COMPUTER SIMULATION OF GEAR CUTTING AND STRENGTH ANALYSIS WITH SPECIAL REGARD TO EXTREMELY LOW TEETH NUMBERS

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ABSTRACT
The involute gear system is widely used by gear manufacturers for spur and helical gear in mechanical power transmissions. This is because of its advantages in standardization, machining, high efficiency and adaptability to load and center distance. For mechanical power transmission, involving high speed a maximum, gear ratio in one step and pinions with minimum possible teeth number are recommended. Usually, extremely low teeth numbers, particularly for rack and pinion, are not treated in detail in the literature. These are of interest for cases of slow moving machines or manually operated mechanisms where lifetime and uniformity of motion is not crucial, but a bigger gearing ratio of single gear set for achieving cheap and small transmissions is important. For such gears with small number of teeth, modifications of parameters like teeth number, profile correction, undercut, head shortening, rack pressure angle and their interactions are investigated. An instructive computer-based modeling of involute gear cutting using the gear generation method is given preference over purely theoretical investigations. A number of gear dimensions are taken from the modeled gears and conclusions are drawn from them. Except mentioning some important features, there is no intention to represent in this article the internationally acknowledged involute gear system and its definitions in detail which can be studied separately.

INTRODUCTION
The involute gear system with its geometrical parameters has been extensively studied by a number of authors and researchers and standards have been established in various literatures. In this paper, the DIN standard is exclusively used.

The determination of gear geometry and necessary parameters are well presented in [1,2,4,5,6] for teeth numbers above 10. In general design procedures, the number of teeth \( z \geq 17 \) is considered. For special cases (starter motor pinions of IC engines) the number of teeth can be 9, 10 or 11 [4]. In [3] the possibility of using minimum number of teeth \( z_{\text{min}}=7 \) is suggested by offsetting the cutting tool or by reducing the addendum. It is possible to have pinions with minimum teeth number of 7 free from undercut and pointed tips for certain positive profile offset. Non-standard gear systems are sometimes applied in cases where investment for special cutting tools warrants mass production [8]. The problem of non-standard gears is noise and impact during operation. Smooth running of these gear sets can, however, be achieved by improving the contact ratio. The gear contact ratio in turn can be improved by increasing the number of teeth and employing a smaller pressure angle. In case of helical gear pinions, it is even possible, theoretically, to reduce the number of teeth down to one [8].

The design of gears for high lifetime considering fatigue on the tooth flank surface (pitting formation) is based on investigation results of reference gears made of different materials and different methods of production. Tooth flank endurance strength values \( \sigma_{\text{fl}} \) and tooth root bending strength values \( \sigma_{\text{f}} \) were determined from these investigation [1,2,6,8]. These values give the base for the design of continuously meshing gear sets of power transmission. Finally, rupture and plastic deformation due to possible overloading, have to be checked.

MODELING OF CUTTER RACK PROFILE
The standard rack profile according to [10] was taken as a base for modeling the cutter rack profile, shown as hatched cross section in Fig. 1. The standard normal module \( m_n \) forms the basic size parameter of the involute gear system for ease of production and ensuring interchangeability. Further, the addendum below the pitch line \( P \) of the cutter
The profile has to be extended and taken as \( m_x(1 + C) \), of gears or increased contact ratio. For such gears a certain tooth root weakening can be acceptable. Fig. 2 shows schematic representations of different offsets of the rack cutting profile with respect to the gear blank. For the same teeth number, the pitch circle diameter \( d = m \times x \) is equal for the gears shown in Fig. 2a, b and c. Positive offset in Fig. 2a leads to greater addendum circle diameter by \( 2^x m \) compared to "zero" gears; whereas negative offset Fig. 2c gears reduces the addendum circle diameter.

![Figure 1](image1.png)

**Figure 1** Model of cutter rack profile

(where \( C = c/m_x \)), to modify the dedendum of the gear, to a depth corresponding to line D.

And the fillet radius of the cutter should be:

\[
r = m_x \frac{C}{(1 - \sin \alpha)}
\]

According to [9] and [10], \( C = 0.25 \) for general cases, and \( C = 0.167 \) according to [9] for higher precision gears to increase tooth root bending strength. The rack profile angle \( \alpha = 20^\circ \) (the rack pressure angle) is most commonly in use. For special cases only \( \alpha = 12^\circ, 14.5^\circ, 15^\circ, 17.5^\circ, 22.5^\circ \) and \( 25^\circ \) are used.

The dedendum of the cutter is provided with an extension \( c' = 0.25m_x \) above line A to provide clearance outside the external gear blank diameter during machining. Therefore, a fillet radius is not necessary for this portion of the cutter.

During gear machining, the pitch line \( P \) of the cutter is tangent to the pitch circle of the gear. By doing this, "zero" gears are generated. Such type of gears with \( x < 14 \) are problematic because undercut becomes significant [1,8]. This undercutting may result in a considerable loss of tooth root width and a reduction of the bending strength.

A remarkable modification of the tooth geometry can be achieved by offsetting the cutting tool during machining. The offset is defined as \( m \times x \) [mm], where \( x \) is a non-dimensional offset factor. Positive offset can avoid or minimize undercut and strengthen the teeth, whereas negative offset (with higher teeth numbers) leads to improved overlapping of the same profile, to be extended and taken as \( m_x(1 + C) \),

\[
\frac{p}{2} = m \frac{\pi}{180}
\]

**Figure 2** Profile offset relative to gear blank

\[
\text{positive offset } x = 0.5
\]

\[
\text{negative offset } x = -0.5
\]
Profile offsetting is, therefore, an important parameter to improve the undercut problem and also to adjust center distance of mating gears.

**SIMULATION OF GEAR GENERATION**

A Pascal program was developed for determining the indicated model of the rack profile, where the following gear parameters are taken as input data:

- rack pressure angle, \( \alpha \)
- dedendum extension factor, \( C \)
- normal module, \( m_n \)

These parameters define completely the shape of the normal rack profile (perpendicular to the flanks of the gear) and are essential for the simulation of spur gear cutting.

The Pascal program is written to create an output file which compiles all necessary commands and input data in the right sequence for AutoCAD to trace the rack profile on a predefined position in an AutoCAD drawing. This procedure corresponds to the automatic generation model described in (1) (see Fig. 3), where "MAKEGEAR.SCR" is the command file and "GEAR1.DWG" and "GEAR2.DWG" are drawing files created by AutoCAD.

To develop a tooth, or even an entire set of a gear teeth, the real cutting principle like gear hobbing can be considered. The continuous straight advance of the helix during the rotation of the gear cutter equals the circular advance of the gear. The pitch line of the cutter rolls theoretically on the pitch circle of the gear which rotates about its fixed center.

An inversion of these relative motions will be when the gear is held fixed and the cutter rolls with its pitch line on the pitch circle of the gear. This inversion is the base for the model to be developed. The involute, which is delineated by the path of the contact point \( C \) of the pitch line of the rack rolling on the pitch circle of the gear, is shown in Fig. 4.
density of the rack positions was applied in order to achieve a better image of the curved tooth geometry shown in Fig. 6. The pinion appears as a negative model which was cut out of the surrounding material.

For cutting helical gears, the center of the cutter has to be inclined at an angle of \( \beta \) to the gear axis, so that the desired helix angle \( \beta \) can be attained. With this helix angle \( \beta \), the transverse pitch \( P_t \) is obtained from

\[
P_t = \pi m_r = \frac{P}{\cos \beta} = \frac{\pi m_n}{\cos \beta} \quad (4)
\]

where \( P \) is the normal pitch.

For modeling with AutoCAD, the unstretched basic rack profile was first traced and defined as a block. During insertion of this block, Fig. 6 the \( x \)-scale factor \( 1/\cos \beta \) was applied whereas the \( y \)-scale factor remained as 1. This stretched rack image was defined again as a new block ("rack2") and inserted at a different defined positions. This will give the image of one tooth in its tangential direction and eventually the image of the front section of the helical gear as shown in Fig. 6.

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Similarly, the resolution and the degree of accuracy of the tooth profile obtained by 45 insertion points of the rack can be observed in Fig. 8. It shows a good approach to the real shape of the enveloped flank shortly above the pitch circle where it is much curved.

**PARAMETER INFLUENCE**

The geometric gear parameters required for calculations were directly determined on the PC screen. Since AutoCAD is a vector based drawing and calculation program, it enables to "measure"
distances and angles with any desired exactness which has to be preselected and set by the number of digits of the magnitudes. The precision was set to a nanometer meter \((10^{-9} \text{ m})\). The rack and pinion (spur gear), as a typical drive in mechanical engineering, was selected as an example for checking the influences of different important parameters such as profile offset, head shortening factor and contact ratio on a gear with low tooth number.

Determination of the least possible number of teeth of a pinion, with acceptable effects of these parameters, is the main objective of this paper. To analyze the effect of these important parameters on pinions with small number of teeth, the following assumptions were considered:

Assumption:
- rack and pinions have the theoretical correct profile,
- no backlash of the flanks,
- teeth are rigid with no deformation under any load.

The values taken for these parameters are:
- Rack module \(m_r = 6 \text{ mm}\)
- Pressure angle \(\alpha_r = 20^\circ\)
- Dedendum extension \(C = 0.25\)
- Helix angle \(\beta = 0^\circ\)

OPTIMUM PROFILE OFFSET

To analyze the profile offset, gears with teeth numbers from 10 down to finally 4 were modified. To avoid undercut completely, the profile offset factor \(x\) has to be taken in such a way that the addendum line of the basic rack profile passes through the tangent point of the pressure line (line of action) and the pinion base circle (see Fig. 9b).

The offset limit that avoids or minimizes undercut from [1,3,8] is

\[
x_{\text{lim}} = 1 - \frac{z}{2} \sin^2 \alpha_p
\]

(5)

For a tolerable undercut, the profile offset factor (for \(x_{\text{lim}} = 14, x = 0, \alpha_p = 20^\circ\)) is given by

\[
x_{\text{opt}} = 0.82 - 0.0586 x
\]

(6)

To avoid undercut, the \(x\) values for small number of teeth will be bigger. This, however, will result in a pointed shape of the tooth profile. To minimize pointed tips of the tooth profile and to decrease the undercut at the same time, an optimum offset factor and head shortening factor \(k\) can be considered.

HEAD SHORTENING FACTOR

Tooth tip pointing can be corrected using Eq. (6) and (7). The pinion for \(x = 12\) (in Fig. 5) has already got a very small top land width where the addendum (gear blank diameter) was taken to be \(2m_r (1+x)\) which is bigger than the pitch circle. The addendum circle of gears with small number of teeth will have pointed tips (see circle “A” in Fig. 6). Therefore, the addendum circle can be modified using Eq. (7) to avoid pointed tips [1,2,6,8].

\[
d_a = \frac{m_r z}{\cos \beta} + 2m_r (1 + x - k)
\]

(7)

where \(k\) is the head shortening factor.

For mating gears \(k\) is usually determined depending on the offset factors and the theoretical and real center distance [1,2,6,8]. Obviously, this does not apply for the rack and pinion drive. In this article, \(k\) was determined so as to keep the top land width of the teeth equal to one third of the normal module i.e \(t_l = 1/3m_r\).

And \(k\) is given by the relation:

\[
k = \frac{(r_h z)_{z=0} - (r_h d_{a,max} - 1/2)}{m_r}
\]

(8)

CONTACT RATIO

For the determination of the transverse contact ratio of mating gears, neither the equations nor the graphs indicated in the literature for gear sets with finite teeth numbers could be used for rack and pinion drive. The length of the contact line \(c_l\) (path of contact, measured from the rack addendum intersecting the pressure line up to the intersection with the real addendum circle) was, therefore, determined principally on the monitor, and the transverse contact ratio \(v_x\) was calculated from Eq. (9).

\[
v_x = \frac{c_l}{m_r \pi \cos \alpha_r}
\]

(9)

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The line of action is exactly placed in AutoCAD. The exactness of the "measured" length $c_l$ of the contact line depends on the possible error introduced during the determination of the addendum radius $r_a$ for the defined top land width which in turn depends on the correctness of the involute represented by various positions of the rack flank generating lines. This problem is originated by nearly parallel and very cramped insertion area of the rack flank lines near the tooth tip and therefore, it is not a stochastic but a principal error due to the density of the rack position.

To analyze this problem, a pinion with $z = 10$ was generated two times with different cutting line densities (with 45 and 90 different rack positions). The addendum circle $r_a = 38.30$ mm was maintained. This gives necessarily the same contact line length $c_l$. The length of the top land was determined $t = 2.010805$ mm for 45 positions and $t = 1.999997$ mm for 90 positions making a relative deviation of 0.54%. 90 rack positions is practically exact enough and was therefore applied for this purpose.

**CONDITIONS FOR UNDERCUT**

Non-offset spur gears with teeth numbers from 14 to 16 are acceptable as undercut free [1,8], although a negligible undercut appears. Generally, undercut occurs when portions of the rack cutter flank (or involute portions of the pinion cutter) passes behind the tangent point $T$ of the line of action on the base circle. This is the case shown in Fig. 9 where the addendum line of the mating rack intersects the extended line of action at $G$ instead of $T$. The two teeth and the rack are positioned in the figure such that the right tooth (assuming the pinion as driver rotating ccw) just starts touching the rack flank, where as the left tooth is about to clear the contact point at $E$ before ending the contact point at $E$. This gives a short period in which two pair of teeth are still in engagement and can be expressed by the contact ratio $e_c$.

Figures 9a through 9d, show the locations of interest with increasing degree of magnification. Points 1 and 2 were added (see Fig. 9c and 9d), to mark the flank of the rack. The undercut begins just outside the base circle down to the root (see Fig. 9c and 9d).

![Figure 9 Mesh of Rack with Undercut Pinion](image)

$z=6, x=0.467867, k=0.17271, e_c=1.02166$

(pinion flank omitted in Fig. "b")

An auxiliary circle was added to the image dividing the involute from the undercut portion on the flank. Point A (Fig. 9b) marks the intersection of this.
auxiliary circle with the line of action and the beginning of the contact. The path of the contact reaches from A to E (Fig. 9b). The external portion on this standardize rack flank from point 1 to point A will not be in contact with the tooth. The rack addendum could therefore be reduced when meshing with undercut teeth.

**STRESS ANALYSIS**

**Maximum Bending Stress**

In comparison to transmission gearing of machines and vehicles, the rack and pinion drive may be considered as a slow moving mechanism with small number of contact cycles during a life time. Fatigue on the flank contact surface or fatigue due to bending does not usually happen with this type of gearing. For such cases (contact cycle numbers <10^3 [8]), the design is usually based on the nominal bending stress only. Fig.10 was adapted from [8] showing schematically the cantilever beam generally taken as base for tooth root bending. Due to the small contact ratio, it is always safe to consider the maximum nominal bending stress for only one tooth. This limiting situation is clearly shown in Fig.9a. If the pinion is assumed to be the driver and rotates in clockwise direction, its left tooth remains in contact with the tooth of the rack just when the next pair of teeth begins to move into contact.

Fatigue fracture caused by bending stress and sudden overloading, starts most likely at the transitional curve from the flank to the tooth root on the tension side [1,4,6,8]. The 30° tangent on the fillet curve was found to be the place where cracks really appear and it is, therefore, usually considered as the critical cross-section. The magnitudes $s_p$ and $b$ of this cross-section are required for calculation of the section modules. The magnitude $b$ is the flank width of the gear. Resolving of the normal force $F_n$ gives the radial and tangential forces $F_r$ and $F_t$ respectively,

$$F_t - F_b \cos \alpha_p$$

$$F_r = F_b \sin \alpha_p$$

The compressive stress due to the radial force, shear stress and friction force at the contact area are usually neglected in this strength analysis. To find out the tooth root thickness $s_p$, and the bending arm $h_{br}$, a triangle is constructed. At times, there may be doubts whether the 30° tangents are appropriate for determining bending arm $h_{br}$ and tooth root thickness $s_p$. Therefore, several sections were traced at the tooth root within the first series of investigation to get both parameters modified. Fig. 11 shows a tooth profile obtained with all the necessary data, including a 60° triangle. A graph was added to this figure which indicates the development of the form factor $Y_p$ at five different root sections. The maximum value of $Y_p$ occurs approximately where the 30° tangent touches the fillet curve (compare Fig.12). Obviously, the deviation of the form factor obtained at 30° tangent from the maxima is negligible for pinions with small number of teeth. The 30° tangent is, therefore, maintained as one of the basis for strength analysis. The basic equation for estimating the nominal bending stress is given by Eq. (12).
Where: \(b = \text{flank width}\)
\(s_p = \text{tooth thickness at critical cross-section}\)
\(h_p = \text{bending arm length}\)

The geometric relation to determine the form factor is obtained from the following equation.

\[
Y_F = \frac{(h_p/m_p) \cdot \cos \alpha_F}{(s_p/m_p)^2 \cdot \cos \alpha_F}
\]  

\[(13)\]

Taking into account possible overloading of the gear, the maximum tangential force \(F_{r \text{, max}}\) is used to determine the maximum bending stress \(\sigma_{b \text{, max}}\). This is to be related to the yield strength \((\sigma_{\text{yield}} - \sigma_0)\) or the tensile strength \(\sigma_t\) for brittle materials. The safety factor guarding against rupture or plastic deformation is obtained from:

\[
S_F = \frac{\sigma_t}{\sigma_{r \text{, max}}} \geq 1.3
\]  

\[(14)\]

The form factor of gear 1, \(Y_{F1}\) and its meshing conditions, generally, depend on the geometry of the mating gear. Furthermore, the form factor of the mating gear \(Y_{F2}\) can be different in magnitude, if its tooth geometry and particular meshing conditions are different from the first. Therefore, the form factor is determined in this case for the rack as well. Whereas, the bearing pressure between of the mating gears is equal for both surfaces in contact. The contact surface strength may vary depending on the material properties alone.

**Optimum Parameters of Rack and Pinion drives**

The method used in this paper can be applied in design of similar rack and pinion drives to determine the geometry and bending strength of gears with different profiles.

Figure 13 shows the influence of the pinion profile offset on geometric and strength parameters of the rack and pinion drive for a 6 teeth pinion. Maximum contact ratio can be attained from the optimum profile offset according to Eq. (6). The pinion teeth become slightly stronger with greater offset. This is accompanied, however, by increased head shortening to maintain the specified top land width which also in turn causes the reduction in the contact path. Therefore, the \(x_{\text{opt}}\) is recommended for any gear design that minimizes the head shortening.

In rack and pinion drives, the rack should be separately checked for strength, since it is obviously weaker than the pinion.

If the optimum profile offset factor is taken, it is possible to attain a maximum transmission ratio of a rack and pinion drive, even for the number of teeth of the pinion down to five (see Fig.14). For this
extreme case, the contact ratio remains still above one. Because of the problem of tooth pointing, the addendum diameter has to be reduced by using increased head shortening factor \( k \). For constant rack tooth geometry, the form factor and bending stress of pinions with smaller tooth number increase due to the decrease in the contact ratio.

\[
Y_{E_{a}}, Y_{p}(\text{rack}), Y_{p}(\text{pinion})
\]

\[
\alpha = 20^\circ, \beta = 0
\]

\[
t/m = 1/3
\]

\[C = 0.25\]

\[\text{pinion: 6 teeth}\]

**Figure 14** Important rack and pinion parameter with optimum profile offset.

This of course makes the contact points of both rack and pinion move toward the tooth tips. The form factor of the rack is always bigger than that of the pinion for teeth number less than 15. For pinions with teeth number \( z \geq 15 \), the pinion form factor is bigger than that of the rack (not shown in Fig 14).

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contact would push towards the rack flanks and may act as scrapers and eventually lead to fast wearing out of the rack or the pinion flanks.

Furthermore, tooth tip angles of those pinions in consideration range from 87.5° for \( z=10 \) to 114.5° for \( z=4 \), if no top land is considered. The pinion tooth top land width is categorized in 3 stages, namely 1/3, 1/6 or 0 of the normal module (see Fig.15).

The head shortening factor \( k \) is also classified accordingly. A pinion with teeth number \( z=4 \), and with no top land having an optimum offset factor, \( x=x_{oo} \), seems to be acceptable because the contact ratio approaches unity. As long as a pair of teeth start touching each other through the undercut portion of the involute just while the other pair loose the contact, the undercut can be tolerated and the respective profile offset factor can be taken as a design parameter.

This makes the pinion tooth tips to apply a pulling force on the rack teeth when they are losing contact at the end of the path of contact. Therefore, a pointed tooth tip may not be considered to be problem in design. However, this could be more difficult when the rack is used as the driver, where the tooth tips in

\[ Y_F, \varepsilon_c, x_{\text{top land}} \]

\[ \varepsilon_c \quad (\text{contact ratio}) \]

\[ X_{\text{top land}} \]

\[ K \]

Figure 15 Characteristic of the top land width for various parameters

0: pointed tips
1: \( t_1 = 1/6m_n \)
2: \( t_1 = 1/3m_n \)

In addition to this, a smaller top land width of the pinion leads to a reduction in bending stress of the rack. This can be observed in Fig.16, where the form factor decreases as the top land width decreases.

Non Standard Rack Profile

The rack profile applied so far corresponds to standard profiles according to [10]. It is obvious that the teeth of a rack are simpler to cut (for example by
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shaping) than those for gears. It is customary to correct positively pinions with extremely low teeth number to avoid undercut. The flanks of the rack cutter used to manufacture these type of pinions are never fully utilized near the dedendum of the pinion due to their short heads. The addendum portion of the rack may have no contact with pinions which are exposed to undercut. This influence of the rack profile on such pinions is not considered in this paper.

A considerable strengthening of the rack tooth root can be achieved by modifying the rack profile individually for every pinion, so that the clearance between pinion teeth and rack dedendum remains constant $C.m_n = 0.25.m_n$. The rack dedendum height $h_n$ in this case, is

$$h_n = m_n (1 + C - k)$$

The head shortening factor $k$ of the pinions with different number of teeth can be taken from Fig.15. Rack addendum height, pitch, pressure angle and transition radius at the root require no modification. In Fig.16, the form factors of the rack with dedendum height according to equation (12) and with different top land width, are shown. The influence of the top land width on the form factor is clearly demonstrated in Fig.16.

CONCLUSION

To attain maximum gear reduction ratio, a small teeth number for pinions and maximum teeth number for gears is required. To achieve such type of speed reduction, a pinion with extreme low number of teeth can be used.

In this paper, gears with low number of teeth (less than 14) are analyzed; undercut, contact ratio and strength are also discussed.

To avoid undercut and to improve contact ratio, a reasonable positive profile offset can be made. The profile offset, which is also an important parameter to increase the strength of the tooth, can also be increased by correcting the profile offset positively. The parameters, bending arm length $h_n$ and tooth thickness at critical cross-section $s_n$ are measured directly from the drawing using AutoCAD and the form factor $Y_r$ is calculated for number of teeth 4 - 10 and represented in graphs which could be applied by gear designers.

REFERENCES


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