

In order to develop a mathematical model that captures the dynamics of changes in the prices of stocks, an approach that is analogous to the method used over the years to formulate models resulting in deterministic ordinary differential equation is used. This involves studying the dynamics of a system of interest for a short time interval Δt after which the information obtained from the short time study of the dynamics of the system were used to formulate the mathematical model of the system. A stochastic system studied for a discrete time interval, results to a discrete stochastic model and for a sufficiently small Δt, the discrete time stochastic model leads to stochastic differential equation model as $\Delta t \to 0$, Allen E.J (2003), This modeling approach has been used widely for formulation stochastic models of some important dynamical systems such as Emmert et.al (2006), Allen E.J (2003). As $\Delta t \to 0$, a continuous time model is obtained from the discrete model which according to Kurtz (1971), has similarities with continuous time Makov chain models. This approach is different from the commonly used approach that is based on the hypothesis that the drift and diffusion coefficient are linear function of the solution.

3.2 Mathematical Model of Dynamics of Change in Three Stock Prices

Consider three stocks $S_1, S_2 and\ S_3$ subjected to random influence by the market forces. We assume that in a small interval of time Δt, a stock price may change by losing one unit (-1) or remain stable (0) or gain one unit (+1) which we represent by $[-1 \quad 0 \quad 1]$. The diagram below illustrates the schematic diagram for the model:

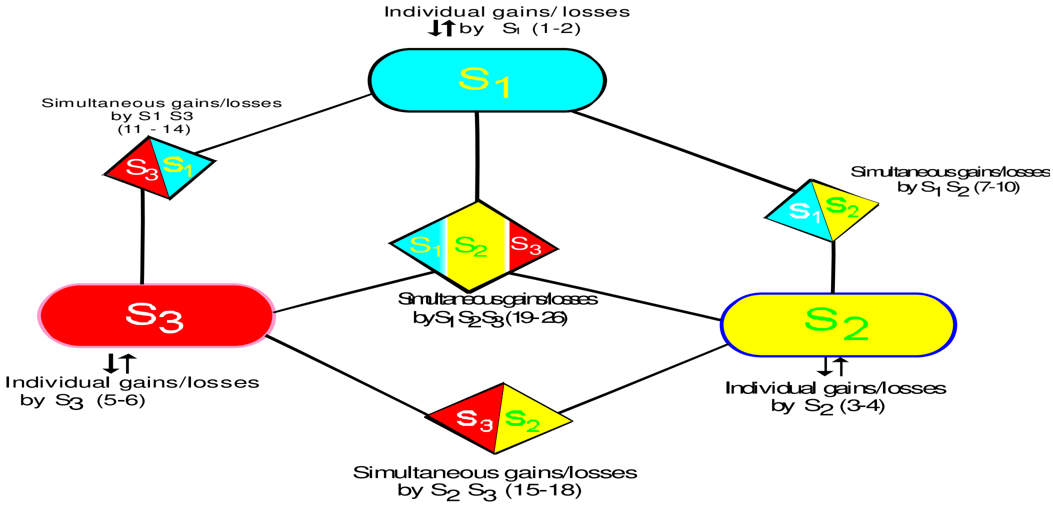

Figure 4.1: *Schematic Diagram for the Stochastic Model for Change in Stock Price*

There are $3^3 = 27$ possibilities by which the stocks $S_1, S_2 and\ S_3$ may vary in the small time interval Δt. These possibilities are indicated in table 3.2 below.

Table 4.1: ***Possible outcome of change in 3 stock price and the probabilities***

S/N	Change in Stock price $\Delta S = [\Delta S_1, \Delta S_2, \Delta S_3]^T$	Probabilities
1	[-1 0 0]	$p_1 = d_1 S_1$
2	[1 0 0]	$p_2 = b_1 S_1$
3	[0 1 0]	$p_3 = b_2 S_2$
4	[0 -1 0]	$p_4 = d_2 S_2$
5	[0 0 -1]	$p_5 = d_3 S_3$
6	[0 0 1]	$p_6 = b_3 S_3$
7	[-1 1 0]	$p_7 = \alpha_{12} S_1 S_2$
8	[-1 -1 o]	$p_8 = \alpha_{11} S_1 S_2$
9	[1 1 0]	$p_9 = \alpha_{22} S_1 S_2$
10	[1 -1 0]	$p_{10} = \alpha_{21} S_1 S_2$
11	[-1 0 1]	$p_{11} = \beta_{12} S_1 S_3$
12	[-1 0 -1]	$p_{12} = \beta_{11} S_1 S_3$
13	[1 0 1]	$p_{13} = \beta_{22} S_1 S_3$
14	[1 0 -1]	$p_{14} = \beta_{21} S_1 S_3$

15	[0 1 1]	$p_{15} = \gamma_{22}S_2S_3$
16	[0 -1 -1]	$p_{16} = \gamma_{11}S_2S_3$
17	[0 -1 1]	$p_{17} = \gamma_{12}S_2S_3$
18	[0 1 -1]	$p_{18} = \gamma_{21}S_2S_3$
19	[-1 1 1]	$p_{19} = \eta_{122}S_1S_2S_3$
20	[-1 1 -1]	$p_{20} = \eta_{121}S_1S_2S_3$
21	[-1 -1 -1]	$p_{21} = \eta_{111}S_1S_2S_3$
22	[-1 -1 1]	$p_{22} = \eta_{112}S_1S_2S_3$
23	[1 -1 -1]	$p_{23} = \eta_{211}S_1S_2S_3$
24	[1 1 -1]	$p_{24} = \eta_{221}S_1S_2S_3$
25	[1 -1 1]	$p_{25} = \eta_{212}S_1S_2S_3$
26	[1 1 1]	$p_{26} = \eta_{222}S_1S_2S_3$
27	[0 0 0]	$p_{27} = 1 - \sum_{j=1}^{26} p_i$

Here, ΔS represent change in stock price. For example, $\Delta S = [1\ 0\ 0]$ represent a gain of 1 unit in stock S_1 while stock S_2 and S_3 remain stable; $\Delta S = [1 - 1\ 1]$ represent a simultaneous gain of 1 unit by stock $S_1$and $S_3$and a loss of 1 unit stock S_2. It is assumed that the change in the stock price is proportional to the price of the stock. For simultaneous gains/losses, we assume that the probability of the change is proportional to the product of the stock prices. This is reasonable as, supposing that the one of the stock price is zero then, the probability of a simultaneous gain is zero. It is also assumed that Δt is sufficiently small so that p_{27} which is the probability that there is no change in the three stocks prices within the time interval Δt is positive.

The parameters $b_i, d_i, i = 1,..,3$, defines the rate at which stocks experience individual gains or losses respectively. The parameter $\alpha_{j,k}, \beta_{j,k}, \gamma_{j,k}\ \eta_{j,k,l} for\ j,k,l = 1,2$ defines the rate at which stocks experience simultaneous gains and/or losses with each parameter depending on t. For example, $b_iS_i\Delta t$ is the probability that stock i gains one unit in the time interval Δt.

The change involving $S_1 and\ S_2$ simultaneously is denoted by $\alpha_{j,k}$. Also, $\beta_{j,k}, \gamma_{j,k}$denotes the changes involving $S_1 and\ S_3$ and $S_2 and\ S_3$ simultaneously. For example, $\alpha_{1,2}$ represent the change involving the two stocks S_1S_2 in which S_1 gains while S_2 losses. Also $\beta_{2,1}$ represent the change involving the two stocks S_1S_3

in which S_1 losses while S_3 gains. Finally, $\eta_{j,k,l}$ denote the change in the three stocks $S_1, S_2 and\ S_3$ simultaneously. In all cases, the subscript 1 or 2 represent loss of one unit or gain of one unit respectively. It should be noted that $\sum_{i=1}^{27} p_i = 1$.

Using the above representations for p_i and ΔS_i the expectation vector is derived as follows:

$$E(\Delta S) = \sum_{i=1}^{27} p_i \Delta S_i = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \qquad (6)$$

$$f_1 = (d_1 + b_1)S_1 + (\alpha_{12} + \alpha_{21} + \alpha_{22} + \alpha_{11})S_1S_2 + (\beta_{12} + \beta_{21} + \beta_{22} + \beta_{11})S_1S_3 + (\eta_{122} + \eta_{121} + \eta_{111} + \eta_{112} + \eta_{211} + \eta_{221} + \eta_{212} + \eta_{222})S_1S_2S_3 \qquad (7)$$

f_1 Represent the totality of the likelihood of change involving stock S_1.

$$f_2 = (d_2 + b_2)S_2 + (\alpha_{12} + \alpha_{21} + \alpha_{22} + \alpha_{11})S_1S_2 + (\gamma_{22} + \gamma_{21} + \gamma_{22} + \gamma_{11})S_2S_3 + (\eta_{122} + \eta_{121} + \eta_{111} + \eta_{112} + \eta_{211} + \eta_{221} + \eta_{212} + \eta_{222})S_1S_2S_3 \qquad (8)$$

.

f_2 Represent the totality of the likelihood of change involving stock S_2

$$f_3 = (d_3 + b_3)S_3 + (\beta_{22} + \beta_{11} + \beta_{22} + \beta_{11})S_1S_3 + (\gamma_{22} + \gamma_{11} + \gamma_{22} + \gamma_{21})S_2S_3 + (\eta_{122} + \eta_{121} + \eta_{111} + \eta_{112} + \eta_{211} + \eta_{221} + \eta_{212} + \eta_{222})S_1S_2S_3 \qquad (9)$$

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f_3 Represent the totality of the likelihood of change involving stock S_3.

Putting

$$dS_1 = (d_1 + b_1)S_1 \qquad (10)$$

$$dS_2 = (d_2 + b_2)S_2, \qquad (11)$$

$$dS_3 = (d_3 + b_3)S_3, \qquad (12)$$

$$dS_1S_2 = (\alpha_{12} + \alpha_{21} + \alpha_{22} + \alpha_{11})S_1S_2, \qquad (13)$$

$$dS_1S_3 = (\beta_{12} + \beta_{21} + \beta_{22} + \beta_{11})S_1S_3, \qquad (3.1.9)$$

$$dS_2S_3 = (\gamma_{12} + \gamma_{21} + \gamma_{22} + \gamma_{11})S_2S_3. \qquad (14)$$

$$dS_1S_2S_3 = (\eta_{122} + \eta_{121} + \eta_{111} + \eta_{112} + \eta_{211} + \eta_{221} + \eta_{212} + \eta_{222})S_1S_2S_3 \qquad (15)$$

Then the covariance matrix is derived as follows:

$$E(\Delta S(\Delta S))^T = \begin{bmatrix} dS_1 & dS_1S_2 & dS_1S_3 \\ dS_2S_1 & dS_2 & dS_2S_3 \\ dS_3S_1 & dS_3S_2 & dS_3 \end{bmatrix} \qquad (16)$$

The covariance matrix represents the volatility coefficient of the SDE. Clearly, the covariance matrix is a positive definite symmetric and hence has a positive definite square root. We then have the SDE

$$dS(t) = \mu(t,S\ ,S_2,S_3)dt + B(t,S\ ,S_2,S_3)dW(t) \qquad (17)$$

where

$$\mu(t,S\ ,S_2,S_3) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \qquad (18)$$

and

$$B(t,S_1,S_2,S_3) = \begin{bmatrix} dS_1 & dS_1S_2 & dS_1S_3 \\ dS_2S_1 & dS_2 & dS_2S_3 \\ dS_3S_1 & dS_3S_2 & dS_3 \end{bmatrix}^{1/2} \qquad (19)$$

Methodology

Prices of stocks are published daily. In other to characterize the drift and volatility which will form the coefficients of the stochastic differential equations, the daily stock prices of three selected stocks were observed for thirty days. The table below showed the daily prices of the selected stocks for the chosen days .

Table 1: *Daily prices of three selected stocks for 30 days*

Day	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
S1	2.50	2.50	2.51	2.50	2.53	2.55	2.55	2.55	2.56	2.57	2.58	2.57	2.58	2.58	2.57
S2	3.00	3.00	3.10	3.00	3.10	3.00	3.15	3.20	3.20	3.20	3.20	3.20	3.30	3.30	3.20
S3	3.70	3.70	3.73	3.73	3.75	3.80	3.80	3.80	3.80	3.70	3.70	3.80	3.90	3.90	3.90

Day	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
S1	2.58	2.57	2.58	2.58	2.59	2.59	2.58	2.59	2.60	2.60	2.60	2.65	2.65	2.67	2.70
S2	3.30	3.30	3.20	3.20	3.30	3.35	3.35	3.35	3.40	3.40	3.41	3.40	3.43	3.50	3.50
S3	3.90	3.80	3.90	4.00	4.10	4.10	4.10	4.10	4.20	4.10	4.20	4.30	4.30	4.30	4.32

From table 1 above, a checklist of instances of loss, gain or unchanged or stable were noted and marked out as shown below in table 3.3 below: The tables shows only the first ten (10) rows of the thirty rows checklist. A stock price could loss (L), gain (G) or remain unchanged or stable (U) in day considered. This was cross checked for each stock and recorded as shown in the table .

Table 2: *A checklist of occurrences of gain, stable or loss in the prices of the first 10 days of the month*

Day	S1			S2			S3			Summary
	L	N	G	L	N	G	L	N	G	
1										
2		✓			✓			✓		0 0 0
3			✓			✓			✓	1 1 1
4	✓			✓				✓		-1-1 0
5			✓			✓			✓	1 1 1
6			✓	✓					✓	1 0 1
7		✓				✓		✓		0 1 0
8						✓		✓		0 1 0
9			✓		✓			✓		1 0 0
10			✓		✓			✓		1 0 0

The summary column captures how the prices of each stock vary in relation to the previous day. For example, (000) in the summary column indicates that the price of stock1, stock2 and stock 3 did not change; (-110) shows that stock1 lost, stock2 gained while stock3 remained stable and so on. From table 3.3 above the probability of each occurrence was determined by

.

$$p_i = \frac{number\ of\ occurence}{total\ umberoftime\ \ the\ events\ could\ occur(\ i.e\ 27)}$$

For example, the event that the change in the stock prices is (010) which occurred 4 times is $\frac{4}{27} = 0.148$. The probabilities for the other events were determined and tabulated as shown in table 3.4 below

Table 3 *Table showing the pattern of change in price and their probabilities*

S/N	Change in Stock price $\Delta S = [\Delta S_1, \Delta S_2, \Delta S_3]^T$	Number of Occurrence	Probabilities
1	[-1 0 0]	1	0.0370
2	[1 0 0]	4	0.1481
3	[0 1 0]	3	0.1111
4	[0 -1 0]	0	0.0000
5	[0 0 -1]	1	0.0370
6	[0 0 1]	3	0.1111
7	[-1 1 0]	0	0.0000
8	[-1 -1 o]	2	0.0470
9	[1 1 0]	3	0.1111
10	[1 -1 0]	0	0.0000
11	[-1 0 1]	0	0.0000

12	[-1 0 -1]	2	0.0740
13	[1 0 1]	1	0.0370
14	[1 0 -1]	1	0.0370
15	[0 1 1]	0	0.0000
16	[0 -1 -1]	0	0.0000
17	[0 -1 1]	0	0.0000
18	[0 1 -1]	0	0.0000
19	[-1 1 1]	0	0.0000
20	[-1 1 -1]	0	0.0000
21	[-1 -1 -1]	0	0.0000
22	[-1 -1 1]	0	0.0000
23	[1 -1 -1]	0	0.0000
24	[1 1 -1]	0	0.0000
25	[1 -1 1]	2	0.0740
26	[1 1 1]	5	0.1852
27	[0 0 0]	1	0.0370

From table 3.4 above relevant variables were computed as follows:

$$f_1 = (d_1 + b_1)S_1 + (\alpha_{12} + \alpha_{21} + \alpha_{22} + \alpha_{11})S_1S_2 + (\beta_{12} + \beta_{21} + \beta_{22} + \beta_{11})S_1S_3 + (\eta_{122} + \eta_{121} + \eta_{111} + \eta_{112} + \eta_{211} + \eta_{221} + \eta_{212} + \eta_{222})S_1S_2S_3$$

= 0.0370 + 0.1481 + 0.0470 + 0.1111 + 0.0740 + 0.0370 + 0.0370 + 0.0740 + 0.1852 = 0.7874

$$f_2 = (d_2 + b_2)S_2 + (\alpha_{12} + \alpha_{21} + \alpha_{22} + \alpha_{11})S_1S_2 + (\gamma_{22} + \gamma_{21} + \gamma_{22} + \gamma_{11})S_2S_3 + (\eta_{122} + \eta_{121} + \eta_{111} + \eta_{112} + \eta_{211} + \eta_{221} + \eta_{212} + \eta_{222})S_1S_2S_3$$

= 0.1111+0.0000+.0470+0.1111+0.0740+0.0370+0.0370+0.0740+0.1852 = 0.5653

Similarly,

$$f_3 = (d_3 + b_3)S_3 + (\beta_{22} + \beta_{11} + \beta_{22} + \beta_{11})S_1S_3 + (\gamma_{22} + \gamma_{11} + \gamma_{22} + \gamma_{21})S_2S_3 + (\eta_{122} + \eta_{121} + \eta_{111} + \eta_{112} + \eta_{211} + \eta_{221} + \eta_{212} + \eta_{222})S_1S_2S_3$$

= 0.5923.

Hence, the expectation vector

$$E(\Delta S) = \sum_{i=1}^{27} p_i \Delta S_i = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{pmatrix} 0.7874 \\ 0.5653 \\ 0.5923 \end{pmatrix}$$

The likelihood of the change in the price occurring with stock S1 only was

$dS_1 = (d_1 + b_1)S_1 = 0.1851$

The likelihood of the change occurring in the price of stock S2 only was

$dS_2 = (d_2 + b_2)S_2 = 0.1110$;

Similarly, $dS_3 = (d_3 + b_3)S_3$ = 0. 1481

The likelihood of change in the price of stock$S_1$and S_2 only is

$$dS_1S_2 = (\alpha_{12} + \alpha_{21} + \alpha_{22} + \alpha_{11})S_1S_2 = 0.1581$$

The likelihood of change in the price of stock$S_1$and S_3 is

$$dS_1S_3 = (\beta_{12} + \beta_{21} + \beta_{22} + \beta_{11})S_1S_3 = 0.1480$$

The likelihood of change in the price of stockS_2 and S_3 is

$$dS_2S_3 = (\gamma_{12} + \gamma_{21} + \gamma_{22} + \gamma_{11})S_2S_3 = 0.0000$$

The likelihood of change in the price occurring in the three stocks concurrently i.e. in S1, S_2

and S_3 is:

$$dS_1S_2S_3 = (\eta_{122} + \eta_{121} + \eta_{111} + \eta_{112} + \eta_{211} + \eta_{221} + \eta_{212} + \eta_{222})S_1S_2S_3 = 0.2962$$

Consequently, the covariance matrix is

$$E(\Delta S(\Delta S))^T = \begin{bmatrix} dS_1 & dS_1S_2 & dS_1S_3 \\ dS_2S_1 & dS_2 & dS_2S_3 \\ dS_3S_1 & dS_3S_2 & dS_3 \end{bmatrix} = \begin{bmatrix} 0.1851 & 0.1581 & 0.1480 \\ 0.1581 & 0.1110 & 0.0000 \\ 0.1480 & 0.0000 & 0.2962 \end{bmatrix}$$

The resulting stochastic differential equation is therefore

$$dS(t) = \mu(t, S\ \ , S_2, S_3)dt + B(t, S\ \ , S_2, S_3)dW(t)$$

$$dS(t) \quad = \quad \begin{pmatrix} 0.7874 \\ 0.5653 \\ 0.5923 \end{pmatrix} dt + \begin{bmatrix} 0.1851 & 0.1581 & 0.1480 \\ 0.1581 & 0.1110 & 0.0000 \\ 0.1480 & 0.0000 & 0.2962 \end{bmatrix} dWt$$

Results

The resulting SDE was solved using the multi-dimensional Euler-Maruyama scheme for SDEs. This was achieved through a MatLab script file using the algorithm below:

Algorithm for the Multi-dimensional Euler Maruyama Scheme for SDEs

1 Start
2. Initialization and declaration of global variable
3. Generate random probabilities (substitute observed probabilities)
4. Initiate loop: for k 1: n
5. Compute change in dS_1 as $dS_1 \leftarrow P(1)+P(2)$
6. Compute in change in dS_2 as $dS_2 \leftarrow P(3)+P(4)$
7. Compute as change in dS_3 as $dS_3 \leftarrow P(5)+P(6)$

Compute simultaneous change in S_1 S_2 as dS_1S_2 as $dS_1S_2 \leftarrow P(7)+P(8)+P(9)+P(10)$
Compute simultaneous change in S_1 S_3 as dS_1S_3 as $dS_1S_3 \leftarrow P(11)+P(12)+P(13)+P(14)$
Compute simultaneous change in S_3 S_2 as dS_2S_3 as $dS_2S_3 \leftarrow P(15)+P(16)+P(17)+P(18)$
Compute simultaneous change in S_1 S_2 S_3 as $dS_1S_2S_3$ as $dS_1S_2S_3 \leftarrow P(19)+P(20)+P(21)+P(22)+P(23)+P(24)+P(25)$
$+P(26)+P(27)$

12. Compute Expectation vector as $E(\Delta S) \leftarrow \sum_{i=1}^{27} P\,\Delta S = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$ where:

$f_1 \leftarrow dS_1 + dS_1S_2 + dS_1S_3 + dS_1S_2S_3$
$f_2 \leftarrow dS_2 + dS_2S_1 + dS_2S_3 + dS_1S_2S_3$
$f_3 \leftarrow dS_3 + dS_3S \ + dS_3S_2 + dS_1S_2S_3$

13. Compute covariance Matrix as $E(\Delta S(\Delta S))^T \leftarrow \begin{bmatrix} \Delta S & \Delta S_1S_2 & \Delta S_1S_3 \\ \Delta S_2S_1 & \Delta S_2 & \Delta S_2S_3 \\ \Delta S_3S_1 & \Delta S_3S_2 & \Delta S_3 \end{bmatrix}$
14. Computer numerical solution using the Euler-Maruyama scheme given by

$$S_{n+1}^k \leftarrow S_n^k + \mu^k \Delta t + \sum_{j=1}^{k} B^{k,j} dW_t^k$$

15. End loop
16 Stop

The algorithm was implemented through a MatLab script and the result of the simulation is as shown by the graph below:

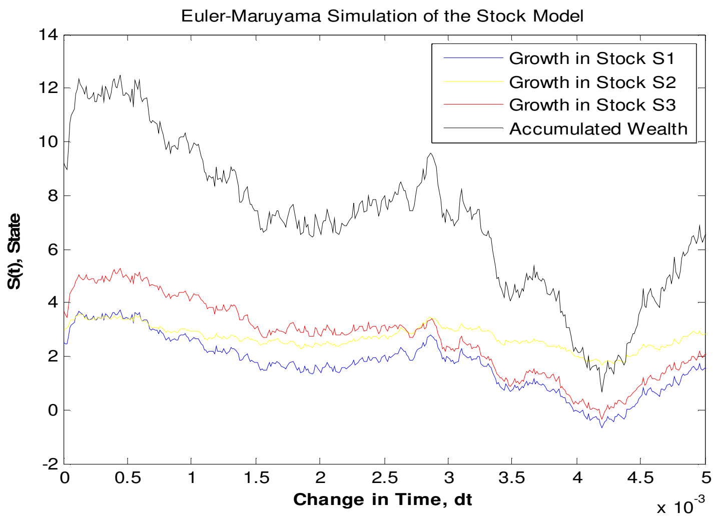

Figure 2: *Euler Maruyama simulation of the stock model showing growth in the*

The figure above shows the growth of the three stocks and the accumulated wealth pattern from the three stocks which is the cumulative sum of growth of the three stocks.

The prices of the stocks observed were compared with the simulated prices. It was simulated prices is found to be sufficiently close to the observed prices and hence could be used for forecasting. It was noted that if the deviation of the simulated prices from the observed increases as the number of days increases. Below is a chart a comparative view of the observed and simulated prices over the period of 30 days

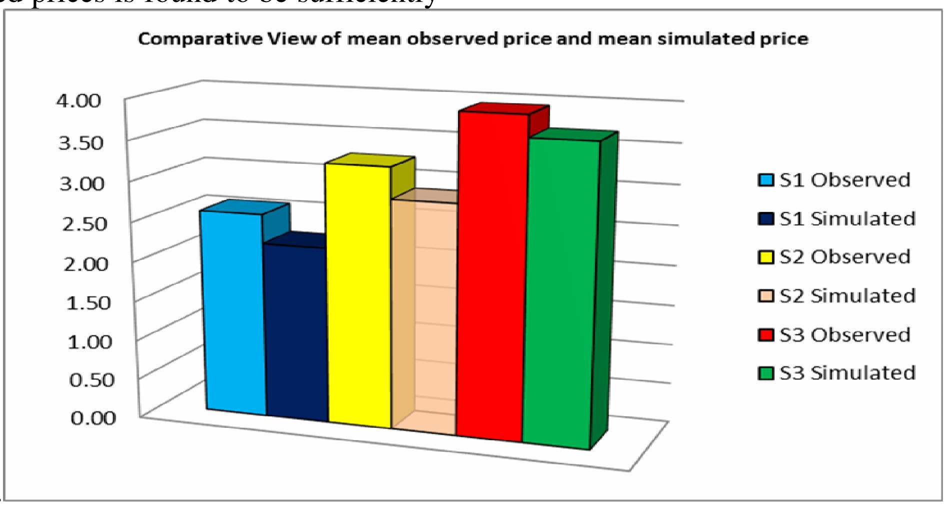

Figure 3: *Comparative View of mean observed price and mean simulated price*

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